POW problem 1 : Solution

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I observed some numbers using Maple, and I found a rule. Here I propose the lemma:

Lemma 1 *Given a positive integer n. If an integer a is greater or equal to n^{n-1}, then $a+1, a+2, \ldots, a+n$ have distinct prime factors, that is, we can choose distinct n primes each divides each of $a+1, a+2, \ldots, a+n$.*

If we choose k numbers among $a+1, a+2, \ldots, a+n$, then we can choose k distinct primes each divides each of them. Hence, we are done if the lemma is true.

(proof) Let $1 \le k \le n$. If $a+k$ has a prime factor $p \ge n$, then

$$\gcd(a+k, a+j) = \gcd(a+k, |j-k|) \le |j-k| < n \le p.$$

Hence, p does not divide any $a+j$, if $j \ne k$. In this case, choose p.

Now, assume that $a+k$ have only prime factors smaller than n, i.e.,

$$a+k = q_1^{e_{k1}} \ldots q_r^{e_{kr}}$$

where $2 = q_1 < q_2 < \ldots < q_r$ are all primes less than n. Notice that

$$n^{n-1} < a+k = q_1^{e_{k1}} \ldots q_r^{e_{kr}}$$

and $r < n-1$. Hence, there is $1 \le s_k \le r$ such that $q_{s_k}^{e_{ks_k}} > n$. Choose q_{s_k}.
($\because$ Product of $r < n-1$ numbers is greater than n^{n-1}.)

We need to show such s_k are distict. If $s_k = s_j$ for some $j \ne k$, then $\min\{q_{s_k}^{e_{ks_k}}, q_{s_k}^{e_{js_k}}\} > n$ divides $\gcd(a+k, a+j)$. So,

$$n < \gcd(a+k, a+j) = \gcd(a+k, |k-j|) < n.$$

It is a contradiction. So $s_j \ne s_k$ for any $j \ne k$.

Hence, we can choose different prime factors from each $a+1, \ldots, a+n$. $\square$