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Let $a_n(i) = \sqrt{1+(i-1)\sqrt{1+i\sqrt{\cdots\ \sqrt{1+n\sqrt{1+n+1}}}}}$

Especially, $a_n(3) = a_n$ which defined in the problem.

Let $b_n(i) = i - a_n(i)$

Let $c_n(i) = \frac{b_n(i)}{i-1}$.

Prop 1. $\lim_{n\to\infty} a_n(i) = i$. i.e. $\lim_{n\to\infty} b_n(i) = 0$.

proof is not hard, left for the readers.

Hint - $3 = \sqrt{1+8} = \sqrt{1+2\sqrt{16}} = \sqrt{1+2\sqrt{1+3\sqrt{25}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}} = \cdots\cdots$

Prop 2. $a_n(n+2) = \sqrt{n+2}$. $c_n(i) \le 1$ for $i = 3, 4, \ldots, n+1, n+2$

pf) by definition and calculation.

Prop 3. if $\lim_{n\to\infty} \frac{3-a_{n+1}}{3-a_n} = \frac{1}{2}$, then $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{a_n - a_{n-1}} = \frac{1}{2}$.

pf) Let $d_n = a_n - a_{n-1}$. Then $\sum_{k=n+1}^{\infty} d_n = 3 - a_n$. If $\lim_{n\to\infty} \frac{3-a_{n+1}}{3-a_n} = \frac{1}{2}$,

$\lim_{n\to\infty} \frac{\sum_{k=n+2}^{\infty} d_n}{d_{n+1} + \sum_{k=n+2}^{\infty} d_n} = \frac{1}{2}$. Both numerater and denominator converges, and both are not zero at any time. so we can reverse it.

$\lim_{n\to\infty} \frac{d_{n+1} + \sum_{k=n+2}^{\infty} d_n}{\sum_{k=n+2}^{\infty} d_n} = \lim_{n\to\infty} \frac{d_{n+1}}{\sum_{k=n+2}^{\infty} d_n} + 1 = 2$. thus $\lim_{n\to\infty} \frac{d_{n+1}}{\sum_{k=n+2}^{\infty} d_n} = 1$.

Similarly, from $\lim_{n\to\infty} \frac{3-a_{n+2}}{3-a_n} = \frac{1}{4}$, we can know $\lim_{n\to\infty} \frac{d_n}{\sum_{k=n+2}^{\infty} d_n} = 2$.

Thus $\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = \frac{1}{2}$, so we get $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{a_n - a_{n-1}} = \frac{1}{2}$. Q.E.D.

Proof of the problem

We could easily know that

$$i+1-b_n(i+1)=a_n(i+1)=\frac{a_n(i)^2-1}{i-1}=\frac{(i-b_n(i))^2-1}{i-1}=i+1+\frac{b_n(i)^2-2ib_n(i)}{i-1}$$

$$b_n(i+1)-2\frac{i}{i-1}b_n(i)+\frac{b_n(i)^2}{i-1}=0$$

$$c_n(i+1)-2c_n(i)+\frac{i-1}{i}c_n(i)^2=0 \ \ldots\ (1)$$

From this equation, we can derive these two inequality.

$$c_n(i+1)=2c_n(i)-\frac{i-1}{i}c_n(i)^2<2c_n(i) \ \ldots\ (2)$$

$$c_n(i+1)-2c_n(i)+c_n(i)^2>0 \ \ldots\ (3)$$

(3) is 2nd-order inequlaity, so we can prove it well known formula.

$0<c_n(i)<1-\sqrt{1-c_n(i+1)}$ ...(4) (Since $c_n(i)<1$)

From (2) we get $c_n(3)>\frac{c_n(4)}{2}>\cdots>\frac{c_n(n+2)}{2^{n-1}}=\frac{b_n(n+2)}{(n+1)2^{n-1}}=\frac{n+2-\sqrt{n+2}}{(n+1)2^{n-1}}$

From (4) we get

$$c_n(3)<1-\sqrt{1-c_n(4)}<1-\sqrt{1-(1-\sqrt{1-c_n(5)}}=1-(1-c_n(5))^{\frac{1}{4}}<\cdots<1-(1-c_n(n+2))^{\frac{1}{2^{n-1}}}$$

$$=1-(1-\frac{1}{2^{n-1}}\times c_n(n+2)-O(\frac{1}{2^{2n-2}}c_n(n+2)^2))=\frac{n+2-\sqrt{n+2}}{(n+1)2^{n-1}}+O(\frac{1}{2^{2n-2}})$$

$0<c_n(n+2)<1$ guarantee that we can put all redundant terms to $O(\frac{1}{2^{2n-2}})$.

Since $b_n(3)=2c_n(3)$,

$$\frac{n+2-\sqrt{n+2}}{(n+1)2^{n-2}}<b_n(3)<\frac{n+2-\sqrt{n+2}}{(n+1)2^{n-2}}+O(\frac{1}{2^{2n-2}})$$

Hence we get $\lim_{n\to\infty}\frac{b_{n+1}(3)}{b_n(3)}=\frac{1}{2}$.

and by Prop 3 with this result, $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{a_n-a_{n-1}}=\frac{1}{2}$.