POW #7 : Solution

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Let $f(x) = 5^{x^2} - 4^{x^2}$ and $g(x) = 4^x - 3^x$. If $x = 0,\ 1$, then $f(x) = g(x)$ obviously. I will show $f(x) = g(x)$ has exactly two real solutions 0, 1.

Claim 1 $0 > x \Longrightarrow f(x) > g(x)$

Proof. If $0 > x$, then $4^x - 3^x < 0$. But $5^{x^2} - 4^{x^2} > 0$.

Claim 2 $x > 1 \Longrightarrow f(x) > g(x)$

Proof. First I will show $5^{x^2} - 4^{x^2} > 5^x - 4^x$.
$5^{x^2} - 4^{x^2} > 5^x - 4^x \Longleftrightarrow 5^x(5^{x(x-1)} - 1) > 4^x(4^{x(x-1)} - 1)$
And it is clear $5^z > 4^z$ and $5^z - 1 > 4^z - 1 > 0$ for $z > 0$.
Now we need to prove $5^x - 4^x > 4^x - 3^x$.
For fixed $x > 1$, let $h(y) = (y+1)^x - y^x$.
$\frac{d}{dy}h(y) = x(y+1)^{x-1} - xy^{x-1} = x((y+1)^{x-1} - y^{x-1})$.
So, $\frac{d}{dy}h(y) > 0$ when $y > 0$ and $h(y)$ is strictly increasing when $y > 0$.

Claim 3 $1 > x > 0 \Longrightarrow g(x) > f(x)$

Proof. First I will show $5^x - 4^x > 5^{x^2} - 4^{x^2}$.
$5^x - 4^x > 5^{x^2} - 4^{x^2} \Longleftrightarrow 5^x(1 - 5^{x(x-1)}) > 4^x(1 - 4^{x(x-1)})$
And it is clear $5^z > 4^z$ for $z > 0$ and $1 - 5^z > 1 - 4^z > 0$ for $z < 0$.
Now we need to prove $4^x - 3^x > 5^x - 4^x$.
For fixed $1 > x > 0$, let $h(y) = (y+1)^x - y^x$.
$\frac{d}{dy}h(y) = x(y+1)^{x-1} - xy^{x-1} = x((y+1)^{x-1} - y^{x-1}) = x\left(\frac{1}{(y+1)^{1-x}} - \frac{1}{y^{1-x}}\right)$.
So, $\frac{d}{dy}h(y) < 0$ when $y > 0$ and $h(y)$ is strictly decreasing when $y > 0$.

Now we can see that $f(x) = g(x)$ has exactly two solutions which are 0 and 1.