Solution for POW #3

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Claim 1 *A square integer matrix X has an inverse with integer entries if and only if* $\det(X) = \pm 1$.

(Proof of **Claim 1**.)

($\Rightarrow$) $\det(X) \cdot \det(X^{-1}) = \det(I) = 1$.
Since $\det(X)$ is an integer, $\det(X) = \pm 1$.

($\Leftarrow$) If $\det(X) = \pm 1$, then $\pm \mathrm{adj} X$, the classical adjoint of X is an inverse of X with integer entries. ◯

To show $A + 4B$ is invertible, I will argue about a more delicate issue.
(a simple generalization of this problem)

Claim 2 *The $n \times n$ matrices A and B are given.*
Let $f(x) = \det(A + xB)$
Assume that we have $2n + 1$ distinct x's which satisfy $f(x) = \pm 1$.
Then $A + xB$ is invertible for every $x \in \mathbb{F}$ where $\mathbb{F}$ is any field.

(Proof of **Claim 2**.)

$f(x)$ is a polynomial of degree at most n such that $f(x) = \pm 1$ for given distinct x's.

By Pigeonhole principle, there are $n + 1$ or more distinct x's which have same values of $f(x)$.

Then f must be a constant function.
($\because$ WLOG, asuume there are $n + 1$ distinct x's such that $f(x) = 1$.
Then the equation $f(x) - 1 = 0$ has $n + 1$ distinct roots.
But the degree of the polynomial f is at most n.
Hence f must be a constant.)

Therefore $f(x) = +1$ for all $x \in \mathbb{F}$ or $f(x) = -1$ for all $x \in \mathbb{F}$ which means $A + xB$ is invertible for every $x \in \mathbb{F}$. ◯

$A + 4B = (A + B) + (A + 3B) - A$ is an integer matrix.
So, from the last sentense of (Proof of **Claim 2**), $A + 4B$ has an inverse with integer entries (by **Claim 1**).