Ph.D. Qualifying Exam: Algebra I
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Let G be a non-abelian group of order 2022. Find the largest normal Sylow subgroup of G.

2. [10 pts] Let R be a ring with 1. Let M be an R-module that is a sum of simple R-submodules (Recall that an R-module is called simple if it has no proper nontrivial R-submodule). Prove that any submodule of M is a direct summand of M.

3. [20 pts] Let G be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by two matrices
$$\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix},$$
where p is a prime integer. Determine whether G is solvable.

4. [20 pts] Let G be a non-abelian group of order p^3, where p is an odd prime integer.
   (a) [10 pts] Show that the center of G is equal to the commutator subgroup of G.
   (b) [10 pts] Show that G has a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

5. [15 pts] Let M be a projective R-module, where R denotes a commutative ring with 1.
   (a) [8 pts] Show that M is torsion-free.
   (b) [7 pts] Show that if R is a PID and M is finitely generated, then M is free.

6. [20 pts] Let H be a subgroup of G. We write K for the quotient group of the normalizer $N_G(H)$ by the centralizer $C_G(H)$.
   (a) [10 pts] Show that K is isomorphic to a subgroup of the automorphism group of H.
   (b) [10 pts] Let H be a cyclic Sylow 3-subgroup of a finite group G, where $|G| = 3^a b$ for some odd integers $a, b > 1$. Prove that K is trivial.

THE END

Ph.D. Qualifying Exam: Algebra II
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let R be a commutative ring with unity.

   (a) [5 pts] State the definition of *(i)* noetherian and *(ii)* artinean R-modules, respectively.

   (b) [10 pts] Show that an R-module M is noetherian if and only if M is finitely generated and $R/\operatorname{ann}(M)$ is a noetherian ring, where $\operatorname{ann}(M)$ is the annihilator of M.

2. [10 pts] State the definition of *(i)* algebraic, *(ii)* separable and *(iii)* normal field extensions, respectively. Make sure to state the definition that can be applied to infinite field extensions as well.

3. Let $K := \mathbb{F}_p(t)$ denote the fraction field of the polynomial algebra $\mathbb{F}_p[t]$ over the prime field with p elements. Decide whether each of the following extensions of K is separable (respectively, normal). Make sure to justify your answer. You may use without proof standard results covered in the graduate-level course.

   (a) [5 pts] An algebraical closure $\overline{K}$ of K.

   (b) [5 pts] A separable closure K^{sep} of K.

   (c) [5 pts] The extension $K(\{t^{1/p^n}\}_{n>0})$ of K generated by a p^nth root t^{1/p^n} of t in $\overline{K}$ for each $n > 0$ that is recursively chosen to satisfy $(t^{1/p^n})^p = t^{1/p^{n-1}}$ for any $n > 0$.

   (d) [5 pts] For a prime $\ell \neq p$, the extension $K(\{t^{1/\ell^n}\}_{n>0})$ of K generated by an ℓ^nth root t^{1/ℓ^n} of t in $\overline{K}$ for each $n > 0$ that is recursively chosen to satisfy $(t^{1/\ell^n})^\ell = t^{1/\ell^{n-1}}$ for any $n > 0$.

4. Let k be a field, and let E and F be (not necessarily finite) field extensions of k. Suppose that E and F are contained in a common field extension K.

   Prove or disprove each of the following claims.

   (a) [6 pts] The compositum EF in K is algebraic over k if and only if both E and F are algebraic over k.

   (b) [6 pts] The compositum EF in K is separable over k if and only if both E and F are separable over k.

   (c) [8 pts] The compositum EF in K is a normal extension of k if and only if both E and F are normal extensions of k.

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5. (a) [15 pts] For any integer $N \geqslant 3$, define the Nth cyclotomic polynomial $\Phi_N(X)$, and show that $\Phi_N(X)$ has integral coefficients and irreducible in $\mathbb{Z}[X]$.

   (b) [20 pts] Let $\mathbb{F}_q$ be a field with q elements, and let K denote the splitting field of $\Phi_N(X)$ over $\mathbb{F}_q$ for some $N \geqslant 3$. Suppose that N and q are coprime. Compute $[K : \mathbb{F}_q]$ and explicitly describe $\mathrm{Gal}(K/\mathbb{F}_q)$.

THE END

Ph.D. Qualifying Exam: Differential Geometry
August 2022

Student ID: Name:

Note: Be sure use English for your answers.

Unless otherwise is stated a smooth manifold means a smooth manifold without boundary.

1. [15 pts] Let M be a smooth n-manifold.
   (a) Define explicitly the natural coordinates and the transition map between such two charts of the tangent bundle TM.
   (b) Show that TM is always orientable.
2. [15 pts] Let M be a smooth n-manifold. Let η be a nowhere zero smooth n-form on M. Show that at every point $p \in M$, there exists a smooth chart $(U, (x_i))$ around p that on U we have
$$\eta = dx_1 \wedge \cdots \wedge dx_n.$$
3. [15 pts] Suppose that M is smooth n-manifold. Let V is a smooth vector field on M and θ is its flow. Given a smooth k-form $\omega \in \Omega^k(M)$ on M, we define the Lie derivative of ω with respect to V, denote by $\mathcal{L}_V\omega$, by
$$(\mathcal{L}_V\omega)_p = \frac{d}{dt}\bigg|_{t=0} (\theta_t^*\omega)_p.$$
Regarding a smooth real-valued function f as 0-form on M, show that $\mathcal{L}_V f = Vf$.
4. [20 pts] Let M be a smooth n-dimensional manifold, $S \subseteq M$ is an embedded submanifold and $p \in S$. Show that as a subspace of T_pM, the tangent space T_pS is characterized by
$$T_pS = \{v \in T_pM : vf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f_{|_S} = 0\}.$$
5. [20 pts] Let M be compact smooth n-manifold with non-empty boundary ∂M. Show that there exists a smooth function $f : M \to [0, \infty)$ such that $f^{-1}(0) = \partial M$ and $df_p \neq 0$ for all $p \in \partial M$.
6. [15 pts] Suppose that M is an oriented compact smooth manifold with non-empty boundary ∂M. Show that there does not exist a smooth function $r : M \to \partial M$ such that $r_{|_{\partial M}}$ the restriction of r on ∂M is the identity map of ∂M. [*hint. Consider an orientation form on* ∂M]

THE END

Ph.D. Qualifying Exam: Algebraic Topology I
August 2022

Student ID: Name:

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. [20 pts] Let (X, A) be a CW pair.
   (a) Show that (X, A) has the homotopy extension property.
   (b) Prove that if A is contractible, then X is homotopy equivalent to X/A.
2. (a) [10 pts] Prove that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^n$ if and only if $n = 2$.
   (b) [10 pts] Prove that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$.
3. Recall that a knot is a smooth embedding of a circle in S^3.
   (a) [10 pts] Does there exist a knot K so that $\pi_1(S^3 \smallsetminus K) \ncong \mathbb{Z}$?
   (b) [10 pts] Does there exist a knot K so that $H_1(S^3 \smallsetminus K) \ncong \mathbb{Z}$?
4. [20 pts] Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.
5. Let n be a positive integer.
   (a) [10 pts] Prove or disprove that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.
   (b) [10 pts] Prove or disprove that every map $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ has a fixed point.

THE END

Ph.D. Qualifying Exam: Algebraic Topology II
August 2022

Student ID: Name:

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (a) [10 pts] Prove or disprove that if there are path connected topological spaces X and Y with $H_*(X) \cong H_*(Y)$ but $H^*(X) \ncong H^*(Y)$.

   (b) [10 pts] Prove or disprove that if there are path connected topological spaces X and Y with $H_*(X) \cong H_*(Y)$ and $H^*(X) \cong H^*(Y)$ but $H^*(X)$ and $H^*(Y)$ are not isomorphic as rings.

2. [10 pts] Let A be a closed subspace of X that is a deformation retract of some neighborhood. Prove that the quotient map $X \to X/A$ induces an isomorphism $H^n(X, A) \to \widetilde{H}^n(X/A)$ for each nonnegative integer n.

3. [20 pts] Compute the cohomology rings of $\mathbb{R}P^n$ and $\mathbb{C}P^n$.

4. (a) [10 pts] Prove or disprove that if a closed orientable $2n$-dimensional manifold M has $H_{n-1}(M)$ torsionfree, then $H_n(M)$ is torsionfree.

   (b) [10 pts] Prove or disprove that there exists a compact manifold that retracts onto its boundary.

5. (a) [15 pts] Recall that there is a homotopy long exact sequence of a pair (X, A). Describe the connecting homomorphism explicitly. Moreover, choose a group from the long exact sequence and show that the sequence is exact at that group.

   (b) [15 pts] Describe the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ and prove that this is a fiber bundle.

THE END

Ph.D. Qualifying Exam: Real Analysis
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] Suppose that $A \subset E \subset B \subset \mathbb{R}$, where A and B are Lebesgue measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is Lebesgue measurable.

2. [15 pts] Prove the following generalization of Lebesgue's dominated convergence theorem: Suppose that $f_1, f_2, \ldots$ are measurable functions on $\mathbb{R}^d$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in \mathbb{R}^d$. Suppose also that $g_1, g_2, \ldots$ are nonnegative, integrable functions such that $|f_k(x)| \leq g_k(x)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ exists for a.e. $x \in \mathbb{R}^d$. Prove that if g is integrable with $\int g = \lim_{n\to\infty} \int g_n$ then $\int f = \lim_{n\to\infty} \int f_n$.

3. Suppose that $F : [a, b] \to \mathbb{R}$ is absolutely continuous and increasing. Let $A = F(a), B = F(b)$. Prove the following:
   (a) [10 pts] If $E \subset [A, B]$ is measurable, the $F^{-1}(E) \cap \{F'(x) > 0\}$ is measurable.
   (b) [10 pts] There exists such an F that is strictly increasing, $F'(x) = 0$ on a set of positive measure, and there is a measurable subset $E \subset [A, B]$ so that $m(E) = 0$ but $F^{-1}(E)$ is not measurable.

4. Let $\mathcal{B}$ be a Banach space.
   (a) [10 pts] Prove that $\mathcal{B}$ is a Hilbert space if and only if
   $$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$$
   for any $f, g \in \mathcal{B}$.
   (b) [10 pts] Prove that $L^p(\mathbb{R}^d)$ $(p \in [1, \infty))$ with the Lebesgue measure is a Hilbert space if and only if $p = 2$.

5. [15 pts] Let μ be a σ-finite measure on a measure space X. Prove that every measurable set of infinite measure in X contains measurable sets of arbitrarily large finite measure.

6. [15 pts] Let S be a set of all complex, measurable, simple functions on a measure space X with a positive measure μ, satisfying that, for any $f \in S$,
   $$\mu(\mathrm{supp}(f)) < \infty.$$
   Prove that S is dense in $L^p(X, \mu)$ for any $1 \leq p < \infty$.

THE END

Ph.D. Qualifying Exam: Complex Analysis
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. [20 pts] Let $\mathbb{C}_\infty$ be the Riemann sphere. Show that if $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is meromorphic, then f is rational.

2. (a) [10 pts] Evaluate
$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\, dx$$
   (b) [10 pts] Check if the integral is integrable. If so, evaluate it.
$$\int_0^\infty \frac{\log x}{x^b - 1}\, dx, \quad b > 1$$

3. [20 pts] Denote $\mathbb{D} = \{z : |z| < 1\}$. Show if $f : \mathbb{D} \to \mathbb{D}$ is analytic, then
$$|f'(z)| \le \frac{1 - |f(x)|^2}{1 - |z|^2}.$$
Moreover, if $f(z)$ is a conformal self-map of $\mathbb{D}$, then the equality holds. (Hint. Use the conformal self-map of $\mathbb{D}$ sending 0 to z_0 and its inverse.)

4. [20 pts] Let $f(z)$ be the Riemann map of a simply connected domain D onto the unit disk $\mathbb{D}$. Suppose $f(z_0) = 0$ and $f'(z_0) > 0$. Show that if $g(z)$ is an analytic function on D such that $|g(z)| \le 1$ for $z \in D$, then Re $g'(z_0) \le f'(z_0)$.

5. (a) [10 pts] Let $\{a_n\} \subset \backslash\{0\}$ be a sequence. Show that $\prod_{n=1}^\infty (1 - \frac{z}{a_n})$ is entire if and only if $\sum_{n=1}^\infty \frac{1}{z-a_n}$is meromorphic.
   (b) [10 pts] Find a meromorphic fuction $f(z)$ which has poles only at $z = n$ for each positive integer n with order n.

THE END

Ph.D. Qualifying Exam: Probability Theory
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. [15 pts] State and prove Kolmogorov's 0-1 Law.

2. [15 pts] Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with $\mathbb{E}|X_i| < \infty$. Define $S_n := X_1 + \cdots + X_n$. Compute $\mathbb{E}(X_1|S_n)$.

3. [15 pts] X is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(X^2) < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Show that $\mathbb{E}(X|\mathcal{G})$ is a minimizer of $\mathbb{E}(X - Y)^2$ over all $\mathcal{G}$-measurable random variables Y.

4. [15 pts] Suppose that events $\{A_n\}_{n=1,2,\cdots}$ are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Show that

$$\mathbb{P}(A_n \text{ infinitely often}) = 1.$$

(Hint: Use the formulation $\{A_n \text{ infinitely often}\} = \cap_{N=1}^{\infty} \cup_{n=N}^{\infty} A_n$ and the inequality $1 - x \leq e^{-x}$ for $x \geq 0$.)

5. [20 pts] Let $\{X_n\}_{n=1,2,\cdots}$ be an i.i.d. sequence of exponential random variables (i.e. the probability density function is given by $f(x) = e^{-x}$ for $x \geq 0$). Show that almost surely,

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1.$$

6. [20 pts] Let $\{X_n\}_{n=0,1,\cdots}$ be a martingale with $X_0 = 0$ such that $\mathbb{E}(X_n - X_{n-1})^2 = 1$ for all $n \geq 1$. Show that almost surely,

$$\frac{X_n}{n} \to 0.$$

(Hint: First show that $\frac{X_{a_n}}{a_n} \to 0$ a.s. along a suitable subsequence $\{a_n\}_{n=0,1,\cdots}$ using Borel-Cantelli lemma. Then, extend it to the full sequence.)

THE END

Ph.D. Qualifying Exam: Advanced Statistics
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. [30 pts] A bent coin has probability p of coming up heads and $q = 1 - p$ of coming up tails, where $0 < p < 1$. The first stage of an experiment consists of tossing this coin a known total of M times and recording X, the number of heads. In the second stage, the coin is tossed until a total of $X + 1$ tails have come up. The number of Y heads observed in the second stage along the way to getting the $X+1$ tails is then recorded. We repeat the experiment a total of n times and record the two counts (X_i, Y_i) for each experiment $i = 1, 2, \ldots, n$

   (a) [10 pts] Give a sufficient statistic for the unknown parameter p, and give its sampling distribution.

   (b) [10 pts] Derive the uniformly minimum variance unbiased estimator for p.

   (c) [10 pts] Give the uniformly most powerful test of H_0: $p = 1/2$ vs. H_1 : $p < 1/2$ and give an expression for calculating the p-value of the test in terms of a standard distribution.

2. [50 pts] An experiment yields two positive measurements X and Y that are modeled as independent Gamma distributed random variables with common, known, shape parameter a and unknown, scale parameters b and c, respectively. We denote this $X \sim \Gamma(a, b)$ and $Y \sim \Gamma(a, c)$ and recall that the density of X, say, is

$$f_X(x) = \frac{b^a x^{a-1} \exp(-bx)}{\Gamma(a)}, x > 0.$$

   (a) [10 pts]Derive the sampling distribution of $T = X/Y$.

   (b) [10 pts] In a Bayesian analysis, we place independent and identical prior distributions on b and c. Both are $\Gamma(a_0, \nu)$ for fixed hyperparameters a_0 and ν. Derive the joint posterior distribution of b and c given data x, y.

   (c) [10 pts] With the same prior as in part (b), derive the posterior density of $\tau = c/b$ given data x, y. One possible Bayes estimator of τ is the posterior mode $\hat{\tau}_B$. Compute $\hat{\tau}_B$.

   (d) [10 pts]Derive the marginal distribution for the pair (X, Y) using the prior in part (b), and call this $p_A(x, y)$.

   (e) [10 pts]A null hypothesis asserts that $b = c$. Prior uncertainty in the common value is represented by a $\Gamma(a_0, \nu)$ distribution. Derive the marginal distribution of (X, Y) under this null hypothesis, and call the result $p_0(x, y)$. Also report the Bayes factor $p_A(x, y)/p_0(x, y)$ which is used in the testing problem.

3. [20 pts] Let $x \in R^k$, $y \in R^k$, and A be a $k \times k$ positive definite matrix.

(a) [10 pts] Suppose that $y^T A^{-1} x = 0$. Show that

$$x^T A^{-1} x = \max_{c \in R^k, c \neq 0, c^T y = 0} \frac{(c^T x)^2}{c^T A c}.$$

(b) [10 pts] Assume a model $X \sim N_n(Z\beta, \sigma^2 I)$, where β is a p-vector of unknown parameters, $\sigma^2 > 0$ is unknown, and Z is an $n \times p$ known matrix of rank $p < n$. Using the result in (a), construct simultaneous confidence intervals (with confidence coefficient 1-α) for $c^T\beta$, $c \in R^p, c \neq 0, c^T y = 0$, where $y \in R^p$ satisfies $Z^T Z y = 0$.
Use the fact that $(\hat{\beta} - \beta)^T (Z^T Z)(\hat{\beta} - \beta)/(p\hat{\sigma}^2)$ has the F-distribution with p and $n - p$ degrees of freedom, where $\hat{\beta}$ is the least squares estimator of β and $\hat{\sigma}^2 = (X - Z\hat{\beta})^T (X - Z\hat{\beta})/(n - p)$.

THE END

Ph.D. Qualifying Exam: Numerical Analysis
August 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let the ordinary Legendre polynomial of degree k be denoted $P_k(x)$ for $k \geq 0$. The associated Legendre polynomials are

$$P_k^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_k(x), \quad m > 0, k \geq m.$$

(Note that despite the name, for odd m they are not actually polynomials.)

(a) [10 points] Consider the interpolation problem of finding coefficients a_k such that

$$\sum_{k=1}^{N} a_k P_k^1(x_i) = y_i, \quad i = 1, \ldots, N.$$

Prove that this linear system of equations for the unknown coefficients a_k is nonsingular whenever the set of interpolation points x_i does not include ± 1, and does not include duplicates.

(b) [6 points] Consider the approximation problem of finding coefficients a_k to minimize the squared approximation error

$$\left\| f(x) - \sum_{k=1}^{N} a_k P_k^1(x) \right\|_2^2,$$

where the L^2 norm is over $x \in [-1, 1]$. Write down a linear system for the unknown coefficients a_k and explain why it is nonsingular. You should give an explicit integral expression for the entries of the coefficient matrix and right hand side, but the expression does not need to be simplified.

(c) [4 points] Let M be the coefficient matrix from (b). Prove that $M_{kj} = 0$ when $k + j$ is odd.

2. [20 points] The integral

$$(f, g) := \int_{-1}^{1} f(x)g(x)\,dx$$

defines a scalar product for functions $f, g \in C[-1, 1]$. Show that if f and g are polynomials of degree less than n, if $x_i, i = 1, 2, \ldots, n$, are the roots of the nth Legendre polynomial, and if

$$\gamma_i := \int_{-1}^{1} L_i(x)\,dx$$

with

$$L_i(x) := \prod_{\substack{k \neq i \\ k=1}}^{n} \frac{x - x_k}{x_i - x_k}, \quad i = 1, 2, \ldots, n,$$

then

$$(f, g) = \sum_{i=1}^{n} \gamma_i f(x_i) g(x_i).$$

3. [15 points] Consider a 2D fixed point iteration of the form

$$\begin{aligned} x_{k+1} &= f(x_k, y_k) \\ y_{k+1} &= g(x_k, y_k). \end{aligned}$$

Assume that the vector-valued function $\vec{h}(x, y) = (f(x, y), g(x, y))^T$ is continuously-differentiable, and the infinity norm of the Jacobian matrix is less than 1 at a unique fixed point (x_∞, y_∞).

Now consider the "nonlinear Gauss-Seidel" version of the iteration:

$$\begin{aligned} x_{k+1} &= f(x_k, y_k) \\ y_{k+1} &= g(x_{k+1}, y_k). \end{aligned}$$

Prove that the "nonlinear Gauss-Seidel" version is convergent, to the same fixed point, for initial conditions sufficiently close to the fixed point.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1/8 & 1/4 \\ 2 & 1 & 1/4 \\ 2 & 2 & 1 \end{bmatrix}.$$

(a) (5 points) Find an orthogonal matrix Q such that QA has a zero in the circled entry.

(b) (10 points) Provide as good a bound as possible on the eigenvalues of A.

5. Consider iterative solvers for computing the solution to $Ax = b$.

(a) [10 points] Write down the steps of the gradient descent method with fixed time step. Show when the method is guaranteed to converge.

(b) [5 points] Write down the steps of the steepest descent method.

(c) [15 points] Write down the steps of the conjugate gradient method (CGM). Explain why CGM converges to the solution in finite time.

THE END