Ph.D. Qualifying Exam: Algebraic Topology I: (February 2022)

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (10 Points)
   (a) (5 Points) Find path connected topological spaces X and Y with $H_*(X) \cong H_*(Y)$ but not homotopy equivalent.
   (b) (5 Points) Find path connected topological spaces X and Y which are homotopy equivalent but not homeomorphic.

2. (30 Points)
   (a) (10 Points) Find a deformation retraction of $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}.
   (b) (10 Points) Show that a topological space X is contractible if and only if the identity map on X is homotopic to a constant map.
   (c) (10 Points) Prove or disprove that if a map is homotopic to a homotopy equivalence, then it is a homotopy equivalence.

3. (20 Points)
   (a) (10 Points) Show that every continuous map $f\colon D^2 \to D^2$ has a fixed point.
   (b) (10 Points) Let X be a topological space obtained by the wedge product of finite copies of $\mathbb{RP}^2$. Prove that any map from X to S^1 lifts to the universal cover.

4. (20 Points)
   (a) (10 Points) Prove or disprove that there is a connected regular (or normal) cover of the wedge of 2-circles where the set of deck transformation forms a non-abelian group under composition.
   (b) (10 Points) Prove or disprove that there is a connected cover of the wedge of 2-circles where the set of deck transformation forms a group isomorphic to $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

5. (20 Points) Let n be an non-negative integer.
   (a) (10 Points) Describe a CW-complex structure on $\mathbb{RP}^n$.
   (b) (10 Points) Compute $H_*(\mathbb{RP}^n; \mathbb{Z})$ and $H_*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$.

Ph.D. Qualifying Exam: Algebraic Topology II: (February 2022)

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (20 Points)
   (a) (10 Points) Prove or disprove that if there are path connected topological spaces X and Y with $H_*(X) \cong H_*(Y)$ but $H^*(X) \ncong H^*(Y)$.
   (b) (10 Points) Prove or disprove that if there are path connected topological spaces X and Y with $\pi_1(X) \cong \pi_1(Y)$ and $H_*(X) \cong H_*(Y)$ but not homotopy equivalent.

2. (20 Points) Let X be a closed 5-dimensional manifold with $\pi_1(X) \cong \mathbb{Z}/3\mathbb{Z}$ and $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/5\mathbb{Z}$.
   (a) (10 Points) Compute $H^*(X;\mathbb{Z})$.
   (b) (10 Points) Prove or disprove that any CW decomposition of X has at least one 4-cell.

3. (20 Points) Let n be an non-negative integer.
   (a) (10 Points) Describe a CW-complex structure on $\mathbb{CP}^n$.
   (b) (10 Points) Compute $H^*(\mathbb{CP}^n;\mathbb{Z})$ and its ring structure.

4. (20 Points)
   (a) (10 Points) Show that a covering map $p\colon X \to Y$ induces isomorphism $p_*\colon \pi_n(X) \to \pi_n(Y)$ for each $n \geq 2$.
   (b) (10 Points) Show that $\pi_n(X)$ is abelian for each $n \geq 2$.

5. (20 Points)
   (a) (10 Points) Suppose $S^k \to S^m \to S^n$ is a fiber bundle. Show that $k = n-1$ and $m = 2n-1$.
   (b) (10 Points) Show that any fiber bundle over $[0,1]$ is trivial.

Ph.D. Qualifying Exam: Algebra I
February 2022

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Please ensure to use English for your answers

1. (15 pts) Let G be a group, and let $G' < G$ be the subgroup generated by elements of the form $g^{-1}h^{-1}gh$ for any $g, h \in G$. Show that G' is a *normal* subgroup of G and G/G' is abelian. Furthermore, show that any group homomorphism $\varphi \colon G \to H$ to an abelian group H factors through G/G'.

2. (20 pts) Let G be a finite group of order n, and let k be a field such that $n \in k^{\times}$. Let kG denote the group ring of G over k. Then for any kG-module V and a kG-submodule $W \subset V$, show that there exists a complement kG-submodule $W' \subset V$ such that $V = W \oplus W'$ as kG-module.

3. (30 pts) Let p be a prime number
   (a) (5 pts) Classify groups of order p^2 up to isomorphism.
   (b) (10 pts) For each group H of order p^2 up to isomorphism, explicitly describe its automorphism group $\mathrm{Aut}(H)$ and compute $|\mathrm{Aut}(H)|$.
   (c) (10 pts) Let q be a prime less than p. Show that any group G of order p^2q can be written as
   $$G \cong P \rtimes Q$$
   where P (respectively, Q) is a p-Sylow subgroup (respectively, a q-Sylow subgroup) of G.
   (d) (5 pts) State and prove a necessary and sufficient condition on p and q for the existence of non-abelian group of order p^2q.

4. (35 pts) Let k be any field. Fix an integer $n > 0$ and let $G := \mathrm{GL}_n(k)$ denote the group of invertible $n \times n$ matrices. Write $V := k^n$.
   (a) (10 pts) Fix $\gamma \in G$, and view V as a $k[T]$-module via $f(T) \cdot v := f(\gamma)(v)$ for any $f(T) \in k[T]$ and $v \in V$. Show that there exists a <u>*unique*</u> sequence of non-constant monic polynomials $f_1, \cdots, f_r$, called the <u>*elementary divisors*</u> of γ, such that we have $V \cong \bigoplus_{i=1} k[T]/(f_i)$ as $k[T]$-modules and we have $f_{i+1}|f_i$ for any $i = 1, \cdots, r-1$. (You may use the fundamental theorem of finitely generated modules over PID without proof, *if* you state it rigorously.)
   (b) (10 pts) For $\delta \in G$, let $g_1, \cdots, g_s$ denote the elementary divisors of δ. Show that γ and δ are conjugate if and only if $r = s$ and $f_i = g_i$ for any $i = 1, \cdots, r$.
   (c) (8 pts) Let $f_1, \cdots f_r \in k[T]$ be monic non-constant polynomials with non-zero constant term, such that $\sum_{i=1}^{r} \deg(f_i) = n$ and $f_{i+1}|f_i$ for any $i = 1, \cdots, r-1$. Show that there exists an element $\gamma \in G$ whose elementary divisors are $f_1, \cdots, f_r$.
   (d) (7 pts) If k is a finite field, then compute the number of conjugacy classes of $\mathrm{GL}_3(k)$ in terms of $|k|$.

THE END

Ph.D. Qualifying Exam: Algebra II
February 2022

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Please ensure to use English for your answers

1. (10 pts) Let B be an integral domain. Suppose that A is a subring of B that makes B a finitely generated A-module. Show that A is a field if and only if B is a field.

2. (15 pts) Given a noetherian ring R, show that $R[[X]]$ is noetherian.
Hint: Mimic the proof of the Hilbert basis theorem.

3. (10 pts) Let k be a field. Let E/k and F/k be finite Galois extensions contained in a common field extension L of k. Let EF denote the compositum of E and F in L. Show that *(a)* EF and $E \cap F$ are Galois extensions of k, and *(b)* the natural map $\phi\colon \operatorname{Gal}(EF/k) \to \operatorname{Gal}(E/k) \times \operatorname{Gal}(F/k)$ induced by the restriction is injective. Finally, *(c)* describe the image of ϕ.

4. (20 pts) Solve the following problems.
   (a) (7 pts) Let K be a field containing an algebraically closed field k. Given any integer $n > 1$ and $a \in K$, state and prove a necessary and sufficient condition for $t^n - a \in K[t]$ to be irreducible. (We do *not* assume that n is prime to the characteristic of K.)
   (b) (3 pts) Show that a UFD is integrally closed.
   (c) (6 pts) Let R be a UFD such that $2 \in R^\times$, and set $K := \operatorname{Frac}(R)$. Given a square-free non-unit element $x \in R$, explicitly describe the normalization of R in $K(\sqrt{x})$.
   (d) (4 pts) Let $f \in \mathbb{C}[X_1, \cdots, X_n]$ be a non-constant *square-free* polynomial. Show that $\mathbb{C}[X_1, \cdots, X_n, Y]/(Y^2 - f)$ is an integrally closed domain.

5. (20 pts) Prove or disprove each of the following claims.
   (a) (5 pts) Let E/F and L/E be finite Galois extensions of fields. Then L/F is also Galois.
   (b) (7 pts) Let R be a commutative ring. Then there exists a natural R-algebras isomorphism $R[\frac{1}{f}] \otimes_R R[\frac{1}{g}] \cong R[\frac{1}{fg}]$, where $R[\frac{1}{f}]$ is the localization of R with respect to $\{1, f, f^2, \cdots\}$.
   (c) (8 pts) Given a field extension K/k, there exists a natural k-algebra isomorphism $K(X_1, \cdots, X_r) \cong K \otimes_k k(X_1, \cdots, X_r)$, where X_i's are algebraically independent variables.

6. (25 pts) Let $f(X) := X^4 + aX^2 + b$ be an *irreducible* polynomial over $\mathbb{Q}$ with roots $\pm\alpha, \pm\beta$. Let K be the splitting field of $f(X)$ over $\mathbb{Q}$.
   (a) (15 pts) Show that $\operatorname{Gal}(K/\mathbb{Q})$ is a subgroup of D_8, the dihedral group of order 8.
   (b) (10 pts) Show that $\operatorname{Gal}(K/\mathbb{Q})$ is *(a)* cyclic of order 4 iff $\alpha/\beta - \beta/\alpha \in \mathbb{Q}$, *(b)* a Klein 4-group iff $\alpha\beta \in \mathbb{Q}$ or $\alpha^2 - \beta^2 \in \mathbb{Q}$, and *(c)* D_8 otherwise.

THE END

Ph.D. Qualifying Exam: Differential Geometry
February 2022

Student ID: Name:

Note: Be sure use English for your answers.

All considered manifolds below are smooth manifolds without boundary.

1. [15 pts] Let $F : M \to N$ be a smooth map between two smooth manifolds. The global differential map $dF : TM \to TN$ is given by $dF(v) = dF_p(v)$ for $v \in T_pM$, where $dF_p : T_pM \to T_{F(p)}N$ is the differential of F (or the tangent map of F) at p. Show that dF is a smooth map.
2. [15 pts] Let M be a compact smooth manifold and let V be a smooth vector field on M. Denote the flow generated by V by $\theta_t : M \to M$, i.e.
$$\frac{d\theta_t}{dt}(p) = V_{\theta_t(p)} \quad \text{for every } p \in M.$$
Given a smooth function $f : M \to \mathbb{R}$, prove that
$$f(\theta_1) - f(\theta_0) = \int_0^1 \theta_t^*(df)(V)dt.$$
3. [20 pts = 10 pts + 10 pts] Let ω be the 2-form on $\mathbb{R}^3 \setminus \{0\}$ defined by
$$\omega = (x^2 + y^2 + z^2)^{-3/2}(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy).$$
   (a) Let $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ be the inclusion map. Let $p \in \mathbb{S}^2$ and v_1, v_2 be an (positive) oriented orthonormal basis of $T_p\mathbb{S}^2 \subset T_p\mathbb{R}^3$ with respect to the standard Euclidean inner product in $T_p\mathbb{R}^3 \cong \mathbb{R}^3$. Show that $(\iota^*\omega)_p(v_1, v_2) = 1$.
   (b) Show that ω is closed but not exact on $\mathbb{R}^3 \setminus \{0\}$.
4. [15 pts] A Riemannian metric on a smooth manifold M is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point, i.e., a smooth section $g : M \to T^*M \otimes T^*M$ such that at each point $p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is a symmetric positive bilinear form. Show that every smooth n-manifold, $n \geq 1$, admits a Riemannian metric.
5. [15 pts] Let M be a smooth m-dimensional manifold and $f : M \to \mathbb{R}$ is a smooth function. Let $N \subset M$ be an embedded smooth $(m-1)$-dimensional submanifold which is defined by $g(x) = 0$, where $g : M \to \mathbb{R}$ is a smooth function for which 0 is a regular value. Suppose the restriction $f_{|_N}$ of f to N attains its maximum or minimum on N at some point $p \in N$. Show that $df_p = \lambda dg_p$ for some real number λ. (*hint. show that* $\ker(dg_p) = T_pN$)
6. [20 pts = 10 pts+10 pts]
   (a) Show that there is a nowhere vanishing smooth vector field on $\mathbb{S}^1$.
   (b) Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the torus (with the standard product smooth structure). Show that its tangent bundle $T\mathbb{T}^2$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$.

THE END

2022 QUALIFYING EXAM - COMPLEX ANALYSIS

Problem 1. (20 pt) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence such that $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ converges. Find an entire function that has zeros only at $\{a_n\}_{n=1}^{\infty}$. (You need to verify that your example is entire.)

Problem 2. (20 pt) Let $f : D \to D$ be analytic in a simply connected domain $D \subsetneq \mathbb{C}$ having a fixed point in D. Show that $|f'(a)| \leq 1$ for all $a \in D$. Show if $|f'(a)| = 1$ for some $a \in D$, then f is bijective on D.

Problem 3. (15pt) Let D be a domain and $f : D \to \mathbb{C}$ be an analytic function with $f'(a) \neq 0$ for some $a \in D$. Show that the derivative $df(a)$ is a composition of rotation and dialation in $\mathbb{C}$. (Here, $df(a)$ is the gradient of f, when one understand $f : D \subset \mathbb{R}^2 \to \mathbb{R}^2$)

Problem 4. (20pt) Let D be a connected domain and $\{f_n\}$ a sequence of injective analytic functions on D. Assume that $\{f_n\}$ converges uniformly on each compact subset of D. Show that the limit function f is either injective or constant.

Problem 5. (15pt) Let f be analytic and satisfy $|f(z)| \leq M$ on $|z - z_0| < R$ for some $M, R > 0$. Show that if $f(z)$ has a zero of order m at z_0, then

$$|f(z)| \leq \frac{M}{R^m}(z - z_0)^m, \qquad |z - z_0| < R.$$

Show that if the equality holds at some point, then $f(z) = C(z - z_0)^m$ for some C.

Problem 6. (10pt) Let D be a domain and $f : D \to \mathbb{C}$ be an analytic function. Assume that $f(a_n) = 0$ for all n, where $\{a_n\}_{n=1}^{\infty} \subset D$ is a convergent sequence in $\mathbb{C}$. Prove or disprove that $f \equiv 0$.

Ph.D. Qualifying Exam: Numerical Analysis
February 2022

1. [10 points each] Let $x_0 < x_1 < \cdots < x_n$ be given real numbers. Let $f(x_0), f(x_1), \cdots, f(x_n)$ be the values at the nodes and $f'(x_0)$ and $f'(x_n)$ be the values of the derivative at the end points of a smooth function $f(x)$.

   (a) Derive a formula for the polynomial of the lowest degree that interpolates this data.

   (b) Derive an error estimate for the interpolating polynomial assuming that the function $f(x)$ has as many derivatives as needed for your analysis.

2. (a) [15 points] Consider $\{p_i(x)\}_{i=0}^{\infty}$, a family of orthogonal polynomials associated with the inner product

   $$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,w(x)dx, \quad w(x) > 0 \quad \text{for} \quad x \in (-1, 1),$$

   where $p_i(x)$ is a polynomial of the degree i. Let $x_0, x_1, \ldots, x_n$ be the roots of $p_{n+1}(x)$. Construct an orthonormal basis in the subspace of polynomials of degree less than or equal to n such that, for any polynomial in this subspace, the coefficients of its expansion into the basis are equal to the scaled values of this polynomial at the nodes $x_0, x_1, \ldots, x_n$.

   (b) [10 points] Discuss in some (qualitative) detail how you would go about obtaining, in an efficient way, an accurate value for

   $$\int_{-1}^{1} \frac{1 + |x^2 - 1/4|}{\sqrt{1 - x^2}}\,dx\,.$$

   Justify your approach, by citing relevant facts about the quadrature rule(s) you are proposing to use.

3. (a) [6 points] Give a locally convergent method for determining the fixed point $\xi = \sqrt[3]{2}$ of $\Phi(x) := x^3 + x - 2$.

   (b) [9 points] The polynomial $p(x) = x^3 - x^2 - x - 1$ has its only positive root near $\xi = 1.839\ldots$. Without using $p'(x)$, construct an iteration function $\Phi(x)$ having the fixed point $\xi = \Phi(\xi)$ and having the property that the iteration converges for any starting point $x_0 > 0$.

4. An algorithm (attributed to Eudoxos) generating a sequence of increasingly accurate approximations to the length of the diagonal of the unit square has been know in ancient Greece. Starting with $p_0 = q_0 = 1$, and defining the iteration as $p_{n+1} = p_n + q_n$, and $q_{n+1} = p_{n+1} + p_n, n = 0, 1, 2, \ldots$, then the ratio q_n/p_n converges to $\sqrt{2}$ as $n \to \infty$.

   (a) [10 points] Prove convergence of this algorithm using the power method.

(b) [5 points] Give an expression that determines how many iterations are needed in order to achieve the accuracy

$$\left| \sqrt{2} - \frac{q_n}{p_n} \right| \leq 10^{-8}\,?$$

5. Consider the real system of linear equations $Ax = b$, where A is nonsingular and satisfies $(x, Ax) > 0$ for all real $x \neq 0$, where $(x, y) = x^T y$ is the Euclidean inner product.

   (a) [5 points] Show that $(x, Ax) = (x, Mx)$ for all real x, where $M = (A + A^T)/2$ is the symmetric part of A.

   (b) [8 points] Prove that $(x, Ax)/(x, x) \geq \lambda_{\min}(M) > 0$, where $\lambda_{\min}(M)$ is the smallest eigenvalue of M.

   (c) [12 points] Consider the iterative sequence $x_{n+1} = x_n + \alpha_n r_n$, where $r_n = b - Ax_n$ is the residual, and α_n is chosen to minimize $\|r_{n+1}\|_2$ as a function of α_n. Prove that

$$\frac{\|r_{n+1}\|_2}{\|r_n\|_2} \leq \left(1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^T A)}\right)^{1/2}.$$

THE END

Ph.D. Qualifying Exam: Real Analysis
February 2022

Student ID: Name:

Note: Be sure to use English for your answers.

1. For a given set $E \in \mathbb{R}^k$, define $\mathcal{O}_n = \{x \in \mathbb{R}^n : d(x, E) < 1/n\}$.
   (a) [10 pts] Show that $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$ if E is compact, where m is the Lebesgue measure.
   (b) [10 pts] Show that the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

2. [15 pts] Show that $f * g$ is uniformly continuous when f is integrable and g is bounded.

3. [15 pts] Suppose that f is integrable on $\mathbb{R}^k$. For each $\alpha > 0$, define $E_\alpha = \{x \in \mathbb{R}^k : |f(x)| > \alpha\}$. Prove that
$$\int_{\mathbb{R}^k} |f(x)|\,dx = \int_0^\infty m(E_\alpha)\,d\alpha.$$
(Here, m is the Lebesgue measure.)

4. [15 pts] Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator. If T is self-adjoint, prove that
$$\|T\| = \sup_{x\in\mathcal{H}}\{|\langle Tx, x\rangle| : \|x\| \le 1\}.$$

5. [15 pts] Suppose that (X, μ) is a measure space such that $\mu(A) > 0 \Rightarrow \mu(A) \ge 1$. Prove that, if $1 \le p \le q \le \infty$, then
$$\|f\|_{L^\infty(X,\mu)} \le \|f\|_{L^q(X,\mu)} \le \|f\|_{L^p(X,\mu)} \le \|f\|_{L^1(X,\mu)}.$$

6. Let $C([a, b])$ be the vector space of continuous functions on the closed and bounded interval $[a, b]$. Prove the following:
   (a) [10 pts] For a given Borel measure μ on this interval with $\mu([a, b]) < \infty$,
$$f \mapsto \ell(f) = \int_a^b f(x)\,d\mu(x)$$
   is a linear functional on $C([a, b])$, which is positive in the sense that $\ell(f) \ge 0$ if $f \ge 0$.
   (b) [10 pts] For any positive linear functional ℓ on $C([a, b])$, there exists a unique finite Borel meausre μ such that
$$\ell(f) = \int_a^b f(x)\,d\mu(x)$$
   for all $f \in C([a, b])$.

THE END

Qualifying Exam in Probability Theory

Spring 2022

Note: use English only for your answers.

1. (10 pts) Let $A_1, A_2, \cdots$ be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_n \text{ infinitely often}) = 0$.

2. (10 pts) Let X be a random variable with mean 0 and variance σ^2. Show that for any $\lambda > 0$,

$$\mathbb{P}(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

(Hint: Consider the function $\phi(x) = (x + \frac{\sigma^2}{\lambda})^2$)

3. (20 pts) Suppose that X and Y are independent random variables with the same exponential density

$$f(x) = \theta e^{-\theta x}, \quad x > 0.$$

Show that the sum $X + Y$ and the ratio X/Y are independent.

4. (20 pts) Let $X_1, X_2, \cdots$ be an i.i.d. sequence of random variables with $\mathbb{E}X_i = 0$ and $\mathrm{Var}X_i = 1$. Show that

$$\limsup_n \frac{X_1 + \cdots + X_n}{\sqrt{n}} = \infty \quad \text{almost surely.}$$

5. (20 pts) Let $X_1, X_2, \cdots$ be an i.i.d. sequence of random variables with $\mathbb{E}X_i = 0$ and $\mathrm{Var}X_i = 1$. Let T be a stopping time with respect to the natural filtration such that $\mathbb{E}T < \infty$. Define $S_n = X_1 + \cdots + X_n$.
(a) (8 pts) Show that both S_n and $S_n^2 - n$ are a martingale with respect to the natural filtration.

(b) (6 pts) Prove that

$$\mathbb{E}S_T = 0.$$

(c) (6 pts) Prove that

$$\mathrm{Var}S_T = \mathbb{E}T.$$

6. (20 pts) Let $X_1, X_2, \cdots$ be an i.i.d. sequence of random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p$, where $\frac{1}{2} < p < 1$. Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$.
(a) (10 pts) Let $\phi(x) = (\frac{1-p}{p})^x$. Prove that $\phi(S_n)$ is a martingale with respect to the natural filtration.

(b) (10 pts) Let $T_x = \inf\{n \geq 1 : S_n = x\}$. Prove that for any positive integer k,

$$\mathbb{P}(T_{-k} < T_k) = \frac{1}{1 + \phi(-k)}.$$