Ph.D. Qualifying Exam: Algebraic Topology I: (August 2021)

Fully support all answers. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts)

1. (20 Points)

   (a) (5 Points) Find path connected topological spaces X and Y with $\pi_1(X) \cong \pi_1(Y)$ but $H_*(X) \ncong H_*(Y)$.

   (b) (15 Points) Find topological spaces X and Y which are not homotopy equivalent but $H_*(X) \cong H_*(Y)$.

2. (25 Points)

   (a) (15 Points) A knot is a smooth embedding of a circle in S^3. Let K be a knot. Compute $H_*(S^3 \smallsetminus K)$.

   (b) (10 Points) Let n be an integer with $n \geq 2$. An n-component link is a smooth embedding of disjoint n circles in S^3. Let L be an n-component link. Compute $H_*(S^3 \smallsetminus L)$.

3. (25 Points)

   (a) (15 Points) Show that every subgroup of a finitely generated free group is free.

   (b) (10 Points) Find a 3-fold regular (or normal) cover and a 3-fold irregular cover of the wedge of 2-circles.

4. (20 Points)

   (a) (10 Points) Let n be an non-negative integer, $\{x_1, \ldots, x_n\}$ be n distinct points in S^2, and $X := S^2/\{x_1, \ldots, x_n\}$. Compute $\pi_1(X)$ and $H_*(X)$.

   (b) (10 Points) Let $f, g\colon X \to S^1 \times S^1$ be two continuous maps. Prove or disprove that f and g are homotopic.

5. (10 Points) Let m be a positive odd integer.

   (a) (5 Points) Prove or disprove that if n is a positive odd integer, then there is a free $\mathbb{Z}/m\mathbb{Z}$ action on S^n.

   (b) (5 Points) Prove or disprove that if n is a positive even integer, then there is a free $\mathbb{Z}/m\mathbb{Z}$ action on S^n.

Ph.D. Qualifying Exam: Algebra I
August 2020

Department of Mathematical Sciences, KAIST

Student ID: Name:

1. (15 pts)
   (a) (6 pts) State the fundamental theorem of finitely generated modules over PID. (You do not have to prove it.)
   (b) (9 pts) Let $\mathbb{F}$ be a finite field. Explicitly describe the group structure of the multiplicative group $\mathbb{F}^\times$ of non-zero elements of $\mathbb{F}$. Make sure to justify your answer.

2. (15 pts) Let R be a UFD. Show that the polynomial ring $R[x]$ is also a UFD.

3. (20 pts) Let G be a group of order 12. If G admits a *non-normal* Sylow 3-subgroup, then show that G is isomorphic to A_4. (**Hint:** Count the number of Sylow 3-subgroups.)

4. (25 pts) Let $n \geq 3$ be an integer, and set $G := (\mathbb{Z}/2\mathbb{Z}) \ltimes (\mathbb{Z}/n\mathbb{Z})$ with respect to the map $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ sending 1 to the multiplication by -1. Let $\rho \in G$ (respectively, $\sigma \in G$) denote the image of the generator of $\mathbb{Z}/n\mathbb{Z}$ (respectively, $\mathbb{Z}/2\mathbb{Z}$).
   (a) (10 pts) Find all conjugacy classes of G. (The answer may depend on n.)
   (b) (5 pts) Find the center Z of G, and describe G/Z via semidirect product of cyclic groups.
   (c) (10 pts) Write $n = 2^r \cdot n'$ where n' is odd and r is non-negative. Find the upper central series of G (i.e., ascending central series $\{e\} \subsetneqq Z \subsetneqq \cdots$). Find a necessary and sufficient condition for G to be *nilpotent*.

5. (25 pts) Let R be a commutative ring with 1, and let k be a field.
   (a) (5 pts) Given R-modules M and N, state the universal property of $M \otimes_R N$ as an R-module.
   (b) (5 pts) For *finite-dimensional* k-vector spaces V and W, compute $\dim_k(V \otimes_k W)$.
   (c) (15 pts) Given k-vector spaces V and W, show that there is a *well-defined* k-linear map

$$\varphi\colon V^* \otimes_k W \to \mathrm{Hom}_k(V, W) \quad \text{s.t. } \varphi(\alpha \otimes w) : v \mapsto \alpha(v) \cdot w,$$

   where $\alpha \in V^*$, $v \in V$ and $w \in W$. Furthermore, show that φ is *injective*, and is *isomorphic* if and only if V is finite-dimensional.

THE END

Ph.D. Qualifying Exam: Algebra II
August 2021

Student ID: Name:

Note: Be sure to write your answers in English. All answers must be supplied with supporting arguments.

1. [10] Prove the Hilbert basis theorem: let R be a noetherian commutative ring with unity. Then the polynomial ring $R[x]$ is also a noetherian ring.

2. [10] Answer the following questions:

   (a) ([5]) Let R be a unique factorization domain. Prove that R is normal, i.e. it is integrally closed in its field of fractions $F = \mathrm{Frac}(R)$.

   (b) ([5]) Let k be a field. For two distinct indeterminates x, y, consider the ring $S = k[x, y]/(y^2 - x^3)$. Prove that it is an integral domain, but it is *not* a unique factorization domain.

3. [25] Answer the following questions on field extensions $F \subset E$:

   (a) ([5]) Give a concrete example of $F \subset E$ such that E is a finite inseparable extension of F, for which the statement of the primitive element theorem fails.

   (b) ([5]) Give a concrete example of $F \subset E$ such that E is a finite separable extension of F, but not a Galois extension.

   (c) ([5]) Give a concrete example of $F \subset E$ such that E is a finite Galois extension of F, such that the Galois group $\mathrm{Gal}(F/E)$ is a simple non-abelian group.

   (d) ([5]) Give a concrete example of $F \subset E$ such that E is a finite Galois extension of F, such that the Galois group $\mathrm{Gal}(F/E)$ is an abelian group that is not cyclic.

   (e) ([5]) Give a concrete example of $F \subset E$ such that E is a finite Galois extension of F, such that the Galois group $\mathrm{Gal}(F/E)$ is a nontrivial cyclic group.

4. [20] Let F be a finite field. Answer the following questions.

   (a) ([5]) Prove that there exists a smallest positive integer $p > 0$ such that $p \cdot 1 := \underbrace{1 + \cdots + 1}_{p} = 0$, and this positive integer is a prime number.

   (b) ([5]) Prove that for the above prime number p, show that there exists a unique positive integer $n > 0$ such that $|F| = p^n$.

   (c) ([5]) Let $F^\times$ be the multiplicative group of nonzero elements of F. Prove that $F^\times$ is a cyclic abelian group.

   (d) ([5]) Let $\mathbb{F}_p \subset F$ be the prime subfield. Prove that any other finite extension $\mathbb{F}_p \subset F'$ of degree n is isomorphic to F.

More problems on the next page.

5. [10] Let k be a field and let A be a k-algebra. Answer the following questions.

   (a) ([5]) Give a concrete example of A that is an integral domain, and a finite extension $k \subset k'$ of fields, such that the ring $k' \otimes_k A$ is no longer an integral domain.

   (b) ([5]) Give a concrete example of A that is an integral domain, such that for any field extension $k \subset k'$, the ring $k' \otimes_k A$ is an integral domain.

6. [15] Let R be a commutative ring with unity. Answer the following questions.

   (a) ([5]) Let $I \subset R$ be an ideal. Consider the ring homomorphism $R \to R/I$ and its induced group homomorphism $\phi : R^\times \to (R/I)^\times$ of units.

   Give a concrete example of the pair (R, I) such that the group homomorphism ϕ is not surjective.

   (b) ([5]) Let $I \subset R$ be an ideal that is contained in the Jacobson radical $J(R)$ of R, i.e. the intersection of all maximal ideals of R. Under this assumption, is ϕ a surjection?

   (c) ([5]) Let R be a local ring and let $I \subset R$ be a proper ideal. What can you say about the surjectivity of ϕ?

7. [10] Answer the following questions.

   (a) ([2]) Give a concrete example of a homomorphism of rings $\phi : R \to S$ and a maximal ideal $M \subset S$ such that $\phi^{-1}(M) \subset R$ is a prime ideal, but not a maximal ideal.

   (b) ([8]) Let R, S be two finitely generated k-algebras over a field k and let $\phi : R \to S$ be a k-algebra homomorphism. Let $M \subset S$ be a maximal ideal. Prove that $\phi^{-1}(M) \subset R$ is a maximal ideal, which is a finite extension field of k.

THE END

Ph.D. Qualifying Exam: Complex Analysis

August 2021

Student ID:

Note: use English only for your answers.

Problem 1 (25 pts (5 each))
Find all singularities of all functions below. (Caution: Some functions might not be well-defined, and some might just look as if they are not well-defined.) Determine the nature of these singularities (i.e. decide whether they are isolated singularities and classify them as removable, pole or essential). In case of isolated singularities, find the radius of convergence of the Laurent series at 0. In case of poles, determine their order.

- $z/\tan^2 z$
- $z\sin(1/z)$
- $ze^z/(z^2+1)$
- $\cos(\sqrt{z})$
- $z/\sin(\sqrt{z})$.

Problem 2 (13 pts) Use a contour integral to prove

$$e^{-\pi u^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x u}\, dx = 1.$$

You may assume $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$

Problem 3 (7+10 pts) Let $G \subset \mathbb{C}$ be a connected open set and $f : G \to \mathbb{C}$ analytic.
a) Show that $g(z) = \overline{f(\overline{z})}$ is analytic in $\overline{G} = \{\overline{z} : z \in G\}$.
b) Show that $h(z) = f(\overline{z})$ is *not* analytic in $\overline{G}$, unless f is constant.

Problem 4 (14 pts)
Suppose that f is a holomorphic function in the unit disk centered at the origin such that $|f(z)| < 1$ for all $|z| < 1$. If $f(0) = 1/2$, how large can $|f'(0)|$ possibly be?

Problem 5 (17 pts)
(Half-plane Poisson integral) Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous, and define for $z = x + iy$,

$$Pf(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} f(t)\, dt.$$

Show that Pf is harmonic in the upper half plane, has a continuous extension to $\mathbb{R}$, and that $Pf = f$ on $\mathbb{R}$.

Problem 6 (14 pts) Prove that $f(z) = -\frac{1}{2}(z + 1/z)$ is a conformal map from the half-disc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane. [Hint: One method notes that $f(z) = w$ becomes $z^2 + 2wz + 1 = 0$, which has two distinct roots in $\mathbb{C}$ whenever $w \neq \pm 1$; however, there are several ways of doing this problem.]

Ph.D. Qualifying Exam: Numerical Analysis
August 2021

1. Consider the equation $e^x = \sin x$.

   (a) [6 points] Show that there is a solution $x^* \in (-\frac{5}{4}\pi, -\pi)$.

   (b) [8 points] Consider the following iterative methods:

      i. $x_{k+1} = \ln(\sin x_k)$,

      ii. $x_{k+1} = \arcsin(e^{x_k})$.

   What can you say about the local convergence of each of the methods for x^* as in (a) and their convergence order? If you use a theorem give its precise statement.

   (c) [6 points] For x^* as in (a) give a method that is quadratically convergent. Justify why the method is quadratically convergent.

2. [15 points] Let function $f \in C^{n+1}[a,b]$, $|f^{(n+1)}(x)| \leq M$ and $E_n(f)$ be the error of its best approximation by a polynomial of degree n. Show that

$$E_n(f) \leq \frac{2M}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}.$$

3. [15 points] By construction, the nth Newton-Cotes formula yields the exact value of the integral for integrands which are polynomials of degree at most n. Show that for even values of n, polynomials of degree $n+1$ are also integrated exactly.

4. The basic QR method for finding the eigenvalues of a real matrix A is

$$A_i = Q_i R_i, \quad A_{i+1} = R_i Q_i, \quad A_1 = A,$$

   where $A_i = Q_i R_i$ is the QR factorization of A_i.

   (a) [6 points] Prove that the eigenvalues of A_i are the same as the eigenvalues of A.

   (b) [8 points] Explain how to construct an upper Hessenberg matrix B that has the same eigenvalues as A.

   (c) [6 points] Prove that if A is upper Hessenberg, then A_i is also upper Hessenberg. (You may assume that A is invertible, and that Q_i is upper Hessenberg whenever A_i is upper Hessenberg.)

5. Prove the following.

(a) [5 points] A is an $n \times n$ symmetric positive definite matrix, prove:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T A x - b^T x$$

is equivalent to:

$$Ax = b\,.$$

(b) [8 points] Suppose a set of vectors spans $\mathbb{R}^n$: $\mathbb{R}^n = \mathrm{span}[v_1, v_2, \cdots, v_n]$, find the explicit solution to the following two problems (x_0 is a given initial data):

- Find, for $k = 1, \ldots, n$ iteratively,

$$\lambda_k^* = \mathrm{argmin}_{\lambda_k} \frac{1}{2}(x_{k-1} - \lambda_k v_k)^T A (x_{k-1} - \lambda_k v_k) - b^T (x_{k-1} - \lambda_k v_k)\,,$$

where $x_{k-1} = x_0 - \sum_{i=1}^{k-1} \lambda_i^* v_i$.

- Find the solution to

$$\min_{\lambda_1, \cdots, \lambda_n} \frac{1}{2} \left(x_0 - \sum_{k=1}^{n} \lambda_k v_k \right)^T A \left(x_0 - \sum_{k=1}^{n} \lambda_k v_k \right) - b^T \left(x_0 - \sum_{k=1}^{n} \lambda_k v_k \right)\,.$$

(c) [5 points] What is the sufficient and necessary condition for the two problems being equivalent? Prove it. (We exclude the possibility of A being diagonal.)

(d) See below for the algorithm of Conjugate Gradient with an initial guess $x_0 = 0$:

$r_0 = b - Ax_0 = b, \quad p_0 = r_0$
for $k = 0, 1, \ldots$ **do**
  $w_k = Ap_k$
  $a_k = (r_k^T r_k)/(p_k^T w_k), \quad x_{k+1} = x_k + a_k p_k$
  $r_{k+1} = r_k - a_k w_k$
  if $\|r_{k+1}\|$ is less than certain tolerance **then**
    stop
  end if
  $b_k = (r_{k+1}^T r_{k+1})/(r_k^T r_k), \quad p_{k+1} = r_{k+1} + b_k p_k$
end for

Show:

i. [4 points] $\mathrm{span}[r_0, \cdots, r_k] = \mathrm{span}[p_0, \cdots, p_k] = \mathrm{span}[p_0, Ap_0, \cdots, A^k p_0]\,.$

ii. [4 points] Assume $p_i^T A p_j = 0$ for $0 \le i < j \le k-1$, show $p_k^T A p_j = 0$ for $j < k - 2$.

iii. [4 points] Assume $r_i^T r_j = 0$ for $0 \le i < j \le k-1$, show $r_k^T r_j = 0$ for $j < k - 1$.

THE END

Qualifying Exam in Probability Theory
(August 2021)

1. (10 pts) Suppose that $\{X_n, n \geq 1\}$ and X are random variables. Show that X_n converges to X almost surely if and only if $\sup_{k \geq n} |X_k - X|$ converges to 0 in probability.

2. (20 pts) Suppose that $\{X_n, n \geq 1\}$ are random variables on a probability space $(\Omega, \mathcal{B}, P)$ and define $S_0 := 0, S_n = \sum_{i=1}^n X_i, n \geq 1$. Let $\tau(\omega) := \inf\{n > 0 : S_n(\omega) > 0\}$ and assume that $\tau(\omega) < \infty$ for all $\omega \in \Omega$. Show that τ and S_τ are random variables where $S_\tau(\omega) = S_{\tau(\omega)}(\omega)$ for $\omega \in \Omega$.

3. (10 pts) If X_n converges to X in distribution and $\{X_n\}$ are uniformly integrable, show that X is integrable and $E[X_n]$ converges to $E[X]$.

4. (10 pts) Let F_n and F be distribution functions. Show that, if F_n converges weakly to F and F is continuous, then $\sup_x |F_n(x) - F(x)|$ goes to 0.

5. (15 pts) Let X, Y, and Z be random variables such that $X \in L^1$, X and Y are independent of Z. Show that $E[X|Y, Z] = E[X|Y]$ almost surely.

6. (20 pts) Let $\{(X_n, \mathcal{G}_n), n \geq 0\}$ be a martingale. Suppose that N is a stopping time for the filtration $\{\mathcal{G}_n, n \geq 0\}$ such that $P\{N < \infty\} = 1$.
   (a) (10 pts) Show that $X_{\min\{N,n\}}$ is a martingale.
   (b) (10 pts) Show that, if $X_{\min\{N,n\}}$ are unformly bounded, then $E[X_N] = E[X_0]$.

7. (15 pts) Let $X \in L^1$, and $\{\mathcal{G}_n, n \geq 0\}$ be an increasing sequence of σ-fields. Prove, as n goes to ∞,
$$E[X|\mathcal{G}_n] \longrightarrow E[X|\mathcal{G}_\infty] \text{ in } L^1$$
where $\mathcal{G}_\infty = \sigma(\cup_n \mathcal{G}_n)$.

Good Luck To You!

Ph.D. Qualifying Exam: Real Analysis
August 2021

Student ID: Name:

Note: Be sure to use English for your answers.

1. Prove the following statements in $\mathbb{R}^n$:
   (a) [10 pts] A countable union of (Lebesgue) measurable sets is (Lebesgue) measurable.
   (b) [10 pts] Closed sets are (Lebesgue) measurable.
2. [15 pts] Suppose that $f : [0, b] \to \mathbb{R}$ is (Lebesgue) integrable. Let
$$g(x) = \int_x^b \frac{f(t)}{t}\, dt$$
for $x \in (0, b]$. Prove that
$$\int_0^b g(x)\, dx = \int_0^b f(t)\, dt$$
3. [15 pts] Construct an increasing function on $\mathbb{R}$ whose set of discontinuties is $\mathbb{Q}$.
4. Prove the following statements:
   (a) [10 pts] If $1 \le p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.
   (b) [10 pts] If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim\limits_{p\to\infty} \|f\|_p = \|f\|_\infty$.
5. [15 pts] Let X be a Banach space, and let A and B be linear operators on X. Assume that A is invertible and $\|B - A\| \cdot \|A^{-1}\| < 1$. Prove that B is invertible.
6. [15 pts] Assume that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite complete measure spaces. Prove that, for any $\mathcal{M} \times \mathcal{N}$-measurable function f on $X \times Y$, if $1 \le q \le p < \infty$, then
$$\left[\int_X \left(\int_Y |f(x,y)|^q d\nu(y)\right)^{p/q} d\mu(x)\right]^{1/p} \le \left[\int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{q/p} d\nu(y)\right]^{1/q}.$$

THE END