Ph.D. Qualifying Exam: Algebra I
August 2023

Student ID: Name:

- **Note:** Be sure to use English for your answers.
- **Common Notation:** For a positive integer d, let C_d denote a cyclic group of order d.
- To "classify groups (with certain conditions) up to isomorphism" means to list all *distinct* isomorphism classes of groups (with certain conditions), constructed either by group presentation or by (semi-)direct product of standard examples (such as cyclic groups). When you define a semi-direct product $H \rtimes_\varphi G$, make sure to write down $\varphi\colon G \to \mathrm{Aut}(H)$.

1. [10 pts]
   (a) [3 pts] State the definition of PID.
   (b) [7 pts] State the fundamental theorem for finitely generated modules over PID. Make sure to state the uniqueness assertion, including the descriptions of the torsion part, in *two* versions.

2. [15 pts] Let p be a prime.
   (a) [6 pts] For a positive integer n, show that $\mathrm{Aut}(C_p^n)$ is isomorphic to $\mathrm{GL}_n(\mathbb{F}_p)$, the group of $n \times n$ invertible matrices over the prime field of characteristic p.
   (b) [9 pts] Compute the order of $\mathrm{GL}_n(\mathbb{F}_p)$.

3. [15 pts] Let H be a group, and suppose that $|\,\mathrm{Aut}(H)| = pm$ for some prime p and a positive integer m that are coprime to each other. Then for any non-trivial group homomorphisms $\varphi, \varphi'\colon C_p \to \mathrm{Aut}(H)$, show that the semidirect products $H \rtimes_\varphi C_p$ and $H \rtimes_{\varphi'} C_p$ are isomorphic.

4. [20 pts] Fix a prime p. Let G be a p-group such that all the non-trivial elements have order p. Let C_p denote a cyclic group of order p.
   (a) [5 pts] Show that $G \cong H \rtimes C_p$ for any maximal subgroup H of G.
   (b) [5 pts] When $p = 2$, classify up to isomorphism 2-groups whose non-trivial elements have order 2.
   (c) [10 pts] When $p > 2$, classify up to isomorphism groups of order p^3 whose non-trivial elements have order p.

CONTINUED TO THE NEXT PAGE

5. [15 pts] Verify the following claims.

   (a) [5 pts] For a positive integer r, find *all* subgroups of $\mathbb{C}^\times$ with order r.

   (b) [10 pts] For a positive integer n, show that any finite-order element $g \in \mathrm{GL}_n(\mathbb{C})$ is diagonalizable. Furthermore, give a necessary and sufficient condition for a diagonalizable matrix $g \in \mathrm{GL}_n(\mathbb{C})$ to have finite order in terms of its eigenvalues.

6. [25 pts] Let k be a field. (We allow k to be infinite.) Let $G := \mathrm{GL}_n(k)$ for some positive integer n.

   (a) [10 pts] Show that the set of conjugacy classes of G is in natural bijection with the set of tuples of monic non-constant polynomials with non-zero constant terms $(f_1, \cdots f_r)$ satisfying $n = \sum_{i=1}^r \deg(f_i)$ and $f_{i+1}|f_i$ for any i. Furthermore, if the conjugacy class of $\gamma \in G$ is associated to $(f_1, \cdots, f_r)$ then show that f_1 is the minimal polynomial of γ and $\prod_{i=1}^r f_i$ is the characteristic polynomial of γ.

   (b) [9 pts] Let $f(T) \in k[T]$ be a degree-n monic polynomial with non-zero constant term. Show that there exists *finitely many* conjugacy classes of G with characteristic polynomial $f(T)$, and there exists a *unique* conjugacy class of G whose *minimal polynomial* is $f(T)$. Furthermore, if $\gamma \in G$ is an element with *minimal polynomial* $f(T)$, then show that the centralizer $C_G(\gamma)$ is the injective image of the homomorphism

   $$(k[T]/(f(T)))^\times \to G; \quad \text{sending } g(T) \mapsto g(\gamma).$$

   (c) [6 pts] Let k'/k be any field extension, and set $H := \mathrm{GL}_n(k')$, which naturally contains G as a subgroup. For $\gamma, \gamma' \in G$, suppose that there exists $\delta \in H$ such that $\gamma' = \delta\gamma\delta^{-1}$. Show that γ and γ' are conjugate by an element of G.

THE END

Ph.D. Qualifying Exam: Algebra II
August 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts]
   (a) [5 pts] State the definition of noetherian ring.
   (b) [5 pts] Show that PID is noetherian.
   (c) [10 pts] Show that PID is UFD, assuming Zorn's lemma.

2. [15 pts] Given a finite group G, explicitly construct a Galois extension L/K with $G \cong \mathrm{Gal}(L/K)$. (Here, L and K are allowed to be any fields.)

3. [15 pts] Let $f(x) \in \mathbb{C}[x]$ be a separable non-constant monic polynomial.
   (a) [5 pts] For a positive integer n, show that $y^n - f(x) \in \mathbb{C}[x,y]$ is irreducible.
   (b) [10 pts] Set $K := \mathbb{C}(x)$ and $L := \mathrm{Frac}\,\mathbb{C}[x,y]/(y^n - f(x))$. Is L/K a finite Galois extension? If so, describe $\mathrm{Gal}(L/K)$ explicitly via standard examples of groups.

4. [25 pts] Let F be a field, and we fix an algebraic closure $\overline{F}$ of F. Let K/F and L/F be finite Galois extensions, and we embed both K and L into $\overline{F}$.
   (a) [9 pts] Show that there is an F-algebra isomorphism $K \otimes_F L \cong KL^{[F':F]}$, where KL is the compositum of K and L in $\overline{F}$ and $F' := K \cap L$ in $\overline{F}$.
   (b) [8 pts] Show that both F'/F and KL/F are Galois extensions. Furthermore, show that $\mathrm{Gal}(KL/F)$ is naturally isomorphic to the following subgroup of $\mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$:
   $$\{(\gamma, \delta) \in \mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F) \text{ such that } \gamma|_{F'} = \delta|_{F'}\}.$$
   (c) [8 pts] Is the group of F-algebra automorphisms of $K \otimes_F L$ isomorphic to $\mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$? If so, then give a proof. If not, then determine exactly when the group of F-algebra automorphisms of $K \otimes_F L$ is isomorphic to $\mathrm{Gal}(K/F) \times \mathrm{Gal}(L/F)$.

CONTINUED TO THE NEXT PAGE

5. [25 pts] Let k be a field, and we fix an algebraic closure $\bar{k}$ of k. Let $f(x) \in k[x]$ be a monic separable irreducible quartic polynomial, with roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \bar{k}$.

   Let K/k be the splitting field extension of $f(x)$, and set $G := \mathrm{Gal}(K/k)$. We view G as a subgroup of S_4 via the natural G-action on $\{\alpha_1, \cdots, \alpha_4\}$.

   Finally, we set

$$\theta_1 := (\alpha_1+\alpha_2)(\alpha_3+\alpha_4), \quad \theta_2 := (\alpha_1+\alpha_3)(\alpha_2+\alpha_4), \quad \theta_3 := (\alpha_1+\alpha_4)(\alpha_2+\alpha_3),$$

   and set $k' := k(\theta_1, \theta_2, \theta_3)$, which is a subextension of K/k.

   (a) [12 pts] Let $g(x) := (x-\theta_1)(x-\theta_2)(x-\theta_3) \in k'[x]$. Show that $g(x) \in k[x]$. Furthermore, show that k'/k is a Galois extension with $\mathrm{Gal}(k'/k) = G/(G \cap V)$ where V is the subgroup of S_4 consisting of all $(2,2)$-cycles and the trivial permutation.

   (b) [5 pts] Suppose that we have $k' = k$ (i.e., $\theta_1, \theta_2, \theta_3 \in k$). Show that $G = V$, where V is the subgroup of all $(2,2)$-cycles (and the trivial permutation) in S_4.

   (c) [8 pts] Suppose that exactly one of $\theta_1, \theta_2, \theta_3$ is an element of k. Then show that G contains a 4-cycle.

THE END

Ph.D. Qualifying Exam: Differential Geometry
August 2023

Student ID: Name:

Note: Be sure use English for your answers.

1. [10 pts] Let M be a smooth compact n-dimensional manifold without boundary, $n \geq 1$. Show that any smooth function $f : M \to \mathbb{R}$ has a critical point.
2. [15 pts] Let $SU(n) = \{A \in \mathrm{Mat}(n, \mathbb{C}) : AA^* = I_n,\ \det A = 1\}$ be the special unitary group, where $A^* = \overline{A^t}$ and $\mathrm{Mat}(n, \mathbb{C})$ is the space of $n \times n$ matrices with complex entries. Show that $SU(2)$ is diffeomorphic to $\mathbb{S}^3$ the unit 3-sphere in $\mathbb{R}^4$.
3. [15 pts = 10 pts + 5pts] Let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand rule rotation around the positive z-axis by t degree and $S_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand rule rotation around the positive x-axis by t degree.
   (a) Find the infinitesimal generators of the flows T_t and S_t, i.e., the vector fields X and Y, respectively, on $\mathbb{R}^3$ whose flows are T_t and S_t.
   (b) Compute the Lie bracket $[X, Y]$.
4. [20 pts] Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$. Let ω be a smooth 1-form on M. Show that
$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$
5. [20 pts = 10 pts +10 pts] Let $B(r) = \{x \in \mathbb{R}^3 : \|x\| \leq r\}$ be a closed ball in $\mathbb{R}^3$; oriented it naturally and give its boundary $S(r)$ the induced orientation. Assume that ω is a smooth 2-form defined in $\mathbb{R}^3 \setminus \{0\}$ such that
$$\int_{S(r)} \omega = a + \frac{b}{r},$$
for each $r > 0$, where a, b are some constants.
   (a) If $d\omega = 0$, what can you say about a and b?
   (b) If $\omega = d\eta$ for some smoothh 1-form η in $\mathbb{R}^3 \setminus \{0\}$, what can you say about a and b?
6. [20 pts = 10pts + 10pts]

   Definition. A **Lie group** is a smooth manifold G without boundary that is also a group in algebraic sense, with the property that the multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$ given by
$$m(g, h) = gh \quad \text{and} \quad i(g) = g^{-1}$$
are both smooth.

   (a) Denote $GL(n, \mathbb{R})$ the set of invertible $n \times n$ matrices with real entries. Consider it as a group under matrix multiplication. Show that it is a Lie group.
   (b) Let G, H be Lie groups. Let $F : G \to H$ be a smooth map such that it is also a group homomorphism, i.e., $F(g_1 g_2) = F(g_1)F(g_2)$ for every $g_1, g_2 \in G$. Show that F has constant rank.

THE END

Ph.D. Qualifying Exam: Algebraic Topology 1 (August 2023)

Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. If answers are not well organized, we will take points off. Here, when we ask you to "compute the group", this means that you have to write down the generators and relations of the the group. When we ask you to "compute the homomorphism", you need to write down the generators and relations of the two groups and finding where the generators go to under the map. (Total 100 pts.)

1. (20 pts.) Let $\mathbf{S}^1$ be the unit circle in $\mathbf{R}^2$. Let T^2 be a torus $\mathbf{S}^1 \times \mathbf{S}^1$. Let $\Delta := \{(x, y)|x = y, x \in \mathbf{S}^1, y \in \mathbf{S}^1\} \subset T^2$.
   (1) (10 pts) Compute the homology groups $H_i(T^2, \Delta)$ for $i = 0, 1, 2$.
   (2) (10 pts) Compute the homology groups $H_i(T^2 - \{(-1, 0) \times (1, 0)\}, \Delta; \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z})$ for $i = 0, 1, 2$.

2. (30 pts.) Let Σ denote a closed orientable surface of genus 2.
   (1) (15 pts.) Find a regular 3-fold cover Σ_1 of Σ by a covering map p_1. Find the corresponding homomorphism $p_{1*} : H_*(\Sigma_1) \to H_*(\Sigma)$.
   (2) (15 pts.) Find a nonregular 3-fold cover Σ_2 of Σ by a covering map p_2. Find the corresponding homomorphism $p_{2*} : H_*(\Sigma_2; \mathbf{Z}_3) \to H_*(\Sigma; \mathbf{Z}_3)$.

3. (25 pts.) Let $X = \mathbf{S}^2 - \{p, q, r, s\}$ for four distinct points p, q, r, s.
   (1) (10 pts.) Show that X is covered by $\mathbf{R}^2 - \mathbf{Z}^2$ with a regular cover p_X, and compute the deck transformation group. (Hint: think about 180 degree rotations about the x-axis for a torus imbedded in $\mathbf{R}^3$ in a standard way.)
   (2) (5 pts.) Show that a linear map L_M from an integral matrix M gives an self-homeomorphism of $\mathbf{R}^2 - \mathbf{Z}^2$ where there is a self-homeomorphism f_M of X with the following commutative diagrm.

$$\begin{array}{ccc} \mathbf{R}^2 - \mathbf{Z}^2 & \xrightarrow{L_M} & \mathbf{R}^2 - \mathbf{Z}^2 \\ p_X \downarrow & & p_X \downarrow \\ X & \xrightarrow{f_M} & X \end{array}$$

   (3) (10pts.) Find $H_*(X)$. Compute $f_{M*} : H_*(X) \to H_*(X)$ for $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

4. (25 pts.) Let $\mathbf{S}^{p+q-1}$ be the unit sphere in $\mathbf{R}^{p+q} = \mathbf{R}^p \times \mathbf{R}^q$ for $2 \leq p, q$, and let $X = \mathbf{S}^{p+q-1} \cap \mathbf{R}^p \times \{O\}$ and $Y = \mathbf{S}^{p+q-1} \cap \mathbf{R}^q$. Let $i_X : S \to \mathbf{S}^{p+q-1}$ and $i_Y : Y \to \mathbf{S}^{p+q-1}$ be the inclusion maps.
   (1) (7 pts.) Show that $\mathbf{S}^{p+q-1} - X$ deformation restracts to Y , and $\mathbf{S}^{p+q-1} - Y$ deformation retracts to X.
   (2) (10 pts) Compute the homology groups $H_*(\mathbf{S}^{p+q-1} - X)$ and $H_*(\mathbf{S}^{p+q-1} - Y)$ respectively.
   (3) (8 pts.) Compute the respective homomorphisms $(i_X)_* : H_*(X) \to H_*(\mathbf{S}^{p+q-1} - Y)$ and $(i_Y)_* : H_*(Y) \to H_*(\mathbf{S}^{p+q-1} - X)$.

Ph.D. Qualifying Exam: Real Analysis
August 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let $A \subset \mathbb{R}$ be a Lebesgue measurable set whose Lebesgue measure is strictly positive. Prove that there exists $B \subset A$ such that B is not Lebesgue measurable.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue integrable function. Prove the following:

   (a) [10 pts] $\lim_{y \to 0} \int f(x+y)dx = \int f(x)dx$

   (b) [10 pts] $\lim_{k \to \infty} \int f(x)e^{-x^2/k}dx = \int f(x)dx$

3. [15 pts] Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying
$$F(x) = \int_a^x f(y)dy$$
for a Lebesgue integrable function f. Prove that F is absolutely continuous (with respect to the Lesbesgue measure).

4. [15 pts] Let $\mathcal{H}$ be a separable Hilbert space and T be a non-zero linear bounded operator on $\mathcal{H}$. Suppose that T is compact and symmetric. Prove that $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

5. Suppose that $\mathcal{M}$ is a σ-algebra in a set X and μ a finite measure on $(X, \mathcal{M})$. We say that a sequence of measurable functions $\{f_n\} \to f$ in measure if for every $\epsilon > 0$
$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0$$
as $n \to \infty$.

   (a) [10 pts] Prove that if $f_n \to f$ almost everywhere (with respect to μ) then $f_n \to f$ in measure.

   (b) [10 pts] Prove that if $f_n \to f$ in measure then $\{f_n\}$ has a subsequence that converges to f almost everywhere (with respect to μ).

6. [15 pts] Assume that μ is a σ-finite measure on S. Suppose that $1 \le p, q \le \infty$ and $1/p + 1/q = 1$. Prove that, for every $f \in L^p(S, \mu)$,
$$\|f\|_p = \sup \left\{ \left| \int_S fg \, d\mu \right| : g \in L^q(X, \mu), \|g\|_q = 1 \right\}.$$

THE END

2023-2 QUALIFYING EXAM - COMPLEX ANALYSIS

Problem 1. (13pt) Let $f(z)$ be entire function such that $|e^{f(z)}| \le |z|$ for $|z| \ge 1$. What can you say about $f(z)$?

Problem 2. (12pt) Find a branch of $\sqrt{z(1-z)}$ so that it becomes a holomorphic (single-valued) function on $\mathbb{C} \setminus [0,1]$.

Problem 3. (15pt)Evaluate the following improper integral

$$\int_0^\infty \frac{\log x}{(1+x^2)(x^2+4)} dx.$$

Problem 4. (15pt) Find a partial fraction decomposition of

$$\frac{\pi}{\cos \pi z}.$$

Problem 5. (15pt) Find a conformal map of the vertical strip $\{-1 < \text{Re } z < 1\}$ onto the unit disc $\{|z| < 1\}$.

Problem 6. (15pt) Supoose that $D \neq \mathbb{C}$ is a simply connected domain. Construct an injective conformal map $f : D \to \{|z| < 1\}$. (Do not quote Riemann mapping theorem. This problem asks a part of its proof.)

Problem 7. (15pt) Let $D \neq \mathbb{C}$ be a simply connected domain. Suppose that $f : D \to D$ a holomorphic function having a fixed point $f(a) = a$. Show that $|f'(a)| \le 1$. Moreover if $|f'(a)| = 1$, then f is a homeomorphism of D.

Ph.D. Qualifying Exam: Probability Theory
August 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [10 points] Let $M_X(t) = \mathbb{E}[e^{tX}]$ be a moment generating function of X. Suppose that $M_X(t)$ is finite in some neighborhood of $t = 0$. Show that there exist constants $a, b > 0$ such that

$$\mathbb{P}(|X| \geq t) \leq ae^{-bt}, \quad \forall t > 0.$$

2. [15 points] Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$.

   (1) (5 points) Show that $X_n \to 0$ in probability if and only if $p_n \to 0$.

   (2) (10 points) Show that $X_n \to 0$ almost surely if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

3. [30 points] Suppose that $\{X_n\}_{n\geq 1}$ and X are (real-valued) random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Prove or provide a counterexample in each case:

   (1) (10 points) If $X_n \to X$ in probability, then $f(X_n) \to f(X)$ in probability.

   (2) (10 points) If $X_n \to X$ in distribution, then $f(X_n) \to f(X)$ in distribution.

   (3) (10 points) If $X_n \to X$ in $L^1(\Omega)$, then $f(X_n) \to f(X)$ in $L^1(\Omega)$.

4. [20 points] Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables such that $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) < \infty$. Show that

$$\frac{2}{n^2} \sum_{1\leq i<j\leq n} X_i X_j \to \mu^2$$

in probability.

5. [25 points] Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 2) = 1/2$. Let $\{\mathcal{F}_n\}_{n\geq 1}$ be the canonical filtration associated to $X_1, X_2, \cdots$. Define $Y_n := \prod_{k=1}^{n} X_k$.

   (1) (10 points) Show that Y_n is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$.

   (2) (15 points) Show that it is NOT possible to find a random variable Z with $\mathbb{E}|Z| < \infty$ such that $Y_n = \mathbb{E}(Z|\mathcal{F}_n)$.

THE END

Ph.D. Qualifying Exam: Advanced Statistics
August 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [30 pts] Suppose X is one observation from a population with beta$(\theta, 1)$ pdf with a pdf of $f(x|\theta) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)} x^{\theta-1} I_{(0,1)}(x)$ and $\theta > 0$.
   (a) [15 pts] Find the most powerful level α test of $H_0 : \theta = 1$ and $H_1 : \theta = 2$.
   (b) [15 pts] Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it. If not, prove so.

2. [20 pts] Consider a hierarchical model
   $Y_n|W_n = w_n \sim N(0, w_n + (1 - w_n)\sigma_n^2)$,
   $W_n \sim$ Bernoulli(p_n).
   Suppose $p_n \to 1$ and $\sigma_n \to \infty$ in such a way $(1 - p_n)\sigma_n^2 \to \infty$.
   (a) [5 pts] Compute $E(Y_n)$ and $Var(Y_n)$.
   (b) [5 pts] What is the limiting variance of Y_n, $\lim_{n\to\infty} Var(Y_n)$?
   (c) [10 pts] What is the limiting distribution of Y_n?

3. [30 pts] Let X be a discrete random variable with $P(X = -1) = 2p(1 - p)$, and $P(X = k) = p^k(1 - p)^{3-k}$, $k = 0, 1, 2, 3$, where $p \in (0, 1)$.
   (a) [15 pts] Is the family of the distributions complete?
   (b) [15 pts] Determine whether there is a UMVUE of $p(1 - p)$.

4. [20pts] Let $X_1, \ldots, X_n$ be iid random variables with the pdf
   $f_\theta(x) = \sqrt{2\theta/\pi} \exp(-\theta x^2/2) I_{[0,\infty)}(x)$, where $\theta > 0$ is unknown. Let the prior of θ be the Gamma distribution $\Gamma(\alpha, \gamma)$ with the pdf $\pi(\theta) = \frac{1}{\Gamma(\alpha)\gamma^\alpha} \theta^{\alpha-1} \exp(-\theta/\gamma) I_{[0,\infty)}(\theta)$. Find the Bayes estimator of θ [10 pts] and its Bayes risk [10 pts] under the loss function $L(\theta, a) = (a - \theta)^2/\theta$.

THE END

Ph.D. Qualifying Exam: Numerical Analysis
August 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. Consider a function $f \in C^\infty$ in a neighborhood of its root $f(x^*) = 0$.
   (a) [7 points] Show that if x^* is a simple root, i.e. $f'(x^*) \neq 0$, then the Newton's method converges quadratically in some neighborhood of x^*.
   (b) [8 points] Determine the rate of convergence of the Newton's method if the root has multiplicity m, where $m \geq 2$.
2. (a) [6 points] Let $f\colon [a, b] \to \mathbb{R}$ be a smooth function. Given the points $a < x_0 < x_1 < \cdots < x_n < b$, show there is a unique polynomial that interpolates the data $(x_i, f(x_i)), i = 0, \dots, n$.
   (b) [8 points] Let $\epsilon > 0$ and consider the three data values $(0, f(0))$, $(\epsilon, f(\epsilon))$ and $(1, f(1))$. Let q be the polynomial that arises as the limit of the second order polynomial interpolant of the data as $\epsilon \to 0$.
      i. What is the degree of q?
      ii. What data (if any) does q interpolate?
      iii. What data (if any) does q' interpolate?
   (c) [6 points] Given the points $a < x_0 < x_1 < \cdots < x_n < b$, denote by Ψ the function
   $$\Psi(x) = \sum_{j=0}^{n} c_j e^{jx}$$
   such that
   $$\Psi(x_i) = f(x_i), \quad i = 0, \dots, n.$$
   Is the choice of interpolation constants $c_0, \dots, c_n$ unique? Provide justification for your answer.
3. Consider the task of numerically approximating
   $$I(f) = \int_a^b f(x)\, dx,$$
   where $f \in C^\infty[a, b]$.
   (a) [6 points] Derive the trapezoidal rule and corresponding error for approximating $I(f)$. Useful information: $\int_a^b (x-a)(x-b)\, dx = -\frac{1}{6}(b-a)^3$.

(b) [6 points] Find the formula for the composite trapezoidal rule using uniform intervals of size $h = \frac{b-a}{n}$ where $n+1$ is the number of quadrature points, i.e. the quadrature points are $x_j = a + jh$ for $j = 0, \ldots, n$.

(c) [8 points] Derive the error for the composite trapezoidal rule.

4. (a) [8 points] Suppose that $\lambda_1, \ldots, \lambda_n$ ($|\lambda_1| > \cdots > |\lambda_n|$) are eigenvalues of a symmetric positive definite matrix A associated with eigenvectors $v_1, \ldots, v_n$. Let x satisfy $x^T v_1 = 1$. Show that

$$B = A - \lambda_1 v_1 x^T$$

has eigenvalues $0, \lambda_2, \ldots, \lambda_n$ and find their corresponding eigenvectors.

(b) [8 points] You first find the first eigenvector v_1 by the power method. Once you have found a good approximation for (λ_1, v_1), can you find the second eigenvector v_2 by the power method? Can you also find all eigenvectors $v_3, \ldots, v_n$ by the power method? Justify your answer.

(c) [9 points] Can you observe numerical instability in the above algorithm? Justify your answer. If you observe the numerical instability, can you state a better way to reduce it in the implementation?

5. Consider a linear system of ODEs where the coefficient matrix is split into two matrices as $A+B$, so the system is

$$\begin{cases} \frac{du}{dt} = Au + Bu \\ u(0) = u_0 \end{cases}$$

(a) [6 points] Show formally that the solution to this ODE is given by

$$u(t) = e^{(A+B)t} u_0$$

(b) [7 points] Suppose that you discretize $U^n \approx u(n\Delta t)$. Then we can advance from U^n to U^{n+1} exactly using the formula above, i.e.,

$$U^{n+1} = e^{(A+B)\Delta t} U^n.$$

However, in many situations, computing the exponential of $A+B$, can be too difficult, but computing the exponential of A and B separately is easier. In this case, we can split the computation in two stages following

$$\begin{aligned} U^* &= e^{B\Delta t} U^n, \\ U^{n+1} &= e^{A\Delta t} U^*, \end{aligned} \tag{1}$$

where the ODE systems $u_t = Au$ and $u_t = Bu$ are solved exactly in each part of the splitting. This is often called an operator splitting method. Derive a formula for the error $U^{n+1} - u(t_{n+1})$, assuming $U^n = u(t_n)$ exactly in Eq. (1).

(c) [7 points] Using your formula, show that, for general A and B, the local truncation error is $O((\Delta t)^2)$.

THE END