Ph.D. Qualifying Exam: Algebra I
February 2023

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. (25 pts) Let G and H be finite groups. We write $\mathrm{Aut}(G)$ for the automorphism group of G.
   (a) (10 pts) Show that $\mathrm{Aut}(G \times H)$ contains a subgroup isomorphic to $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$. Show also that if G and H have relatively prime orders, then $\mathrm{Aut}(G \times H) \simeq \mathrm{Aut}(G) \times \mathrm{Aut}(H)$.
   (b) (15 pts) Prove or disprove the following statement: If $\mathrm{Aut}(G)$ is cyclic, then so is G.

2. (25 pts) Let p and q be prime numbers and let G be a nontrivial p-group.
   (a) (10 pts) Show that the center of G is nontrivial. Show also that if G is non-abelian, then the size of each conjugacy class in G is at most p^{n-2}.
   (b) (5 pts) Show that G is solvable.
   (c) (10 pts) Show that every group of order p^2q is solvable.

3. (25 pts) Let N be a normal subgroup of a finite group G and let P be a Sylow p-subgroup of N for a prime number p. Let K denote the normalizer of P in G.
   (a) (7 pts) Prove that if P is normal in N, then P is normal in G.
   (b) (8 pts) Prove that $G = NK$.
   (c) (10 pts) Prove that K contains a Sylow p-subgroup of G.

4. (15 pts) Let $n = p^3q^2r$, where p, q, and r denote distinct prime numbers. Let R be the polynomial ring in two variables with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Find all nilpotent elements in R.

5. (10 pts) Let F be a field with 3 elements and let $n \geq 1$ be an integer. Determine up to similarity all 4×4 matrices A with entries in F such that $A^{8n} = I$ and $A^2 \neq 2I$, where I denotes the identity matrix.

THE END

Ph.D. Qualifying Exam: Algebra II
February 2023

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. (20 pts) Let $R[x]$ denote the polynomial ring in one variable over a domain R.
   (a) (10 pts) Prove that if $R[x]$ is a unique factorization domain, then so is R.
   (b) (10 pts) Prove that $R[x]$ is a principal ideal domain if and only if R is a field.

2. (20 pts) Show that $\mathbb{Q}(\sqrt[3]{7}) \cap \mathbb{Q}(\sqrt[3]{5}) = \mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers.

3. (20 pts) Let R denote the polynomial ring in one variable over $\mathbb{Q}$.
   (a) (10 pts) For each positive integer $n \leq 4$, show that every polynomial in R of degree n is solvable by radicals.
   (b) (10 pts) For each positive integer $n \geq 5$, provide a concrete example of a polynomial in R of degree n that is not solvable by radicals.

4. (20 pts) Let $E = \mathbb{Q}(\alpha)$, where α denotes a root of the polynomial $x^6 - 6x^3 + 6$. Determine the group of all $\mathbb{Q}$-automorphisms of E.

5. (10 pts) Let I be a nonzero proper ideal of a Noetherian domain R. Show that there exist prime ideals $P_1, \ldots, P_n$ of R satisfying $\prod_{i=1}^{n} P_i \subset I \subset \bigcap_{i=1}^{n} P_i$.

6. (10 pts) Let F be a field with 5 elements. Let $f(x) = x^4 + ax + 1 \in F[x]$ for some $a \in F$. Find the splitting field of $f(x)$ over F.

THE END

Ph.D. Qualifying Exam: Differential Geometry
February 2023

Student ID: Name:

Note: Be sure use English for your answers.

Unless otherwise is stated a smooth manifold means a smooth manifold without boundary.

1. [10 pts] Give a counterexample to show that a finite product of smooth manifolds with boundary need not be a smooth manifold with boundary.
2. [20 pts] Let M be a connected smooth manifold. Prove that for any $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. [*hint.* first consider the case $p, q \in B(0, 1)$ the open unit ball in $\mathbb{R}^n$.]
3. [30 pts= 15pts +15pts] Prove (without using deRham's theorem) the following:
   (a) Every closed 1-form α on the unit 2-sphere $\mathbb{S}^2$ is exact.
   (b) A smooth 2-form η on $\mathbb{S}^2$ is exact if and only if
   $$\int_{\mathbb{S}^2} \eta = 0.$$
4. [20 pts] Let M be a smooth manifold and $\pi : E \to M$ be a smooth line bundle (i.e., vector bundle of rank 1). Show that E is a trivial line bundle (i.e., there is a global trivialization $\phi : E \to M \times \mathbb{R}$) if and only if it admits a smooth global section that is nowhere vanishing.
5. [20 pts = 15pts + 5pts] Suppose M, N are smooth manifolds and $F : N \to M$ is a smooth map. Let $S \subseteq M$ be an embedded submanifold.

   Definition. We say that F is **transverse to** S if for every $x \in F^{-1}(S)$, the spaces $T_{F(x)}S$ and $dF_x(T_xN)$ together span $T_{F(x)}M$, where we consider T_pS as a subspace of T_pM for every $p \in S$.

   (a) Prove that if F is transverse to S, then $F^{-1}(S)$ is an embedded submanifold of N whose codimension is equal to the codimension of S in M.
   (b) Let $S' \subseteq M$ be an embedded submanifold together with the inclusion map $\iota : S' \hookrightarrow M$. Assume that ι is transverse to S (in this case we also say that S' intersects transversely to S). Conclude that $S \cap S'$ is an embedded submanifold of M whose codimension is equal to the sum of codimensions of S and S'.

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
February 2023

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (a) [10 pts] Show that a topological space is contractible if and only if its identity map is homotopic to a constant map.
   (b) [10 pts] Show that S^∞ is contractible.
2. (a) [10 pts] Prove that if $p_1 \colon \widetilde{X}_1 \to X_1$ and $p_2 \colon \widetilde{X}_2 \to X_2$ are covering maps, then $p_1 \times p_2 \colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ is also a covering map.
   (b) [10 pts] Let G be a finitely generated abelian group. Construct a $K(G, 1)$-space.
3. (a) [10 pts] Describe all the connected 2-sheeted covering spaces of
   $$S^1 \vee S^1 \vee \mathbb{R}P^2.$$
   (b) [10 pts] Construct two nonnormal covering spaces of the Klein bottle which are not homeomorphic to each other.
4. Let n be a positive integer.
   (a) [10 pts] Suppose $f \colon S^n \to S^n$ is a continuous function with $f(x) = f(-x)$ for each $x \in S^n$. Prove that f has even degree.
   (b) [10 pts] Prove or disprove that every continuous map $f \colon S^n \to S^n$ can be homotoped to have a fixed point.
5. Compute the homology groups of the following spaces:
   (a) [10 pts] A space obtained by gluing the Möbius band to a torus $S^1 \times S^1$ via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ of the torus.
   (b) [10 pts] Let n and m be positive integers. A space obtained by gluing two copies of $S^n \times B^m$ along their boundaries via the identity map.

THE END

Ph.D. Qualifying Exam: Algebraic Topology II
February 2023

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

Every spaces are CW complexes and $H_*(X)$ is $H_*(X;\mathbb{Z})$.

1. (a) [10 pts] Prove or disprove that if there are path connected topological spaces X and Y with $\pi_*(X) \cong \pi_*(Y)$ but $H_*(X)$ and $H_*(Y)$ are not isomorphic.
   (b) [10 pts] Prove or disprove that if there are path connected topological spaces X and Y with $\pi_*(X) \cong \pi_*(Y)$ and $H_*(X) \cong H_*(Y)$ but $H^*(X)$ and $H^*(Y)$ are not isomorphic as rings.
2. Let X be a oriented closed 4-dimensional manifold with
$$\pi_1(X) \cong \mathbb{Z}^2 * \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} \qquad \text{and} \qquad H_2(X;\mathbb{Z}) \cong \mathbb{Z}.$$
   (a) [10 pts] Compute $H^*(X;\mathbb{Z}/3\mathbb{Z})$.
   (b) [10 pts] Prove that any CW complex structure of X has at least four 3-cells.
3. (a) [10 pts] Prove or disprove that if two spaces have isomorphic cohomology groups with $\mathbb{Z}$ coefficients then they have isomorphic cohomology groups with $\mathbb{Z}/p\mathbb{Z}$ coefficients for each prime p.
   (b) [10 pts] Prove or disprove that if two spaces have isomorphic cohomology rings with $\mathbb{Z}$ coefficients then they have isomorphic cohomology rings with $\mathbb{Z}/p\mathbb{Z}$ coefficients for each prime p.
4. (a) [10 pts] Compute the cohomology ring for $S^2 \times S^2$.
   (b) [10 pts] Prove or disprove that every continuous map $S^4 \to S^2 \times S^2$ induces the trivial homomorphism $H_4(S^4) \to H_4(S^2 \times S^2)$.
5. (a) [10 pts] Let X and Y are topological spaces with contractible universal covers. Prove or disprove that if a continuous map $f\colon X \to Y$ induces an isomorphism on their fundamental groups then X and Y are homotopy equivalent.
   (b) [10 pts] Prove or disprove that every closed simply connected 3-manifold is homotopy equivalent to S^3.

THE END

Ph.D. Qualifying Exam: Real Analysis
February 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous mapping. Prove that, if A is a Borel subset of $\mathbb{R}^n$, then $f^{-1}(A)$ is a Borel subset of $\mathbb{R}^m$.

2. Prove the following:

   (a) [10 pts] There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but $\limsup_{x\to\infty} f(x) = \infty$.

   (b) [10 pts] If f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

3. [15 pts] Suppose that $a, b > 0$. Let

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{if } 0 < x \le 1\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

   Prove that f is of bounded variation in $[0, 1]$ if and only if $a > b$.

4. For a bounded linear operator T on a Hilbert space $\mathcal{H}$, we say that T is an isometry if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$.

   (a) [10 pts] Prove that $T^*T = I$ if T is an isometry.

   (b) [10 pts] Prove that if an isometry T is surjective then it is unitary and $TT^* = I$.

5. [15 pts] Suppose that $\mathcal{M}$ is a σ-algebra in a set X and μ a (positive) measure on $(X, \mathcal{M})$. For $f \in L^1(\mu)$, define a signed measure λ on $(X, \mathcal{M})$ by $\lambda(E) = \int_E f\, d\mu$ for $E \in \mathcal{M}$. Prove that

$$|\lambda|(E) = \int_E |f|\, d\mu.$$

6. [15 pts] Let F be an increasing function on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$. Let μ be a Borel measure defined by $\mu((a, b)) = F(b^-) - F(a^+)$ and $\mu(\{0\}) = \mu(\{1\}) = 0$. Suppose that the function F satisfies a Lipschitz condition $|F(x) - F(y)| \le A|x - y|$ for some $A > 0$. Prove that $\mu \ll m$, where m is the Lebesgue measure on $[0, 1]$.

THE END

Ph.D. Qualifying Exam: Complex Analysis
February 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let $f(z)$ is holomorphic in a connected domain D. Assume that $f(z)$ is constant on a curve $C \subset D$. Show that $f(z)$ is constant in D.

2. [15 pts] Evaluate the following improper integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2} dx$$

3. [15 pts] Prove that the following infinite product converges and evaluate it

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{n}\right)$$

4. [20 pts] Denote the upper half plane by $\mathbb{H} = \{\text{Im } z > 0\}$. Find most general form of linear fractional transforms that maps $\mathbb{H}$ onto $\mathbb{H}$. Show that any conformal self-map of $\mathbb{H}$ is of that form.

5. [20 pts] Find poles and their principal parts of $\frac{1}{\sin^2 z}$. Verify the partial fraction formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}.$$

From this deduce that

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{k \neq 0} \left(\frac{1}{z-k} + \frac{1}{k} \right).$$

6. [15 pts] Construct an entire function that has simple zeros at the points n^2, for each $n \in \mathbb{N}$ and no other zeros.

THE END

Ph.D. Qualifying Exam: Probability Theory
February 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] State and prove the central limit theorem.
   (You can use the fact " $c_n \to c \in \mathbb{C} \Rightarrow (1+\frac{c_n}{n})^n \to e^c$ " without a proof.)
2. [15 pts] Let $\{X_n\}_{n=1,2,\cdots}$ and X be (real-valued) random variables. Suppose that X_n converges to X in probability. Prove that for any continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(X_n)$ also converges to $f(X)$ in probability.
3. [15 pts] Let $\{X_n\}_{n=1,2,\cdots}$ be independent random variables such that
   $$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1-\frac{1}{n} \\ -1 & \text{with probability } \frac{1}{2n} \end{cases}.$$
   Let $Y_1 := X_1$ and
   $$Y_n = \begin{cases} X_n & \text{if} \quad Y_{n-1} = 0 \\ nY_{n-1}|X_n| & \text{if} \quad Y_{n-1} \neq 0 \end{cases}.$$
   Show that $(Y_n)_{n\geq 1}$ is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$.
4. [15 pts] Let $\{X_n\}_{n=1,2,\cdots}$ be i.i.d. random variables with $\mathbb{P}(X_n = -1) = \mathbb{P}(X_n = 1) = 1/2$. Set $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$ for $n \geq 1$. For positive integers a, b, define
   $$\tau := \inf\{n \geq 1 : S_n = -a \text{ or } S_n = b\}.$$
   Compute $\mathbb{E}\tau$. (Hint: Consider a sequence $S_n^2 - n$.)
5. [20 pts] Let $\{X_n\}_{n=1,2,\cdots}$ be i.i.d. random variables with $\mathbb{E}X_1 = 0$. Let $\alpha > 0$ be a constant. Show that the following two statements are equivalent.
   (a) $\lim_{n\to\infty} \frac{X_n}{n^{1/\alpha}} = 0$ almost surely.
   (b) $\mathbb{E}|X_1|^\alpha < \infty$.
6. [20 pts] Let $\{X_n\}_{n=1,2,\cdots}$ be an i.i.d. sequence of standard normal random variables. Show that almost surely,
   $$\lim_{n\to\infty} \frac{\max\{X_1, \cdots, X_n\}}{\sqrt{\log n}} = \sqrt{2}.$$
   Hint: If X is a standard normal random variable, then for any $t > 1$,
   $$\frac{1}{2\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2} \leq \mathbb{P}(X > t) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2}.$$

THE END

Ph.D. Qualifying Exam: Advanced Statistics

February 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ be independently distributed as Exponential distribution $E(a, b)$ and $E(c, b)$, respectively. Here $a \in R, b > 0$ and $c \in R$ are unknown parameters.

   Note that the pdf of Exponential distribution $E(a, b)$ is given by $b^{-1}e^{-(x-a)/b}$ when $x > a$ and 0 otherwise. The parameters are $a \in R$ and $b > 0$.

   (a) [15 pts] Show that $X_{(1)}$, $Y_{(1)}$, $\sum_{i=1}^{m}(X_i - X_{(1)}) + \sum_{i=1}^{n}(Y_i - Y_{(1)})$ jointly are sufficient and complete.

   (b) [15 pts] Find the uniformly minimum variance unbiased estimators of b and $(a - c)/b$.

2. Let (X, U) be a pair of real random variables, where U has a uniform$[0, \theta]$ density, for some $0 < \theta < \infty$, and where X given $U = u$ has the exponential density ue^{-ux}. Let the resulting marginal density of X be denoted $f_\theta(x)$. Also let $X_1, X_2, \dots, X_n$ be a sample of independent and identically distributed random variables from the density $f_{\theta_0}(x)$, where θ_0 is unknown.

   (a) [10 pts] Derive the formula for $f_\theta(x)$.

   (b) [10 pts] Derive the formula for the expectation of X^t, for $0 < t < 1$. (Hint: first compute $E(X^t|U = u)$.) What happens when $t \geq 1$?

   (c) [10 pts] Derive the formulas for the expectation and variance of $\log(X)$. You may use the fact that $\dot{\Gamma}(1) = -\gamma$ and $\ddot{\Gamma}(1) = \gamma^2 + \pi^2/6$, where $\dot{\Gamma}$ and $\ddot{\Gamma}$ are the first and second derivatives, respectively, of the gamma function $\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}dy$, and where $\gamma \approx 0.5772$ is Euler's constant.

   (d) [10 pts] Find an estimator $\hat{\theta}_n$, based on $X_1, X_2, \dots, X_n$, such that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is asymptotically mean zero normal with variance $\sigma_0^2 < \infty$, and give the formula for the variance.

   (e) [10 pts] Derive a consistent estimator $\hat{\sigma}_n^2$ of σ_0^2.

3. [20 pts] Let $X_1, X_2, \dots, X_n$ be i.i.d. from the Pareto distribution $Pa(\gamma, \theta)$, where $\theta > 0$ and $\gamma > 0$ are unknown. Show that a likelihood test for $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$ rejects H_0 when $Y < c_1$ or $Y > c_2$, where $Y = \log(\prod_{i=1}^{n} X_i / X_{(1)}^n)$ and c_1 and c_2 are positive constants. Find values of c_1 and c_2 so that this likelihood ratio test has size α.

   Note that the pdf of the Pareto distribution $Pa(\gamma, \theta)$ is given by $\theta\gamma^\theta x^{-(\theta+1)}$ when $x > \gamma$ and 0 otherwise. The parameters are $\gamma > 0$ and $\theta > 0$.

THE END

Ph.D. Qualifying Exam: Numerical Analysis
February 2023

Student ID: Name:

Note: Be sure to use English for your answers.

1. [5 points each] Consider Newton's method for solving the equation $\sin x = 0$ in the interval $(-\pi/2, \pi/2)$ starting with the initial approximation x_0 such that $\tan x_0 = 2x_0$ (Note that $x_0 \approx \pm 1.1656$).

   (a) What is the result of this iteration?

   (b) What is the result of the iteration if the initial approximation $\tilde{x}_0$ satisfies $|\tilde{x}_0| < |x_0|$?

   (c) What is the result of the iteration if the initial approximation $\tilde{x}_0$ satisfies $|\tilde{x}_0| > |x_0|$?

2. [5 points each] Let $f \in L^2[0, 2\pi]$, be a real-valued function that is squared integrable on $[0, 2\pi]$. We periodically extend it onto the whole real line. Its Fourier coefficients are then defined as

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ikx} dx, \quad \forall k \in \mathbb{Z}.$$

   We define the truncated Fourier series:

$$S_N = \sum_{k=-N}^{N} \hat{f}(k) e^{ikx}.$$

   We assume $f \in C^{m-1}$, i.e. f and its first $m-1$ derivatives are continuous functions, and we assume $f^{(m)}(x) \in L^2[0, 2\pi]$.

   (a) Show that

$$\|f - S_N\|_2^2 = 2\pi \sum_{|k|>N} |\hat{f}(k)|^2.$$

   (b) Show that

$$\|f^{(m)}\|_2^2 = 2\pi \sum_{-\infty}^{\infty} |k|^{2m} |\hat{f}(k)|^2,$$

   where the term on the left is the square of L^2 norm of m-th derivative of function $f(x)$:

$$\|f^{(m)}\|_2^2 = \int_0^{2\pi} |f^{(m)}|^2 dx.$$

(c) Based on the results above, prove that there exists a constant C such that

$$\|f - S_N\|_2 \le CN^{-m}\|f^{(m)}\|_2.$$

(d) Show to following L^∞ bound:

$$\|f(x) - S_N(x)\|_\infty = \max_{x\in[0,2\pi]} |f(x) - S_N(x)| \le CN^{-(m-1/2)}\|f^{(m)}\|_2.$$

(e) Explain and interpret the results on this questions. What does it says about the convergence of S_N and its convergence rate?

3. For functions defined on a closed interval $[0, 1]$, we want to compute the definite integral

$$I[f] = \int_0^1 f(x)\ln(1/x)\,dx$$

that is, with a logarithmic weight function $\ln(1/x) = -\ln(x)$. It is possible to find a basis of orthogonal polynomials $P_n(x)$ for the corresponding weighted inner product, and use them to derive n-point Gaussian quadratures with nodes x_k^n and weights ω_k^n. In this problem, you will explore these polynomials and techniques for creating the corresponding Gaussian quadrature.

(a) [5 points] Let $P_0(x) = 1$. Use Gram-Schmidt process or another technique to find $P_1(x)$. Find the corresponding quadrature node x_1^1 and weight ω_1^1 for the 1-point Gaussian quadrature rule.

(b) [5 points] The family P_n satisfies a recursion formula of the form:

$$P_{n+1}(x) = (x - \delta_{n+1})P_n(x) - \gamma_{n+1}^2 P_{n-1}(x),$$

where $P_{-1}(x) := 0$. Consider the normalized family $Q_n(x) = P_n(x)/h_n$, with $h_n = \left(\int_0^1 P_n(x)^2\ln(1/x)dx\right)^{1/2}$. Given that $\gamma_{n+1} = h_n/h_{n-1}$, find a recursive formula for $Q_n(x)$ of the form:

$$\gamma_{n+2}Q_{n+1}(x) = (x - \delta_{n+1})Q_n(x) - \gamma_{n+1}Q_{n-1}(x)$$

(c) [10 points] Consider the recursive formula above for $n = 0, 1, 2, 3$. Show that $x = \lambda$ is a node of the 4 point Gaussian quadrature if and only if it is an eigenvalue of a symmetric, tridiagonal matrix with diagonal $[\delta_1, \delta_2, \delta_3, \delta_4]$ and super / sub diagonal $[\gamma_2, \gamma_3, \gamma_4]$. Indicate why deriving this eigenvalue problem using the normalized polynomials might be preferable to solving the eigenvalue problem with the polynomials from the original recursive formula.

4. (a) [5 points] State and apply the Gerschgorin theorem to the matrix

$$A = \begin{bmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ \epsilon & \epsilon & 2 \end{bmatrix}$$

with $\epsilon << 1$.

(b) [10 points] It is often possible to improve the Gerschgorin bounds on the eigenvalue estimate by first applying a similarity transformation to the matrix A involving a diagonal matrix $D_n = \mathrm{diag}(d_1, \ldots, d_n)$. Prove that a reduction in the error bound of one eigenvalue for the matrix above must occur at the expense of relaxing the bounds for the remaining eigenvalues.

(c) [5 points] Let $D_3 = \mathrm{diag}(1, k\epsilon, k\epsilon)$, show that the bounding radius for $\lambda_1 \sim 1$ can be reduced from $\rho_1 = 2\epsilon$ to $2\epsilon^2$.

(d) [5 points] How would you use the Gerschgorin circle for $\lambda_1 \sim 1$ to compute a numerical approximation for the eigenvalue in that ball?

5. (15 points) Consider the following iteration for solving $Ax = b$, which is called the weighted Jacobi method:

$$x^{(i+1)} = \omega(D^{-1}(E + F)x^{(i)} + D^{-1}b) + (1 - \omega)x^{(i)},$$

where D is the diagonal matrix whose diagonal entries are those of A, $-E$ is the strictly lower triangular part of A, and $-F$ is the strictly upper triangular part of A. Note that the method reduces to Jacobi method for the case $\omega = 1$. Suppose that A is symmetric and positive definite, and $D = aI$ for a positive constant a. Find the condition for ω, in the form $p < \omega < q$, which guarantees the convergence of the weighted Jacobi method.

THE END