Ph.D. Qualifying Exam: Algebra I

February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Classify the group structure of $\mathrm{GL}_2(\mathbb{F}_2)$ which is the group of 2×2 invertible matrices over $\mathbb{F}_2$.

2. [15 pts] Let G be a finite p-group and H be a normal subgroup of order p in G. Prove that H is contained in the center of G.

3. [15 pts] Let $m \geq 3$ be an odd integer. Prove that $D_{2m} \cong D_m \times \mathbb{Z}/2\mathbb{Z}$, where D_n denotes the dihedral group of order $2n$.

4. [10 pts] Prove or disprove: Let F be an algebraically closed field of characteristic 0 Then, there exists an injective ring homomorphism $g : F[x] \to F[x]$ which is not surjective.

5. [20 pts] Let R be a principal ideal domain.
   (a) [10 pts] Prove or disprove: Every projective R-module is free.
   (b) [10 pts] Prove or disprove: For R-modules E and F_i for $i \in I$,
   $$E \bigotimes \left(\prod_{i \in I} F_i \right) \cong \prod_{i \in I} \left(E \bigotimes F_i \right).$$

6. [15 pts] Let R be a principal ideal domain and let $E \neq 0$ be a finitely generated torsion R-module. Prove that E is the direct sum
   $$E = \bigoplus_p E(p),$$
   where the sum is taken over all primes p such that $E(p) \neq 0$, and $E(p)$ denotes the submodule of E consisting of all elements $x \in E$ such that $p^r x = 0$ for some $r \geq 1$.

7. [15 pts] Prove or disprove: If two finite groups have the same character tables, then they are isomorphic.

THE END

Ph.D. Qualifying Exam: Algebra II

February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let R be a commutative ring with identity. Prove or disprove each of the following statements:

   (a) [10 pts] If M is a finitely generated R-module, then every surjective R-linear map $\varphi\colon M \to M$ is an isomorphism.

   (b) [10 pts] If A, B, C are finitely generated R-modules, $f\colon A \to B$ is an injective R-linear map, $g\colon B \to A$ is a surjective R-linear map such that $g \circ f = \mathrm{id}_A$, and $h\colon B \to C$ is a surjective R-linear map, then there is a surjective R-linear map $k\colon C \to A$ such that $k \circ h \circ f = \mathrm{id}_A$.

2. [20 pts] Let R be a local ring with the maximal ideal $\mathfrak{m}$.

   (a) [10 pts] If M is a finitely generated R-module and $\mathfrak{m}M = M$, then prove that $M = 0$.

   (b) [10 pts] Prove that every finitely generated projective R-module is a free R-module.

3. [10 pts] Let R be an integrally closed domain, F be the fraction field of R, and K be an algebraic extension field of F. For $u \in K$, prove that u is integral over R if and only if $p(x) \in R[x]$, where $p(x)$ is the minimal polynomial of u over F.

4. [20 pts] Let F be a field, K be an extension field of F, and L be an extension field of K. Prove or disprove each of the following statements:

   (a) [10 pts] If $u \in K$ is an element such that $F(u) = F(u^2)$, then u is algebraic over F.

   (b) [10 pts] If L is a radical extension of F, then K is a radical extension of F.

5. [15 pts] For an integer $n \geq 1$, let K be the n-th cyclotomic extension field of $\mathbf{Q}$, and $L := \mathbf{Q}(\sqrt[n]{2})$. If $m := [K \cap L : \mathbf{Q}]$, then prove that $m = 1$ or 2, and $m = 2$ if and only if $8 \mid n$.

6. [15 pts] Compute the Galois group of $x^4 - 2 \in \mathbf{Q}[x]$ over $\mathbf{Q}$ and the Galois group of $x^4 - 2 \in \mathbf{F}_5[x]$ over $\mathbf{F}_5$, where $\mathbf{F}_5$ is a finite field with 5 elements.

THE END

Ph.D. Qualifying Exam: Differential Geometry

February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] For $F : \mathbb{R}^4 \to \mathbb{R}$ given by $F(x, y, z, w) = xy + e^{zw}$, show that the level set $f^{-1}(3)$ is a smooth manifold.

2. [15 pts]

   (a) [10 pts] For M, N smooth manifolds without boundary, show that the tangent bundle $T(M \times N)$ is diffeomorphic to the product bundle $TM \times TN$.

   (b) [5 pts] Determine whether $T\mathbb{S}^n = \mathbb{S}^{n-1} \times \mathbb{R}$ and explain why.

3. [15 pts] Let M be a compact smooth manifold of dimension n and $f : M \to \mathbb{R}^{n+1}\setminus\{0\}$ a differentiable map. Show that there exists a straight line through the origin in $\mathbb{R}^{n+1}$ that intersects $f(M)$ only finitely many times.

4. [15 pts] Show that for any Lie group G, the rank of the exponential mapping $\exp : T_eG \to G$ is equal to $\dim G$.

5. [20 pts] For the vector fields $X = (-x + y)\frac{\partial}{\partial x} + (x - y)\frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on the plane, compute their flows and determine whether the flows commute.

6. [20 pts]

   (a) [5 points] Give the statement of Stokes' theorem.

   (b) [15 points] Evaluate the integral

   $$\int_{T^2} x^2 \, dy \wedge dz,$$

   where $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is the 2-torus embedded in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ as the subset $\{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, z^2 + w^2 = 1\}$, given the product orientation coming from the standard orientation of each $\mathbb{S}^1$.

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
February 2025

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (a) [10 pts] Prove or disprove: Two 2-manifolds are homotopy equivalent if and only if they are homeomorphic.

   (b) [10 pts] Prove or disprove: Two 3-manifolds are homotopy equivalent if and only if they have the same homology groups.

2. (a) [10 pts] Prove or disprove: The covering of a covering is a covering.

   (b) [10 pts] Prove or disprove: Given a covering space $p : \widetilde{X} \to X$ and a map $f : Y \to X$, if two lifts $f_1, f_2 : Y \to \widetilde{X}$ of f agree at one point in Y, and Y is connected, then f_1 and f_2 agree on all of Y.

3. (a) [10 pts] Let X be a connected CW complex and G a group such that every homomorphism $\pi_1(X) \to G$ is trivial. Prove or disprove that every map $X \to K(G, 1)$ is nullhomotopic.

   (b) [10 pts] Construct a $K(\mathbb{Z}_2, 1)$-space.

4. (a) [10 pts] Construct a simply connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter.

   (b) [10 pts] Construct a simply connected covering space of the space $X \subset \mathbb{R}^3$, which is the union of a sphere and a circle intersecting it at two points.

5. (a) [10 pts] Prove or disprove: There exists a space X such that $H_i(X; \mathbb{Z}) = 0$ but $H_i(X; \mathbb{Z}_2) \neq 0$ for some $i > 0$.

   (b) [10 pts] Prove or disprove: An odd map $f : S^n \to S^n$, satisfying $f(-x) = -f(x)$ for all x, must have odd degree.

THE END

Ph.D. Qualifying Exam: Algebraic Topology II
February 2025

Student ID: Name:

Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. If answers are not well organized, we will take points off. Here, when we ask you to "compute the group", this means that you have to write down the generators and relations of the the group. When we ask you to "compute the homomorphism", you need to write down the generators and relations of the two groups and finding where the generators go to under the map. (Total 100 pts.)

1. (45 pts.) Let M be a compact orientable n-manifold with boundary ∂M.

   (a) (10 pts) Show that the boundary map $H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends a generator to a class $[\partial M]$ of $H_{n-1}(\partial M)$ so that $(i_X)_*([\partial M])$ is a generator of $H_{n-1}(X)$ for each component X of ∂M under the inclusion map $i_X : X \to \partial M$.

   (b) (35 pts) Recall that the following diagram commutes up to $\pm$I.

$$\begin{array}{ccccccc} H^{k-1}(\partial M) & \to & H^k(M,\partial M) & \to & H^k(M) & \to & H^k(\partial M) \\ \downarrow [\partial M]\cap & & \downarrow [M]\cap & & \downarrow [M]\cap & & \downarrow [\partial M]\cap \\ H_{n-k}(\partial M) & \to & H_{n-k}(M) & \to & H_{n-k}(M,\partial M) & \to & H_{n-k-1}(\partial M) \end{array}$$

   Let X be $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. Let B_1 and B_2 be two disjoint compact 3-balls embedded in X. Let M be the 3-manifold $X - B_1^o - B_2^o$. Compute each group and each homomorphism of the sequences for all n, k. Describe some cycles representing $[M]$ and $[\partial M]$. (hint: 20 points for the groups, 10 points for the homomorphisms, and 5 points for $[M]$ and $[\partial M]$.)

2. (40 pts.) Let X be $\mathbf{S}^1 \times \mathbf{S}^2 \times \mathbf{S}^3$.

   (a) (10 pts.) Compute the homology groups of $H_*(X)$.

   (b) (15 pts.) Compute the cohomology ring of $H^*(X)$.

   (c) (15 pts.) Compute $\cap : H^k(X) \times H_m(X) \to H_{m-k}(X)$ for all $m \geq k$.

3. (15 pts.) Let $\mathbf{S}^n$ be the n-dimensional sphere. Let ι be the involution given by $\vec{x} \to -\vec{x}$. Let X be the space $\mathbf{S}^3 \times I/\sim$ where $(x, 0) \sim (\iota(x), 1)$.

   (a) (5 pts.) Show that X is a fiber bundle over a circle $\mathbf{S}^1$ with fibers diffeomorphic to $\mathbf{S}^n$.

   (b) (10 pts.) Write down the exact homotopy sequence of the fiber bundle $\mathbf{S}^n \to X \to \mathbf{S}^1$, and compute each term and each homomorphism for every $n \geq 1$.

THE END

Ph.D. Qualifying Exam: Real Analysis
February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Define the Lebesgue measurable sets on $\mathbb{R}$ and the Lebesgue measure m on the Lebesgue measurable sets on $\mathbb{R}$.
2. [20 pts] Let $f \in L^1(\mathbb{R})$ and define $F(x) = \int_0^x f(t)dt$. Prove that there exists a measure zero set $E \subset \mathbb{R}$ such that F is differentiable and $F'(x) = f(x)$ for $x \in \mathbb{R} \setminus E$.
3. [15 pts] Prove that the point-wise limit f of Lebesgue integrable functions $\{f_n\}$ is Lebesgue measurable.
4. [15 pts] For any $L^1(\mathbb{R}^n)$, find a sequence $\{f_n\} \subset C_c^\infty(\mathbb{R}^n)$ such that
$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |f(x) - f_n(x)|dx = 0.$$
5. [15 pts] Let $f \in L^1([0,1])$. Prove that $\lim_{p\to 0} \|f\|_p = \exp(\int_0^1 \ln|f|dx)$ and $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.
6. [15 pts] Prove that the Fourier transform is an isomorphism from $\mathcal{S}$ to $\mathcal{S}$, where $\mathcal{S}$ is the class of functions $f \in C^\infty(\mathbb{R}^n)$ such that $\sup_{x\in\mathbb{R}^n}(|x|+1)^m|D^\alpha f(x)| < \infty$ for any positive integer m and $\alpha \in \mathbb{N}^n$.

THE END

Ph.D. Qualifying Exam: Complex Analysis
February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let Ω be a connected open set of $\mathbb{C}$, and let f, g be holomorphic functions on Ω. Suppose $\overline{f}g$ is also holomorphic on Ω. Prove that either f is constant or g is identically zero.

2. [15 pts] Let f be an entire function. Suppose there exist constants $M, R > 0$ and an integer $n \geq 1$ such that

$$|f(z)| \leq M|z|^n \quad \text{for } |z| \geq R.$$

Determine all such entire functions f.

3. [15 pts] Evaluate the integral

$$\int_0^\infty \frac{\log x}{(1+x)^3}\, dx$$

by applying the residue theorem. You may consider the function $f(z) = \frac{(\log z)^2}{(1+z)^3}$.

4. [15 pts] Derive the Hadamard product, which is an infinite product representation, for

$$\frac{\sin(\pi z)}{\pi}.$$

You may use (without proof) the summation formula for the cotangent:

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} \left(= \lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+n} \right), \quad z \in \mathbb{C} \setminus \mathbb{Z}.$$

5. [10 pts] Suppose $\mathcal{F}$ is a normal family of holomorphic functions on $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Show that the family

$$\mathcal{F}' := \{f' \,|\, f \in \mathcal{F}\}$$

is also a normal family on $\mathbb{D}$.

6. (a) [10 pts] State and prove the open mapping theorem for holomorphic functions.

   (b) [10 pts] Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. Let f be bounded on $\Omega \setminus \{z_0\}$. Prove that z_0 is a removable singularity.

   (c) [10 pts] Prove that the punctured disk $\mathbb{D} \setminus \{0\} = \{z \in \mathbb{C} \,|\, 0 < |z| < 1\}$ and the annulus $\{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$ are not conformally equivalent.

THE END

Ph.D. Qualifying Exam: Probability Theory
February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. Prove the following:

   (a) [10 pts] If a, b are fixed and $0 < b \leq a$, then there exists a random variable X such that $\mathbb{E}[X^2] = b^2$ and $\mathbb{P}(|X| \geq a) = b^2/a^2$.

   (b) [10 pts] For a random variable X, if $0 < \mathbb{E}[X^2] < \infty$, then

   $$\lim_{a\to\infty} a^2 \mathbb{P}(|X| \geq a) = 0.$$

2. [20 pts] Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}[X_i] = \mu$ and $\mathbb{E}[X_i^4] = \kappa < \infty$ for any $i = 1, 2, \ldots$. Define $S_n = X_1 + X_2 + \cdots + X_n$. Prove that $\dfrac{S_n}{n} \to \mu$ almost surely.

3. Let μ be a probability measure and φ be its characteristic function. Suppose that $|\varphi(t) - 1| \leq C|t|^\alpha$ for some constants $C > 0$ and $\alpha \in (0, 2)$.

   (a) [10 pts] Prove that $\displaystyle\int |x|^\beta \mu(dx) < \infty$ for any $0 < \beta < \alpha$.

   (b) [10 pts] Find an example of μ such that $\displaystyle\int |x|^\alpha \mu(dx) = \infty$.

4. Let $\{X_n\}_{n\in\mathbb{Z}^+}$ be a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{Z}^+}$ satisfying $\mathbb{E}[|X_n|] < \infty$.

   (a) [10 pts] Prove that $\mathbb{E}[|X_n| \,|\mathcal{F}_j]$ converges in L^1 as $n \to \infty$.

   (b) [10 pts] Prove that there are two non-negative martingales $\{Y_n\}_{n\in\mathbb{Z}^+}$ and $\{Z_n\}_{n\in\mathbb{Z}^+}$ (with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{Z}^+}$) such that $X_n = Y_n - Z_n$.

5. [20 pts] Let $\{X_n\}_{n\in\mathbb{Z}^+}$ be a martingale (with respect to a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{Z}^+}$) such that $Y_n = X_{n+1} - X_n$ is square integrable. Prove that $\mathbb{E}[Y_n Y_m] = 0$ for $n \neq m$.

THE END

Ph.D. Qualifying Exam: Advanced Statistics
February 2025

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let $X_1, \ldots, X_n$ be a random sample from $N(\mu_x, \sigma^2)$, and let $Y_1, \ldots, Y_m$ be an independent random sample from $N(\mu_y, 4\sigma^2)$, where μ_x, μ_y, and σ^2 are unknown parameters. You may use the following facts:

   - The probability density function of $N(\mu, \sigma^2)$ is given by:
   $$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$
   - If $V \sim \chi^2(k)$, then $\mathbb{E}(V) = k$ and $\mathrm{Var}(V) = 2k$.

   (a) [5 pts] Find a complete sufficient statistic.

   (b) [5 pts] Determine the UMVU estimator of σ^2.
   Hint: Find a linear combination of $S_x^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$ *and* $S_y^2 = \frac{\sum_{j=1}^m (Y_j - \bar{Y})^2}{m-1}$.

   (c) [5 pts] Does the UMVU estimator of σ^2 you found in part (b) achieve the Cramér-Rao lower bound (CRLB)?

   (d) [5 pts] Find the UMVU estimator of $(\mu_x - \mu_y)^2$.

2. [30 pts] Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. with $X_i \sim N(0, 1)$ and $Y_i | X_i = x \sim N(x\theta, 1)$. Answer the following questions.

   (a) [5 pts] Find the maximum likelihood estimate $\widehat{\theta}$ of θ.

   (b) [5 pts] Find the limiting distribution of $\sqrt{n}(\widehat{\theta} - \theta)$.

   (c) [5 pts] Find the *exact* distribution of $\sqrt{\sum X_i^2}(\widehat{\theta} - \theta)$.

   (d) [5 pts] Find the Bayes estimator of θ under the squared error loss function with a prior $\theta \sim N(0, \tau^2)$.

   (e) [5 pts] Show that the MLE $\widehat{\theta}$ is a minimax estimator for θ.

   (f) [5 pts] Derive the uniformly most powerful (UMP) test for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$.

3. [25 pts] Let X_i be ith subject's group membership which takes value 1 or 2 with probability p or $1-p$. If $X_i = 1$, $Y_i \overset{iid}{\sim} N(\theta_1, \sigma_1^2)$ and if $X_i = 2$, $Y_i \overset{iid}{\sim} N(\theta_2, \sigma_2^2)$. Assume $p \sim \mathrm{Beta}(a, b)$, $\theta_j \sim N(\mu_0, \tau_0^2)$ and $1/\sigma_j^2 \sim \mathrm{Gamma}(\nu_0/2, \nu_0\sigma_0^2/2)$ for both $j = 1$ and $j = 2$, independently.

   (a) [15 pts] Derive the full conditional distributions of $(X_1, \ldots, X_n)$ and p to implement the MCMC sampling for posterior inference.

(b) [10 pts] Explain the procedure for approximating the marginal posterior predictive distribution for Y.

4. [25 pts] Consider a hierarchical model as follows.

$$\theta_i|\mu, \tau^2 \overset{iid}{\sim} N(\mu, \tau^2) \text{ for } i = 1, \ldots, m$$
$$y_{ij}|\theta_i, \sigma^2 \overset{iid}{\sim} N(\theta_i, \sigma^2) \text{ for } j = 1, \ldots, n_i$$

(a) [10 pts] Compute the following six quantities.

$$\text{Var}[y_{ij}|\theta_i, \sigma^2],\ \text{Var}[\bar{y}_{i\cdot}|\theta_i, \sigma^2],\ \text{Cov}[y_{ij}, y_{ij'}|\theta_i, \sigma^2]$$
$$\text{Var}[y_{ij}|\mu, \tau^2],\ \text{Var}[\bar{y}_{i\cdot}|\mu, \tau^2],\ \text{Cov}[y_{ij}, y_{ij'}|\mu, \tau^2]$$

(b) [2 pts] Compare $\text{Var}[y_{ij}|\theta_i, \sigma^2]$ and $\text{Var}[y_{ij}|\mu, \tau^2]$. Which one is bigger and why?

(c) [2 pts] Compare $\text{Cov}[y_{ij}, y_{ij'}|\theta_i, \sigma^2]$ and $\text{Cov}[y_{ij}, y_{ij'}|\mu, \tau^2]$. Which one is bigger and why?

(d) [6 pts] Assume a prior distribution for μ as $p(\mu)$. Then, show the following equation.

$$p(\mu|\theta_1, \ldots, \theta_m, \sigma^2, \tau^2, \mathbf{y}_1, \ldots, \mathbf{y}_m) = p(\mu|\theta_1, \ldots, \theta_m, \tau^2)$$

where $\mathbf{y}_i = (y_{i1}, \ldots, y_{i,n_i})$. Provide an interpretation.

(e) [5 pts] If you plot $\bar{y}_{i\cdot}$ against $E[\theta_i|\mathbf{y}_1, \ldots, \mathbf{y}_m]$ for $i = 1, \ldots, m$, how would it look like and why? (Provide an exemplary scatterplot)

THE END