Solution for POW #5

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Remind the notion of the principle of duality in projective geometry. (I will omit some other properties for example 'Every two distinct lines must have exactly one common point.'.)

Property 1 (The two-dimentional principle of duality) *Every defintion remains significant, and every theorem remains true, when we interchange point and line (consquently also certain other pairs of words such as join and meet, collinear and concurrent etc.).*

So we can restate the problem in its dual form.

> *Given finite set of lines in the real projective plane, each colored either red or blue and not all concurrnet, there must be a monochromatic point i.e. there is an intersection point incident with only lines of one color.*

By assumption, there must be more than two lines. Now suppose there are no such monochromatic points. If no three lines are concurrent, then any two lines of same color intersects in a monochromatic points. Hence there is an intersection point p incident with at least three of lines.

Let l_1, l_2 and l_3 be the lines through p. Since the lines through p are not all of the same color, we may assume l_1 and l_2 have color red and l_3 has color blue WLOG. If the lines which are not incident with p has all same color red, then any one of them must intersect l_1 and that point would be a monochromatic point of color red. Hence there is a line l' of color blue not incident with p. If we ignore other lines, then there are two triangles $l_1 l_3 l'$ and $l_3 l_2 l'$ which is cut by l_3 (share an edge which is the part of the line l_3). We can get a larger merged triangle removing the line l_3. [I] Call this triangle as a 'characteristic triangle' of the arrangement. Any other triangle of this form, where l_1' and l_2' have the same color blue and l_3' and l'' have the same color red (and same as previous argument, l_1', l_2' and l_3' are incident with point p' i.e. concurrent) we also refer to as a characteristic triangle.

[I] Remind that the real projective plane $\mathbb{R}\mathbf{P}^2 = \mathbf{S}^2/x \sim -x$. Lines are like great circles on the sphere. There are some triangles which is maid by three lines. And there are triangles that we want.

Since we already know the fact that there can be only finitely many characteristic triangles ($\because$ There are only finitely many lines an also finitely many intersection point.), there must be one which is minimal (which means there is no other characteristic triangle contained in it.). Suppose l_1, l_2, l_3 and $l^\star$ forms this minimal characteristic triangle $l_1l_2l^\star$ which is cut into two triangles by l_3.

Claim 1 *The intersection point q of l_3 and $l^\star$ is monochromatic.*

Suppose not. Since l_3 and $l^\star$ have the same color red, there exist a line l_c of color blue passing through q. l_c must cut either triangle $l_1l_3l^\star$ or $l_3l_2l^\star$. WLOG, assume l_c cut $l_1l_3l^\star$. But then the for lines l_c, l_1, l_3 and $l^\star$ bears a characteristic triangle $l_1l_3l^\star$ which is contained in $l_1l_2l^\star$. Contradiction.

Q.E.D.