Ph.D. Qualifying Exam: Algebra I
February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let G be a group, $Z(G)$ be the center of G, and $\mathrm{Aut}(G)$ be the automorphism group of G.
   (a) [10 pts] Prove that if $G/Z(G)$ is cyclic, then G is abelian.
   (b) [10 pts] Prove that if $\mathrm{Aut}(G)$ is cyclic, then G is abelian.

2. [10 pts] Let G be a finite group, and N be a subgroup of G. Prove that if the index $[G : N]$ of N in G is the smallest prime divisor of the order $|G|$ of G, then N is a normal subgroup of G.

3. [20 pts] Suppose that a finite group G acts on a finite nonempty set A. For an element $a \in A$, let G_a be the stabilizer of a, and $G \cdot a$ be the orbit of a.
   (a) [10 pts] Prove that $|G \cdot a| = [G : G_a]$.
   (b) [10 pts] Prove that the number of all distinct orbits is $(1/|G|) \sum_{a \in A} |G_a|$.

4. [20 pts] Let p, q be prime numbers with $p < q$.
   (a) [10 pts] Prove that if G is a group of order pq and $p \nmid q - 1$, then G is cyclic.
   (b) [10 pts] Prove that if H is a group of order q and K is a group of order p with $p \mid q - 1$, then there is a unique nontrivial group homomorphism $\varphi\colon K \to \mathrm{Aut}(H)$ and $H \rtimes_\varphi K$ is nonabelian.

5. [20 pts] Let R be an integral domain.
   (a) [10 pts] Prove that if R is a unique factorization domain, then for every infinite chain
   $$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
   of principal ideals of R, there is an integer $n \geq 1$ with $I_m = I_n$ for every integer $m \geq n$.
   (b) [10 pts] Prove that R is a field if and only if for every infinite chain
   $$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$
   of ideals of R, there is an integer $n \geq 1$ with $I_m = I_n$ for every integer $m \geq n$.

6. [10 pts] Let R be a commutative ring with identity, and M be a finitely generated R-module. Prove that if
   $$f\colon M \longrightarrow R^n = \underbrace{R \oplus \cdots \oplus R}_{n \text{ times}}$$
   is a surjective R-module homomorphism for an integer $n \geq 1$, then the kernel of f is a finitely generated R-module.

THE END

Ph.D. Qualifying Exam: Algebra II
February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let A be a nonzero commutative ring with unity. Answer the following questions.
   (a) [5 pts] Suppose A is a PID. Prove that A is a noetherian UFD.
   (b) [5 pts] Suppose A is a UFD. Prove that A is normal, i.e. integrally closed in the field $K = \mathrm{Frac}(A)$ of fractions of A.
   (c) [5 pts] Prove that the converse of (b) is false by constructing an example of the integral closure of $\mathbb{Z}$ in a finite extension field of $\mathbb{Q}$. You should justify your answer.

2. Let $A \subset B$ be two nonzero commutative rings with unity and suppose B is an integral extension of A. Answer the following questions.
   (a) [10 pts] Prove that A is a field if and only if B is a field.
   (b) [10 pts] Let $P \subset A$ be a prime ideal. Prove that there exists a prime ideal $Q \subset B$ such that $P = A \cap Q$.

3. [15 pts] Let R be a nonzero commutative ring with unity. Let $J \subset R$ be the set of elements $x \in R$ such that x is nilpotent.

   Prove that J is equal to the intersection of all prime ideals of R.

4. Let $F = \mathbb{Q}$, the field of rational numbers. Fix the algebraic closure $\overline{F} \subset \mathbb{C}$.

   Can we construct two finite proper cyclic Galois extensions F_1, F_2 of F in $\overline{F}$ such that the compositum F_1F_2 is ...
   (a) [5 pts] a cyclic Galois extension of F? If possible, present an example (F_1, F_2) with a justification. If not possible, prove why it is impossible.
   (b) [5 pts] an abelian Galois extension of F that is not cyclic? Do as in (a).
   (c) [5 pts] a nonabelian Galois extension of F? Do as in (a).

Continued on the next page

5. Let $F := \mathbb{F}_p$ be the field with p elements for a prime number $p > 0$. Answer the following questions.

   (a) [15 pts] Prove that for each integer $r \geq 2$, there exists a field K that extends F such that $[K : F] = r$, and that any two such extensions K are isomorphic to each other as fields.

   Furthermore, prove that K a Galois extension of F and compute the Galois group $\mathrm{Gal}(K/F)$.

   (b) [5 pts] Prove that there exists an irreducible polynomial of degree r in $F[x]$. For $r = 2$ and 3, write concrete examples, respectively.

6. Let A be a nonzero commutative ring with unity and let $p > 0$ be a prime number. Via the natural ring homomorphism $\mathbb{Z} \to A$, regard p as an element of A as well. For $a, b \in A$ and an integer $r \geq 1$, let's write

$$a \equiv b \mod p^r$$

if $a - b \in I_r := (p^r)$, the ideal generated by p^r in A.

   (a) [5 pts] Suppose that $a \equiv b \mod p$. Then for each integer $r \geq 1$, prove that

$$a^{p^r} \equiv b^{p^r} \mod p^{r+1}.$$

   (b) [5 pts] Suppose that A is an integral domain such that A/p is perfect (i.e. the Frobenius homomorphism $F : A/p \to A/p$ is a ring isomorphism) and A is p-adically complete (i.e. the natural homomorphism $A \to \widehat{A} := \varprojlim_{r} A/p^r$ that sends a to $(\bar{a})_r \in \prod_r A/p^r$ is an isomorphism of rings.

   Using (a), prove that there is a natural set map

$$\varphi : A/p \to A$$

   such that its composition with the projection $A \to A/p^r$ is given by the well-defined set map $A/p \to A/p^r$ sending $\bar{a}$ to $\overline{a^{p^{r-1}}}$.

   (c) [5 pts] Continue to suppose that A is a p-adically complete integral domain such that A/p is perfect. Prove that every member $x \in A$ has an expression as a convergeant p-adic power series

$$x = \varphi(a_0) + \varphi(a_1)p + \varphi(a_2)p^2 + \varphi(a_3)p^3 + \cdots$$

   for some $a_0, a_1, a_2, \cdots \in A/p$.

THE END

Ph.D. Qualifying Exam: Differential Geometry
February 2024

Student ID: Name:

Note: Be sure to use English for your answers. All manifolds are smooth and without boundary unless otherwise stated.

1. [15 pts] For a smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$ and $p \in \mathbb{R}^n$ a regular point of f, show that there is a neighborhood of p in $\mathbb{R}^n$ that does not contain a point other than p whose image under f is $f(p)$.

2. [10 pts] For your choice of n, give an example of an n-dimensional manifold that cannot be smoothly embedded in $\mathbb{R}^{2n-1}$ and explain why. (You don't have to give examples for all n!)

3. [25 pts]

   (a) [15 pts] Prove or disprove: a smooth orientable vector bundle over an orientable manifold is always trivial.

   (b) [10 pts] Denote by $\mathbb{RP}^n$ the n-dimensional real projective space. Show that the tautological bundle

   $$T = \{([p], v) \mid p \in \mathbb{R}^{n+1} \backslash \{0\}, v = \lambda p \text{ for some } \lambda \in \mathbb{R}\}$$

   is not trivial.

4. [20 pts] Let X, Y be vector fields on $\mathbb{R}^3$ given by

   $$X(x, y, z) = -|(x, y, z)|^3 (x, y, z),$$

   $$Y(x, y, z) = (y, z, x).$$

   (a) [10 pts] Show that the flow of X exists for all time.

   (b) [10 pts] Calculate the Lie derivative $\mathcal{L}_X Y$.

5. [15 pts] A *Riemannian metric* on a manifold is a smooth, symmetric, positive-definite covariant two-tensor field. Show that every smooth manifold of positive dimension admits a Riemannian metric.

6. [15 pts] For M^2 a compact two-dimensional manifold with boundary ∂M, show that for any smooth function $f : M \to \mathbb{R}$,

   $$\int_{\partial M} f \, df = 0.$$

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
February 2024

Student ID: Name:

Note: Be sure to use English for your answers. Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. Here, when we ask you to "compute the group", this means that you have to write down the generators and relations of the the group. When we ask you to "compute the homomorphism", you need to write down the generators and relations of the two groups and finding where the generators go to under the map. (Total 100 pts.)

1 [(25 pts.)] Let P_{xy}, P_{yz}, P_{zx} denote the xy-plane, the yz-plane, and the zx-plane in $\mathbf{R}^3$ respectively. Let Z be the union of these planes. Let B_1 be a closed 3-ball of radius 1 with the center O.

(a) (15 pts.) Compute the homology groups $H_*(B_1 \cap Z, B_1 \cap Z - \{(x, y, z)\}; G)$ for an abelian group G for each case of $(x, y, z) = (0, 0, 0), (1/2, 0, 0), (1/2, 1/2, 0)$. Do this for every finitely generated abelian group G.

(b) (10 pts.) Prove that under any self-homeomorphism $f : Z \to Z$, the origin, the union of the axes are mapped to themselves respectively.

2 [(25 pts.)] Let S denote a closed orientable surface of genus g with two points p, q for $g \geq 2$. Let X denote the space obtained by identifying p and q.

(a) (12 pts.) Find a 4-fold non-regular cover $\hat{X}$ of X with the covering map g and describe $g_* : \pi_1(\hat{X}) \to \pi_1(X)$ using generators and relations.

(b) (13 pts.) Compute the homomorphisms $g_* : H_*(\hat{X}) \to H_*(X)$ using the generators and relations for all dimensions.

3 [(25 pts.)] Let S be a closed surface of genus 2 embedded in $\mathbf{R}^3$ so that it meets the z-axis at two points and is symmetric about the z-axis. There is a standard involution ι on $\mathbf{R}^3$ about the z-axis acting on S. Let X be the space $S \times I / \sim$ where $(x, 0) \sim (\iota(x), 1)$.

(a) (12 pts) Find a finite presentation of the fundamental group of X.

(b) (13 pts.) Compute the ranks and torsion subgroups of homology groups of X for all dimensions.

4 [(25 pts.)] Let $\mathbf{R}^n$ be a Euclidean space of dimension $n \geq 3$. Let $\mathbf{S}^m$ be a sphere of dimension $m, 2 \leq m < n$. Let $f : \mathbf{S}^m \to \mathbf{R}^n$ be an embedding.

(a) (10 pts.) For every pair (n, m), find the number of components of $\mathbf{R}^n - f(\mathbf{S}^m)$.

(b) (15 pts.) For every pair (n, m), compute $H_*(\mathbf{R}^n - f(\mathbf{S}^m))$ for all dimensions.

THE END

Ph.D. Qualifying Exam: Algebraic Topology II
February 2024

Student ID: Name:

Note: Be sure to use English for your answers. Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. Here, when we ask you to "compute the group", this means that you have to write down the generators and relations of the the group. When we ask you to "compute the homomorphism", you need to write down the generators and relations of the two groups and finding where the generators go to under the map. When we use $*$, you need to do for all dimensions. (Total 100 pts.)

1. [30 pts.] Let X be the product of 3 unit circles $\mathbf{S}^1$. Let ι be the involution $X \to X$ given by $(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$. Let Y be the quotient space of X given by identifying every point with its image under ι. Let $q : X \to Y$ denote the quotient map.

   (a) [10 pts.] Compute the cohomology groups of X for all dimensions.

   (b) [10 pts.] Compute the cohomology groups of Y for all dimensions.

   (c) [10 pts.] Compute the homomorphism $q^* : H^*(Y) \to H^*(X)$.

   (Hint: Y is actually a mapping torus of a map of a torus.)

2. [40 pts.] Let Σ be a closed surface of genus 2 with the interior of a closed disk removed. Hence, $\partial\Sigma$ is homormorphic to a circle. Let X denote the wedge sum of a torus and a torus with the interior of a closed disk removed. Let ∂X denote the boundary of the disk here. There is an obvious quotient map $f : \Sigma \to X$ of degree 1 collapsing a simple closed curve α in the interior to a point.

   (a) [10 pts.] Compute the cohomology ring of $(X, \partial X)$.

   (b) [10 pts.] Compute the cohomology groups of $(\Sigma, \partial\Sigma)$ for all dimensions.

   (c) [10 pts.] Compute the group homomorphisms $f^* : H^*(X, \partial X) \to H^*(\Sigma, \partial\Sigma)$.

   (d) [10 pts.] Compute the cohomology ring of $(\Sigma, \partial\Sigma)$.

3. [30 pts.] Let P^2 denote the projective plane.

   (a) [15 pts.] Compute the groups $H_*(P^2; \mathbf{Z}_2)$ and $H^*(P^2; \mathbf{Z}_2)$.

   (b) [15 pts.] Compute all cup and cap products of elements of these groups.

   (Hint: use a cellular decomposition or a triangulation of P^2.)

THE END

Ph.D. Qualifying Exam: Real Analysis

February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Prove that the set of $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that
$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3}$$
is a set of (Lebesgue) measure zero.

2. Suppose that f and g are measurable functions on $\mathbb{R}^d$. Prove the following statements:
   (a) [10 pts] If f is integrable and g is bounded, then $f * g$ is uniformly continuous.
   (b) [10 pts] If f and g are integrable, and g is bounded, then $(f * g)(x) \to 0$ as $|x| \to \infty$.

3. Prove the following statements:
   (a) [10 pts] If $1 \leq p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.
   (b) [10 pts] If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

4. [15 pts] Prove that $L^p(\mathbb{R}^d)$ $(p \in [1, \infty))$ with the Lebesgue measure is a Hilbert space if and only if $p = 2$.

5. [15 pts] For a signed measure ν, prove that its total variation $|\nu|$ is a (positive) measure that satisfies $\nu \leq |\nu|$.

6. [15 pts] Let μ and ν be σ-finite measures on the Borel sets of the positive real line $[0, \infty)$. Suppose that $\phi(t) := \nu([0, t))$ is finite for every $t > 0$. Prove that for any μ-measurable function $f : [0, \infty) \to [0, \infty)$,
$$\int_0^\infty \phi(f(x)) d\mu(x) = \int_0^\infty \mu(\{x : f(x) > t\}) d\nu(t).$$

THE END

Ph.D. Qualifying Exam: Complex Analysis
February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1.[15 points] Prove that $\sum_{n=1}^{\infty} e^{-n^2} z^n$ is an entire function.

2.[15 points] Find all entire functions f such that $f(n\pi) = 0$ for any $n \in \mathbb{Z}$ and $|f(x+iy)| \leq Ce^{|y|} < \infty, x, y \in \mathbb{R}$ for some $C > 0$.

3.[15 points] Find all entire functions f which satisfies the property that for some $R, C > 0$, $|f(z)| \geq C$ when $|z| \geq R$.

4.[15 points] Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Prove that f is entire if f^2 is entire and f is continuous.

5.[20 points] Evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5+3\cos\theta} d\theta$.

6.[20 points] Show that a polynomial $f(z) = z^5 + 2z^3 + 1$ has no zero in $D(0; 2/3)$, three zeros in $D(0,1) \setminus \overline{D(0,2/3)}$ and two zeros in $D(0,2) \setminus \overline{D(0,1)}$

THE END

Ph.D. Qualifying Exam: Probability Theory
February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Assume that $\{X_n\}_{n\geq 1}$ is uniformly integrable and $X_n \to X$ in distribution as $n \to \infty$. Prove that
$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

2. [20 pts] Let $X_1, X_2, \cdots$ be independent and identically distributed random variables, uniformly distributed in the interval $[-1, 1]$. Show that
$$\frac{\sum_{k=1}^{n} kX_k}{n^{3/2}}$$
converges in distribution as $n \to \infty$. Also determine the limiting distribution. (Hint: You can use $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.)

3. [20 pts]

   (a) [5 pts] Let X be a random variable such that $\mathbb{E}|X|^k < \infty$ for some $k \geq 1$. Then, show that
   $$\lim_{x\to\infty} x^k \mathbb{P}(|X| > x) = 0.$$

   (b) [15 pts] Let $X_1, X_2, \cdots$ be independent and identically distributed random variables such that $\mathbb{E}X_1^2 = 1$. Using (a), prove that
   $$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \max_{1\leq i\leq n} |X_i| = 0 \qquad \text{in probability.}$$

4. [20 pts] Let $X_1, X_2, \cdots$ be independent and identically distributed random variables, and set $S_n := X_1 + \cdots + X_n$. Assume that there exists some $\delta > 0$ such that $\mathbb{E}[e^{\delta X_1}] = 1$.

   (a) [5 pts] Prove that $\{e^{\delta S_n}\}_{n\geq 1}$ is martingale w.r.t. canonical filtration.

   (b) [15 pts] Prove that for any $x > 0$,
   $$\mathbb{P}(S_k \geq x \text{ for some } k \geq 1) \leq e^{-\delta x}.$$

5. [20 pts] Let $X_1, X_2, \cdots$ be independent and identically distributed random variables such that $\mathbb{P}(X_1 = 1) = 2/3$ and $\mathbb{P}(X_1 = -1) = 1/3$. Set $S_n := X_1 + \cdots + X_n$. Define $\tau := \inf\{n : S_n = 10\}$. Compute $\mathrm{Var}[\tau]$.

THE END

Ph.D. Qualifying Exam: Advanced Statistics
February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. [30 pts] Let $Y_1, \ldots, Y_n$ constitute a random sample of size $n(>1)$ from $N(\mu, \sigma^2)$. Let $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$ be the sample mean and sample variance. Let Y_{n+1} be a random variable from $N(\mu, \sigma^2)$ independently from $Y_1, \ldots, Y_n$.

   (a) [5 pts] Consider two cases; (1) σ^2 is known and (2) σ^2 is unknown. In each case, derive an explicit expression for exact $100(1-\alpha)\%$ prediction interval for Y_{n+1}.

   (b) [7 pts] Assume σ^2 is known as 200 and prior information for μ is available with some uncertainty as $\mu \sim N(\mu_0, \tau_0^2)$. Derive an explicit expression for $100(1-\alpha)\%$ posterior predictive interval for Y_{n+1}.

   (c) [8 pts] Now, suppose σ^2 is unknown. To estimate σ^2, consider an estimator kS^2 where k is a positive constant. Find the value of k, say k^*, that minimizes $\text{MSE}(kS^2, \sigma^2)$. Derive an explicit expression for the ratio $\text{MSE}(k^*S^2, \sigma^2)/\text{MSE}(S^2, \sigma^2)$ for any finite $n(>1)$, and evaluate the limit of this ratio as $n \to \infty$.

   (d) [10 pts] Now, suppose prior information for σ^2 is available with some uncertainty as $\sigma^2 \sim IG(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2})$ in addition to prior information for μ as $\mu \sim N(\mu_0, \sigma^2/\kappa_0)$. Derive a marginal posterior mean $E(\sigma^2|Y_1, \ldots, Y_n)$ as a Bayes estimator $\hat{\sigma}_B^2$ for σ^2.

2. [20 pts] Data (x_i, Y_i) were collected for $i = 1, \ldots, n$. Assume that $Y_i|x_i \overset{ind}{\sim} N(\alpha + \beta(x_i - \bar{x}), \sigma^2)$ where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, and $x_1, ..., x_n$ are known constants.

   (a) [5 pts] Derive the maximum likelihood estimator (MLE), $\hat{\alpha}_M$ and $\hat{\beta}_M$ for α and β, respectively. Furthermore, prove that $\hat{\alpha}_M$ and $\hat{\beta}_M$ are uncorrelated.

   (b) [5 pts] Determine the exact distributions for $\hat{\alpha}_M$, $\hat{\beta}_M$, and $\hat{\alpha}_M + \hat{\beta}_M(x_i - \bar{x})$.

   (c) [10 pts] Suppose you want to predict the value of Y_{n+1} given the information x_{n+1} in addition to the information in $(Y_1, x_1), \ldots, (Y_n, x_n)$. With an appropriate probability assumption for Y_{n+1} given x_{n+1}, derive an exact $100(1-\alpha)\%$ prediction interval for Y_{n+1}.

3. [20 pts] Data (x_i, Y_i) were collected for $i = 1, \ldots, n$. Assume that $Y_i|x_i \overset{ind}{\sim} Poisson(\theta x_i)$ where $\theta(>0)$ is an unknown parameter and $x_1, \ldots, x_n$ are known constants.

(a) [10 pts] Provide what you consider to be the best choice for a $100(1-\alpha)\%$ confidence interval for the unknown parameter θ. Also, provide what you consider to be the best choice for $100(1-\alpha)\%$ credible interval when prior information is available as $\theta \sim Gamma(\alpha, \beta)$.

(b) [10 pts] If $\sum_{i=1}^{n} x_i = 0.82$, what is the power of the uniformly most powerful (UMP) test of an approximate size α for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$ when $\theta = 5$?

4. [30 pts] Let $X_1, \ldots, X_n$ constitute a random sample of size n from Binomial(2, π). It is of interest to find the best estimator $\hat{\theta}_{best}$ for $\theta = \pi^2$ under the restriction that $E(\hat{\theta}_{best}) = \theta$ for any finite n(> 1).

   (a) [10 pts] Find an explicit expression for the Cramer-Rao Lower Bound (CRLB) for the variance of any unbiased estimator for θ.

   (b) [10 pts] Develop an explicit expression for the minimum variance unbiased estimator (MVUE) as $\hat{\theta}_{best}$ for θ.

   (c) [5 pts] Find the MLE $\hat{\theta}_{mle}$ for θ and compare it with $\hat{\theta}_{best}$ obtained in (b) when $n \to \infty$.

   (d) [5 pts] Comment on the asymptotic efficiency of the estimator $\hat{\theta}_{best}$ obtained in (b) relative to the CRLB obtained in (a).

THE END

Ph.D. Qualifying Exam: Numerical Analysis
February 2024

Student ID: Name:

Note: Be sure to use English for your answers.

1. Consider the root finding problem $f(x) = 0$ for $f \in C^2(\mathbb{R})$. For an initial guess x_0 near the single root α, consider the iteration scheme:

$$x_{k+1} = x_k + q_k,$$

where q_k is an approximation of $p_k = -f(x_k)/f'(x_k)$. This method is called an inexact Newton iteration.

   (a) [4 points] Write down an expression for the error of the inexact Newton method $e_{k+1} = x_{k+1} - \alpha$ at the $(k+1)$th step in terms of the error of the (exact) Newton method at the $(k+1)$th step.

   (b) [10 points] Assume the difference between the updates in the Newton and inexact Newton iterations at step $k+1$ satisfy

$$|p_k - q_k| \leq \eta_k |f(x_k)|,$$

   where η_k is small; i.e., your approximate update is close to the Newton update. With this bound and your solution from (a), derive an upper bound for the error e_{k+1} of the inexact Newton step. Determine the convergence rate of this method.

   (c) [6 points] Take $q_k = -f(x_k)/f'(x_0)$. Then what can you say from (b)?

2. [10 points each] Gaussian quadratures that approximate as follows

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{k=0}^{N} w_k f(x_k),$$

where the quadrature nodes include the endpoints (i.e. $x_0 = -1$ and $x_N = 1$) are called *Gauss-Legendre-Lobatto* quadratures.

   (a) Suppose that the interior nodes $x_1, \ldots, x_{N-1}$ in the quadrature are the roots of $P_N'(x)$ where $P_N(x)$ is the N-th degree Legendre polynomial and that

$$w_k = \int_{-1}^{1} \prod_{\substack{j=0 \\ j \neq k}}^{N} \frac{x - x_j}{x_k - x_j}\, dx.$$

Show that the quadrature is exact for polynomials up to degree $2N-1$.
Hint: The three term recursion for Legendre polynomials is given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_k(x) = \frac{2k-1}{k} x P_{k-1}(x) - \frac{k-1}{k} P_{k-2}(x).$$

Then the following recurrence relation is true:

$$(x^2 - 1) P_N'(x) = N[x P_N(x) - P_{N-1}(x)].$$

(b) Find the 4-point Gauss-Legendre-Lobatto quadrature (nodes and weights) for approximating the integral $\int_{-1}^{1} f(x)\,dx$.

3. [15 points] Estimate the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 10^{-3} & 10^{-4} \\ 10^{-3} & 2 & 10^{-3} \\ 10^{-4} & 10^{-3} & 3 \end{bmatrix}$$

as accurately as possible.

4. [10 points each] Suppose that the solution $y(t)$ to the initial-value problem

$$\begin{cases} y' = f(t,y), & a \le t \le b \\ y(a) = \alpha \end{cases}$$

has 3 continuous derivatives. Consider the modified Euler method

$$\begin{aligned} w_0 &= \alpha, \\ w_{i+1} &= w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\right] \quad i = 0, \ldots, N-1. \end{aligned}$$

(a) Show that it agrees with the Taylor series method of order two.

(b) Find the local truncation error of the modified Euler method.

5. Assume that an $n \times n$ matrix A is symmetric, and B is an $m \times n$ matrix with $m < n$. Consider the so-called saddle point system of size $(n+m) \times (n+m)$:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}. \tag{1}$$

Let us consider the following equivalent system for some $\gamma > 0$

$$\begin{bmatrix} A + \gamma B^T B & B^T \\ -\gamma B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b + \gamma B^T c \\ -\gamma c \end{bmatrix}. \tag{2}$$

Assume that $A + \gamma B^T B$ is nonsingular. An iterative method for (2) is given by

$$\begin{aligned} x^{(k+1)} &= (A + \gamma B^T B)^{-1}(b + \gamma B^T c - B^T y^{(k)}), \\ y^{(k+1)} &= \gamma B x^{(k+1)} + y^{(k)} - \gamma c. \end{aligned} \tag{3}$$

(a) [4 points] Prove that (1) and (2) are equivalent.

(b) [4 points] Find the split $L - R$ of the coefficient matrix in (2) that gives (3).

(c) [8 points] Find the iteration matrix $M = L^{-1}R$ for the iterative method in (3).

(d) [9 points] By examining the spectral radius $\rho(M)$, show that the iteration (3) is convergent provided that all the eigenvalues of the matrix $B(A/\gamma + B^T B)^{-1} B^T$ lie within the interval $(0, 2)$.
Hint: Use the fact that the set of eigenvalues for a block triangular matrix (with square diagonal blocks) consists of those of its diagonal blocks.

THE END