Solution of Problem 2008-2

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2008/09/17

> Prove that if x is a real number such that $0 < x \le \frac{1}{2}$ then x can be represented as an infinite sum
>
> $$x = \sum_{k=1}^{\infty} \frac{1}{n^k}$$
>
> where each n_k is an integer such that $\frac{n_{k+1}}{n_k} \in \{3, 4, 5, 6, 8, 9\}$.

Solution
Let us remind the definition of decimical expansion; $x = \sum_{k=1}^{\infty} \frac{d^k}{10^k}$(for $0 \ge x \ge 1$). For example, when $x = 1/3$,

$$x \times 10 - 3 > 0, x \times 10 - 4 < 0 \text{ so, } d_1 = 3$$
$$(x - \frac{3}{10}) \times 100 - 3 > 0, (x - \frac{3}{10}) \times 100 - 4 < 0 \text{ so, } d_2 = 3$$
$$\cdots$$
$$(x - \sum_{k=1}^{n-1} \frac{3}{10^k}) \times 10^n - 3 > 0, (x - \sum_{k=1}^{n-1} \frac{3}{10^k}) \times 10^n - 4 < 0 \text{ so, } d_n = 3$$

I'll use similar argument to solve this problem. For convience, let $d_1 \overset{def}{=} n_1, d_k \overset{def}{=} \frac{n_k}{n_{k-1}} \in \{3, 4, 5, 6, 8, 9\}$(so, $n_k = \prod_{i=1}^{k} d_i$) for $k > 1$. For some $d_1 \in \mathbb{N}$, $x - \frac{1}{d_1} > 0$. Likewise, for some $d_2 \in \{3, 4, 5, 6, 8, 9\}$ such that $(x - \frac{1}{d_1}) \times d_1 - \frac{1}{d_2} > 0$. Reapeating this, for some $d_k \in \{3, 4, 5, 6, 8, 9\}$ such that $\underbrace{(\cdots((}_{k-1} x - \frac{1}{d_1}) \times d_1 - \frac{1}{d_2}) \times d_2 - \frac{1}{d_3}) \cdots - \frac{1}{d_{k-1}}) \times d_{k-1} - \frac{1}{d_k} > 0$ and repeat this process as $k \to \infty$. Let $x_k \overset{def}{=} \sum_{i=1}^{k} \frac{1}{n_i} = \sum_{i=1}^{k} \prod_{j=1}^{i} \frac{1}{d_j}$, $a_1 \overset{def}{=} x, a_k \overset{def}{=} (x - x_{k-1}) \times n_{k-1} = \underbrace{(\cdots((}_{k-1} x - \frac{1}{d_1}) \times d_1 - \frac{1}{d_2}) \times d_2 - \frac{1}{d_3}) \cdots - \frac{1}{d_{k-1}}) \times d_{k-1}$.

Then, these process is reduced by

$$a_k - \frac{1}{d_k} > 0 \text{ for some } d_k, a_{k+1} = (a_k - \frac{1}{d_k}) \times d_k = a_k d_k - 1$$

Using these equations, we will prove that any number in $(0, \frac{1}{2}]$ can be represented as 'Strange representation' by proving some lemmas. Note that $\frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{3^k}, \frac{1}{8} = \sum_{k=1}^{\infty} \frac{1}{9^k}$ can be represented clearly.

Lemma 1. *For all $\frac{1}{8} \leq a_k \leq \frac{1}{2}$, there is $d_k \in \{3, 4, 5, 6, 8, 9\}$ such that $\frac{1}{8} \leq a_{k+1} = a_k d_k - 1 \leq \frac{1}{2}$*

Proof.

$$\frac{1}{8} \leq a_k d_k - 1 \leq \frac{1}{2}$$

is equivalent to

$$\frac{9}{8}\frac{1}{d_k} \leq a_k \leq \frac{3}{2}\frac{1}{d_k}.$$

So, define the closed intervals I_n such that $I_n = [\frac{9}{8}\frac{1}{n}, \frac{3}{2}\frac{1}{n}]$ for $n = 3, 4, 5, 6, 8, 9$. Then,

$$I_9 = [\frac{1}{8}, \frac{1}{6}], I_8 = [\frac{9}{64}, \frac{3}{16}], I_6 = [\frac{3}{16}, \frac{1}{4}], I_5 = [\frac{9}{40}, \frac{3}{10}], I_4 = [\frac{9}{32}, \frac{3}{8}], I_3 = [\frac{3}{8}, \frac{1}{2}].$$

So, it implies that if $a_k \in I_n$ for some I_n, then $\frac{1}{8} \leq a_{k+1} = a_k n - 1 \leq \frac{1}{2}$. But, since $\sup I_9 = \frac{1}{6} > \inf I_8 = \frac{9}{64}$, $\frac{3}{16} = \frac{3}{16}$, $\frac{1}{4} > \frac{9}{40}$, $\frac{3}{10} > \frac{9}{32}$, $\frac{3}{8} = \frac{3}{8}$, $\cup I_n = [\frac{1}{8}, \frac{1}{2}]$. That is, there is at least one n $\in \{3, 4, 5, 6, 8, 9\}$ such that $a_k \in I_n$ for all $\frac{1}{2} \leq a_k \leq \frac{1}{2}$ and it implies that for every $\frac{1}{2} \leq a_k \leq \frac{1}{2}$, there is $d_k \in \{3, 4, 5, 6, 8, 9\}$ such that $\frac{1}{8} \leq a_k d_k - 1 \leq \frac{1}{2}$ and it proves the lemma. □

Lemma 2. *For all $\frac{1}{8} \leq x \leq \frac{1}{2}$, x can be represented as 'Strange representation'.*

Proof. By lemma 1, there is $d_1 \in \{3, 4, 5, 6, 8, 9\}$ such that $a_2 \in [\frac{1}{8}, \frac{1}{2}]$ since $x = a_1 \in [\frac{1}{8}, \frac{1}{2}]$. Likewise we can prove that for every $k \in \mathbb{N}$ there is $d_k \in \{3, 4, 5, 6, 8, 9\}$ such that $a_k \in [\frac{1}{8}, \frac{1}{2}]$ since since $a_{k-1} \in [\frac{1}{8}, \frac{1}{2}]$(by using induction). We also need to prove wheter $\lim_{k\to\infty} x_k = x$. That is, for $\forall\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$k \geq N \Rightarrow |x - x_k| < \epsilon.$$

By definition, $a_k = (x - x_k) \times n_{k-1}$. So $|x - x_k| = \frac{a_k}{n_k}$. Since a_k are bounded above($\leq \frac{1}{2}$), and $n_k = \prod_{i=1}^{k} d_i \geq 3^k$ so we can make $\frac{a_k}{n_k}$ be arbitrary small.

So $\lim_{k\to\infty} x_k = x$ and $\forall x \in [\frac{1}{8}, \frac{1}{2}]$ can be represented as 'Strange representation'.

□

Now we can prove the original problem.

Theorem 3. *For all $0 < x \le \frac{1}{2}$, x can be represented as 'Strange representation'.*

Proof. If $\frac{1}{8} \le x \le \frac{1}{2}$, we proved the statement by Lemma 2. If $0 < x < \frac{1}{8}$, we will find $d_0 \in \mathbb{N}$ such that

$$\frac{1}{8} \le xd_0 \le \frac{1}{2}.$$

It is equivalent to

$$\lfloor \frac{1}{8x} \rfloor \le d_0 \le \lceil \frac{1}{2x} \rceil$$

and since

$$\lceil \frac{1}{2x} \rceil - \lfloor \frac{1}{8x} \rfloor > \frac{1}{x}(\frac{1}{2} - \frac{1}{8}) > 8 \times \frac{3}{8} = 3 > 1$$

, there must be an integer $d_0 \in [\lceil \frac{1}{2x} \rceil, \lfloor \frac{1}{8x} \rfloor]$ as required. Now define $x' = xd_0 \in [\frac{1}{8}, \frac{1}{2}]$ then x' can be represented as 'Strange representation'. Let's denote $x' = \sum_{k=1}^{\infty} n'_k = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{1}{d_i}$, then $x = \frac{x'}{d_0}$ is represented as $x = \sum_{k=1}^{\infty} \frac{1}{d_0 n'_k} = \sum_{k=1}^{\infty} \prod_{i=0}^{k} \frac{1}{d_i}$ and this is the 'Strange representation' of x. □

Remark. Unlike the continued fraction representation of the given number x, the 'Strange representation' is generally not unique. (If we ignore the trailing 1 in the continued fraction representation) For example,

$$\frac{3}{8} = \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{8} = \frac{1}{3} + \sum_{k=2}^{\infty} \frac{1}{3 \cdot 9^{k-1}}$$
$$\frac{3}{8} = \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4} + \sum_{k=2}^{\infty} \frac{1}{4 \cdot 3^{k-1}}$$

are two distinct 'Strange representation' of $\frac{3}{8}$.