Ph.D. Qualifying Exam: Algebra I
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 pts] Let $\{A_i\}_{i\in I}$ be a family of abelian groups A_i. Prove that there exist an abelian group A together with a group homomorphism $\lambda_i : A_i \to A$ for each $i \in I$ such that;

   for any abelian group B together with a group homomorphism $f_i : A_i \to B$ for each $i \in I$, there exists a unique group homomorphism $f : A \to B$ such that $f \circ \lambda_i = f_i$ for each $i \in I$.

   (Please give an explicit description of such an A and λ_i.)

2. [10 pts] Let q be a prime and G be a finite group whose order is divisible by q. Let H be a normal subgroup of G and Q be a Sylow q-subgroup of G. If Q is a normal subgroup of H, prove that Q is normal in G.

3. [15 pts] Let G be a group of order $357 (= 3 \cdot 7 \cdot 17)$. Prove that the center $Z(G)$ of G is non-trivial (i.e., $|Z(G)| \neq 1$).

4. [20 pts] Let R be an integral domain.

   (a) [10 pts] Prove or disprove: If M_i are R-modules and $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then for any R-module N, $0 \to M_1 \otimes_R N \to M_2 \otimes_R N \to M_3 \otimes_R N \to 0$ is also exact.

   (b) [10 pts] Prove or disprove: Every projective R-module is divisible.

5. [45 pts] Let D_8 be the dihedral group of order 8.

   (a) [10 pts] Find a presentation $\langle X \mid R \rangle$ of D_8, where X is a generating set and R is a complete set of relations for D_8. Justify your answer.

   (b) [10 pts] Find all conjugacy classes of D_8 and the class equation of D_8. Justify your answer.

   (c) [10 pts] Find all non-isomorphic (or non-equivalent) 1-dimensional irreducible representations of D_8 over $\mathbb{C}$. Justify your answer.

   (d) [15 pts] Complete the character table of all non-isomorphic (or non-equivalent) irreducible representations of D_8 over $\mathbb{C}$. Justify your answer.

THE END

Ph.D. Qualifying Exam: Algebra II
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

In what follows, all rings are commutative rings with 1.

1. [15 pts] Let $p > 0$ be a prime. Let $F = \mathbb{F}_p$ be a field with p elements. Let $a \in F^\times$.

   Prove that the splitting field E of the polynomial $x^p - x + a \in F[x]$ is a cyclic Galois extension of F of degree p.

2. Answer the following questions.

   (a) [10 pts] Prove the Hilbert basis theorem, i.e. for a noetherian ring R, the polynomial ring $R[x_1, \cdots, x_n]$ in the variables $x_1, \cdots, x_n$ is again a noetherian ring.

   (b) [10 pts] Let k be an algebraically closed field. Let $I \subset k[x_1, \cdots, x_n]$ be an ideal and let $V = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in I\}$.
   Prove that there exists a finite set of polynomials $g_1, \cdots g_N \in I$ such that V is exactly equal to the set of simultaneous solutions $x \in k^n$ of the system of equations $\{g_1(x) = 0, \cdots, g_N(x) = 0\}$.

3. For a ring R, the notation $R^\times$ denotes the multiplicative group of the units of R. Answer the following questions.

   (a) [10 pts] Give a concrete example (R, I) consisting of a ring R and an ideal $I \subset R$ such that the canonical homomorphism $R^\times \to (R/I)^\times$ is not surjective.

   (b) [10 pts] Suppose that R is a local ring. Prove that for any proper ideal $I \subset R$, the canonical homomorphism $R^\times \to (R/I)^\times$ is surjective.

4. [15 pts] Let k be a field and let $M \subset k[x_1, \cdots, x_n]$ be a maximal ideal. Prove that the composition of the ring homomorphisms $k \hookrightarrow k[x_1, \cdots, x_n] \to F := k[x_1, \cdots, x_n]/M$ gives a finite extension $k \subset F$ of fields.

5. [15 pts] Let k be a field, and let $R = k[x, y]/(y^2 - x^3)$. Prove that R is an integral domain, that is not normal (i.e. it is not integrally closed in its field of fractions).

6. [15 pts] Let $I \subset \mathbb{Z}$ be a nonzero proper ideal, and let M be an abelian group. By finding a projective resolution of $\mathbb{Z}/I$, prove that

$$\mathrm{Tor}_0^{\mathbb{Z}}(M, \mathbb{Z}/I) \simeq M/IM,$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/I) \simeq \{m \in M \mid Im = 0\},$$
$$\mathrm{Tor}_i^{\mathbb{Z}}(M, \mathbb{Z}/I) = 0 \ \text{ for } i \geq 2.$$

THE END

Ph.D. Qualifying Exam: Differential Geometry
August 2024

Student ID: Name:

Note: Be sure use English for your answers.

1. [15 pts] Let $F : M \to N$ be a diffeomorphism between two smooth manifolds without boundary. Show that $dF : TM \to TN$ is also a diffeomorphism.
2. [15 pts = 10 pts + 5 pts] Let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be a smooth map. Let
$$L = \{p \in \mathbb{S}^2 : \operatorname{rank}(df_p) \leq 1\}.$$
   (a) Show that L is non-empty.
   (b) Show that $f(L) \subset \mathbb{R}^2$ has empty interior.
3. [25 pts=5 pts + 10 pts+10 pts] Let S be the cylinder in $\mathbb{R}^3$ given by
$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1; 0 \leq z \leq 1\}.$$
   (a) Show that this is a smooth 2-manifold with non-empty boundary ∂S and it is an embedded submanifold with boundary in $\mathbb{R}^3$.
   (b) Choose yourself an orientation on S and consider the inner product inherited from $\mathbb{R}^3$. Describe the unit normal vector field outward pointing corresponding to this orientation of S.
   (c) Describe the unit tangent vector field of ∂S corresponding to the induced orientation of the boundary ∂S from the orientation of S.
4. [15 pts] Let M be a compact smooth manifold of dimension $n \geq 1$ without boundary. Assume that M is orientable. Show that there exists a smooth n-form ω which is nowhere zero on M.
5. [10pts] Let M be a compact connected manifold of dimension $n \geq 1$ and assume ∂M has two component C_1 and C_2. Let ι_1 and ι_2 be the inclusions of C_1 and C_2 into M. Suppose α is a smooth k-form and β is a smooth $(n-k-1)$-form on M satisfying $\iota_1^*\alpha = 0$ and $\iota_2^*\beta = 0$. Show that
$$\int_M d\alpha \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge d\beta.$$
6. [20 pts =10 pts + 10 pts] Let M be a smooth manifold of dimension n without boundary. Let V be a smooth vector field and A a smooth covariant k-tensor on M. Then, the Lie derivative $\mathcal{L}_V A$ is a smooth covariant k-tensor on M which satisfies that for any smooth vector fields $X_1, ..., X_k$,
$$\begin{aligned}(\mathcal{L}_V A)(X_1, ..., X_k) &= V(A(X_1, ..., X_k)) - A([V, X_1], X_2, ..., X_k) \\ &\quad - \cdots - A(X_1, ..., X_{k-1}, [V, X_k]).\end{aligned}$$
   (a) Prove Cartan's magic formula: for any smooth vector field V and smooth differential form ω,
$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner\, \omega).$$
   (You only need to provide the proofs for ω of degree either 0 or 1.)
   (b) Show that for a smooth vector field V, the Lie derivatives $\mathcal{L}_V$ commutes with the exterior derivative d when acting on smooth differential forms.

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
August 2024

Student ID: Name:

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (a) [10 pts] Prove or disprove that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.
   (b) [10 pts] Prove or disprove that S^∞ is contractible.
2. (a) [10 pts] Prove or disprove that $\pi_1(X, x_0) = 0$ for all $x_0 \in X$ if and only if every map $S^1 \to X$ extends to a map $D^2 \to X$.
   (b) [10 pts] Prove or disprove that every homomorphism $\pi_1(X) \to \pi_1(X)$ can be realized as the induced homomorphism of a continuous map $X \to X$.
3. (a) [10 pts] Prove or disprove that the complement of a closed discrete subspace of $\mathbb{R}^3$ is simply connected.
   (b) [10 pts] Prove or disprove that the complement of an embedding of S^1 in $\mathbb{R}^3$ is abelian.
4. (a) [10 pts] Determine the covering space of $S^1 \vee S^1$ corresponding to the subgroup of $\pi_1(S^1 \vee S^1)$ generated by the cubes of all elements.
   (b) [10 pts] Determine the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$.
5. (a) [10 pts] Compute the homology groups $H_n(X, A)$ when X is $S^1 \times S^1$ and A is a finite set of points in X.
   (b) [10 pts] Let n be a positive integer. Prove or disprove that for a finite CW complex X and $p\colon \widetilde{X} \to X$ an n-sheeted covering space, we have that $\chi(\widetilde{X}) = n\chi(X)$.

THE END

Ph.D. Qualifying Exam: Real Analysis
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let $f \in L^1(\mathbb{R})$. For any $\epsilon > 0$, Construct a continuous and piecewise linear function g satisfying $\|f-g\|_{L^p(\mathbb{R})} \leq \epsilon$. (This problem asks a proof of a density theorem. Do not use other density theorem without a proof.)

2. [20 pts] Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x)\sum_{n=1}^{\infty} 4^{-n} \sin(16^n \pi x).$$

Note that $f(x)$ is well-defined as the series converges absolutely. Show that f is continuous, but nowhere differentiable. (Hint. Estimate $f(\frac{j+\frac{1}{2}}{16^n}) - f(\frac{j-\frac{1}{2}}{16^n})$.)

3. [15 pts] Let $1 \leq p < q \leq \infty$. Show that $L^q(\mathbb{R}) \not\subseteq L^p(\mathbb{R})$, and $L^q(\mathbb{R}) \not\supseteq L^p(\mathbb{R})$. Also, show that $L^q([0,1]) \subsetneq L^p([0,1])$ and $\ell^p(\mathbb{Z}) \subsetneq \ell^q(\mathbb{Z})$.

4. [20 pts] Consider a family of functions,

$$\{f_n : (-\infty, 0] \to \mathbb{R} : f_n \text{ is differentiable, } |f_n'(x)| \leq g(x) \text{ and } \lim_{x\to-\infty} f_n(x) = 0\}$$

for some continuous function g satisfying $g \in L^1((-\infty, 0])$. Show that there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ uniformly on each compact interval in $(-\infty, 0]$.

5. [15 pts] Denote the Schwartz class by $\mathcal{S}(\mathbb{R})$,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}}(1+|x|^N)|\frac{d^m}{dx^m}f(x)| = \|f\|_{N,m} < \infty,\ \forall N, m \in \mathbb{N}_0\}.$$

Show that $\mathcal{S}(\mathbb{R})$ is closed under the convolution operation, i.e. if $f, g \in \mathcal{S}(\mathbb{R})$, then $f * g \in \mathcal{S}(\mathbb{R})$, where $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$.

6. [15 pts] Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Show that f is discontinuous at most countable set. Show that f is differentiable a.e. (Hint. You may use the Radon-Nikodym theorem without a proof.)

THE END

Ph.D. Qualifying Exam: Complex Analysis
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1. [20 pts] Find an explicit formula for $f(z) = \int_0^{2\pi} \frac{1}{1+z\sin\theta}\, d\theta$, $|z| < 1$, by using the residue theorem.
2. [10 pts] Let a be a complex number with $0 < |a| < 1$. Determine all holomorphic function f in $\mathbb{D}$ satisfying that $|f(z)| < 1$ on $\mathbb{D}$, $f(\pm a) = 0$ and $|f(0)| = |a|^2$.
3. [10 pts] Determine all entire function f satisfying that $|f'(z)| < |f(z)|$.
4. [20 pts] Set $f_1(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ and $f_2(z) = \frac{\pi^2}{\sin^2(\pi z)}$. Show that f_1 is a meromorphic function which has the same singularities as f_2. Also, show that $f_1(z) = f_2(z)$, except the singular points of the functions.
5. [20 pts] Set $a_n = 1 - \frac{1}{n^2}$ and $f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}$. Show that f defines a holomorphic function on $\mathbb{D}$.
6. Let Ω be a connected open set in $\mathbb{C}$ and f is a function on Ω. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on Ω that converges uniformly on every compact subset of Ω to f.
   (a) [10 pts] Prove that if $f_n(z) \neq 0$ for all $n \geq 1$ and $z \in \Omega$, then either f is identically zero in Ω or $f(z) \neq 0$ for all $z \in \Omega$.
   (b) [10 pts] Prove that if f_n is injective on Ω for all $n \geq 1$, then f is either injective or constant.

THE END

Ph.D. Qualifying Exam: Probability Theory
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Compute the following limit

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \left(\frac{x_1 + \cdots + x_n}{n}\right)^{2024} dx_1 \cdots dx_n$$

   with a validation as rigorous as possible.

2. [20 pts] Let $X_1, \ldots, X_n$ be independent and identically distributed square-integrable random variables with their variances σ^2. Define the sigma field generated by the random variable $X_1 + \cdots + X_n$ as $\mathcal{F}$, that is, $\mathcal{F} = \sigma\big(X_1 + \cdots + X_n\big)$. Find the variance of

$$\mathbb{E}\big[X_1 \big| \mathcal{F}\big] .$$

3. [20 pts] Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with finite mean, μ, and let $S_n = X_1 + \cdots + X_n,\ n \geq 1$ be the sequence of partial sums. Suppose that τ is a stopping time with $\mathbb{E}[\tau] < \infty$. Show that

$$\mathbb{E}[S_\tau] = \mu \cdot \mathbb{E}[\tau] .$$

4. [20 pts] Prove or disprove the following statements:

   (a) [10 pts] Let $\{X_t, \mathcal{F}_t, t \in [0,T] \subset \mathbb{R}\}$ be a supermatingale for a positive constant T. Assume that $\mathbb{E}\big[X_0\big] = \mathbb{E}\big[X_T\big]$. Then, X is a martingale.

   (b) [10 pts] Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$. Then

$$\frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{k=1}^n X_k^2}} \Rightarrow N(0,1) .$$

5. [20 pts] Let $\mathbb{Q} \ll \mathbb{P}$ holds for two probability measures $\mathbb{Q}$ and $\mathbb{P}$ on a σ-algebra $\mathcal{F}$ with a Radon-Nikodym derivative, $\rho = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Prove that for any $\mathcal{F}$-measurable non-negative random variable ξ and a sub σ-algebra $\mathcal{G} \subset \mathcal{F}$,

$$\mathbb{E}_{\mathbb{Q}}\big[\xi \big| \mathcal{G}\big] = \frac{1}{\mathbb{E}_{\mathbb{P}}\big[\rho \big| \mathcal{G}\big]} \mathbb{E}_{\mathbb{P}}\big[\xi\rho \big| \mathcal{G}\big]$$

   holds $\mathbb{P}$-a.s..

THE END

Ph.D. Qualifying Exam: Advanced Statistics
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [30 pts] Let $X_1, X_2, \ldots, X_n$ be i.i.d from the gamma distribution with parameters θ and γ, where $\theta > 0$ and $\gamma > 0$ are unknown. The gamma pdf is given by $f(x;\theta,\gamma) = \frac{1}{\Gamma(\theta)\gamma^\theta} x^{\theta-1}\exp(-x/\theta) I_{(0,\infty)}(x)$.

   (a) [10 pts] Find the limiting distribution of $(\bar{X}, \bar{X^2})$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$ and $\bar{X^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$.

   (b) [10 pts] Let $T_n = \frac{n\sum_{i=1}^n X_i^2}{(\sum_{i=1}^n X_i)^2}$. Find the limiting distribution of T_n.

   (c) [10 pts] Using T_n, find an asymptotically correct test for $H_0 : \theta = 1$ vs $H_1 : \theta \neq 1$.

2. [20 pts] Suppose that $X_i = \rho t_i + \epsilon_i$, $i = 1, \ldots, n$, where $\rho \in R$ is an unknown parameter, $t_i's$ are known and in (a, b), and $0 < a < b$ are constants, and ϵ_i's are independent random variables satisfying $E(\epsilon_i) = 0$ and $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and $\mathrm{Var}(\epsilon_i) = \sigma^2 t_i$ with unknown $\sigma^2 > 0$.

   (a) [5 pts] Find the least squares estimator for ρ.

   (b) [5 pts] Find the minimizing the variance estimator among linear unbiased estimators of ρ.

   (c) [10 pts] Find the asymptotic distribution of estimators in (a) and (b), and compute the asymptotic relative efficiency of the best linear unbiased estimator with respect to least squares estimator.

3. [20 pts] Let $X_1, X_2, \ldots, X_n$ be i.i.d random variables with the pdf $f_\theta(x) = \sqrt{\frac{2\theta}{\pi}}\exp(-\theta x^2/2) I_{[0,\infty)}(x)$, where $\theta > 0$ is unknown. Let the prior of θ be the gamma distribution with parameters α and γ with known α and γ. Find the Bayes estimator of $f_\theta(0)$ and its Bayes risk under the loss function $L(\theta, a) = (a - \theta)^2/\theta$.

4. [30 pts] Consider a model given below:

   $X_i|\mu_i \sim N(\mu_i, 1)$, and $\mu_i \sim N(0, 1)$, $i = 1, 2, \ldots, p$,where μ_i's are independent.

   In this case, consider two estimators: $\hat{\mu}_i^{(1)} = X_i$ and $\hat{\mu}_i^{(2)} = (1 - \frac{p-2}{\sum X_i^2})X_i$

   (a) (10 pts) For any estimator $\hat{\mu}_i$ verify
   $(\hat{\mu}_i - \mu_i)^2 = (X_i - \hat{\mu}_i)^2 - (X_i - \mu_i)^2 + 2(\hat{\mu}_i - \mu_i)(X_i - \mu_i)$.

   (b) (10 pts) Show that $cov_\mu(\hat{\mu}_i, X_i) = E_\mu\left\{\frac{\partial \hat{\mu}_i}{\partial X_i}\right\}$ (Hint: use the integration by parts).

(c) (10 pts) Using (a) and (b), verify $E_\mu(\sum(\hat{\mu}_i^{(1)} - \mu_i)^2) \geq E_\mu(\sum(\hat{\mu}_i^{(2)} - \mu_i)^2)$.

Note: Here E_μ represents the expectation of random variables X_i's given μ. The normal pdf $(\mu, \sigma^2))$ is given by $f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$ for all $x \in R$.

THE END

Ph.D. Qualifying Exam: Numerical Analysis
August 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let $f \in C[a,b]\backslash\mathcal{P}$, and $\mathcal{P}$ the set of all real polynomials. Let n be a nonnegative integer and $p \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$, where $\mathcal{P}_n$ is the set of all real polynomials of degree $\leq n$.
   (a) [10 pts] Prove that there exist $n+1$ distinct points $x_i \in [a,b]$ such that $p(x_i) = f(x_i)$ for $i = 1, \ldots, n+1$.
   (b) [10 pts] Assume $f \in C^{n+1}[a,b]$ and $f^{(n+1)}(x) > 0$ for all $x \in [a,b]$. Prove that $f - p$ reaches its maximum in magnitude with alternative signs at exactly $n+2$ distinct points in $[a,b]$.

2. The midpoint rule for numerical integration is given below,
$$\int_{x_{-1}}^{x_1} f(x)dx \approx 2hf(x_0),$$
where $f \in C^2[x_{-1}, x_1]$, $x_{-1} < \xi < x_1$, and $x_0 - x_{-1} = x_1 - x_0 = h > 0$.
   (a) [8 pts] Derive the error formula.
   (b) [7 pts] Determine the values of h (and n) required to approximate $\int_0^2 e^{2x} \sin 3x \, dx$ to within 10^{-4}, using the composite midpoint rule with uniform intervals of size $h = \frac{2}{n}$.

3. Assume $f \in C^2[a,b]$ and consider the approximate Newton method
$$x_{k+1} = x_k - \left(\frac{1}{f'(x_k)} + r_k\right) f(x_k)$$
with a small deviation term $|r_k| \ll 1$.
   (a) [10 pts] Show that if x_k remain nearby a simple zero $x_* \in [a,b]$ of f, then $e_k := |x_k - x_*|$ satisfies
$$e_{k+1} \leq C_1 |r_k| e_k + C_2 e_k^2,$$
for constants C_1 and C_2 depending on $\max_{\xi \in [a,b]} |f'(\xi)|^{-1}$, $\max_{\eta \in [a,b]} |f''(\eta)|$.
   (b) [15 pts] Consider the two-step Newton method
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})}{f[x_n, x_{n+1}]}, \quad f[x,y] := \frac{f(x) - f(y)}{x - y}.$$

Prove that there exists a constant C such that the following holds,

$$e_{n+2} \leq C e_n^3, \quad n = 0, 2, \ldots,$$

by expressing the second step of the two-step Newton method as an approximate Newton method with appropriate deviation term r_{n+1}.

4. [20 pts] Let A be an $n \times n$ matrix. Prove (by induction) Schur decomposition theorem: $A = UTU^H$, where U is a unitary matrix, and T is an upper triangular matrix.

5. The general form of linear explicit two-step methods for solving the initial-value problem $y' = f(t, y)$, $y(0) = \alpha$ is

$$w_{i+1} = aw_i + bw_{i-1} + h(cf(t_i, w_i) + df(t_{i-1}, w_{i-1})).$$

   (a) [10 pts] Find the coefficients a, b, c and d that maximize the order of this method. What is this order?

   (b) [10 pts] Determine whether the resulting method is stable or not.

THE END