# Ph.D. Qualifying Exam: Algebra I
## February 2021

Department of Mathematical Sciences, KAIST

Student ID: Name:

1. (20 pts) Let $p$ be a prime integer and let $G$ be the group of $n \times n$ matrices of determinant 1 over the finite field of order $p$.
   (a) (5 pts) Compute the order of $G$.
   (b) (15 pts) Determine the number of Sylow $p$-subgroups of $G$.

2. (20 pts) Let $S_n$ denote the symmetric group on $n$ letters with $n \geq 2$. Show that the order of any conjugacy class of $S_n$ is less than or equal to $n!/(n-1)$.

3. (20 pts) Let $R = \{a + b\sqrt{p} \mid a, b \in \mathbb{Z}\}$, where $p \in \{\pm 3, \pm 5\}$. For each integer $p$, determine whether $R$ is a principal ideal domain.

4. (20 pts) Let $G$ be the dihedral group of order $2n$ with $n \geq 1$.
   (a) (10 pts) Calculate the order of the group of automorphisms of $G$.
   (b) (10 pts) Find all $n$ such that $G$ is nilpotent.

5. (20 pts) Let $I_1, \ldots, I_n$ be ideals of a commutative ring $R$ with $n \geq 2$.
   (a) (10 pts) Assume that $I_i + I_j = R$ for all $1 \leq i \neq j \leq n$. Prove that for any integers $k_1, \ldots, k_n \geq 1$ we have
   $$\bigcap_{i=1}^{n} I_i^{k_i} = \prod_{i=1}^{n} I_i^{k_i}.$$
   (b) (10 pts) Assume that $I_1, \ldots, I_n$ are prime ideals. Show that if an ideal $J$ is not contained in $I_i$ for all $1 \leq i \leq n$, then it is not contained in the union of $I_1, \ldots, I_n$.

**THE END**

# Ph.D. Qualifying Exam: Algebra II
## February 2021

Department of Mathematical Sciences, KAIST

Student ID: Name:

**Note:** Please ensure to use English for your answers

1. (20 pts) Let $p$ be a prime number, and let $n > 1$ be an integer. Let $K$ be a splitting field of the polynomial $x^n - p$ over $\mathbb{Q}$. Then *explicitly* describe *(a)* the field $K$ and *(b)* its Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

2. (20 pts) Let $p$ be an *odd* prime number, and let $\mathbb{F}_p$ denote *the* prime field with $p$ elements. For any positive integer $n > 2$ not divisible by $p$, let $\mathbb{F}_p(\zeta_n)$ denote a splitting field of $x^n - 1$ over $\mathbb{F}_p$. Then explicitly compute *(a)* the degree $[\mathbb{F}_p(\zeta_n) : \mathbb{F}_p]$, and *(b)* the group $\mathrm{Aut}(\mathbb{F}_p(\zeta_n))$ of field automorphisms of $\mathbb{F}_p(\zeta_n)$.

3. (15 pts) Let $R := \mathbb{Z}[\sqrt{-5}]$ denote the smallest subring of $\mathbb{C}$ containing the roots of $x^2 + 5$.
   (a) (5 pts) Find all units in $R$.
   (b) (10 pts) Is $R$ a principal ideal domain? Justify your answer. (**Hint:** Try to factorize 6.)

4. (20 pts) Let $R$ be a commutative *Noetherian* ring with 1. Prove or disprove the following claims.
   (a) (10 pts) Any finitely generated commutative $R$-algebra with 1 is Noetherian.
   (b) (10 pts) Any subring of $R$ (containing 1) is Noetherian.

5. (25 pts) Let $k'/k$ be a finite extension of fields with degree $d$.
   (a) (10 pts) If $K$ is a field extension of $k'$ such that $\left|\mathrm{Hom}_{k\text{-alg}}(k', K)\right| = d$. Then construct a natural $K$-algebra isomorphism $k' \otimes_k K \xrightarrow{\sim} \prod_{\mathrm{Hom}_{k\text{-alg}}(k',K)} K$.
   (b) (5 pts) Show that if $k'/k$ is *separable*, then $k' \otimes_k k'$ is *reduced.*
   (c) (10 pts) Show that if $k'/k$ is *not* separable, then $k' \otimes_k k'$ is *not* reduced.

**THE END**

# Ph.D. Qualifying Exam: Differential Geometry
## February 2021

Student ID: Name:

Note: Be sure use English for your answers.

1. [10 pts] Let $F : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}^{n+1}\setminus\{0\}$ be a smooth map, and suppose that for some $d \in \mathbb{Z}$, $F(tx) = t^d F(x)$ for every $t \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Show that the map $\tilde{F} : \mathbb{RP}^n \to \mathbb{RP}^n$ defined by $\tilde{F}([x]) = [F(x)]$ is well-defined and smooth. Here $[y]$ denotes the corresponding point $[y_0 : ... : y_n] \in \mathbb{RP}^n$ of $y = (y_0, ..., y_n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

2. [5pts + 10 pts] Let $M$ be a compact smooth manifold of dimension $n > 0$. Let $f : M \to \mathbb{R}$ be a smooth map.
   (a) Show that $f$ has at least two critical points.
   (b) Let $a$ be a critical point of $f$. Let $(U, \varphi)$ be a local coordinate at $a$, where $\varphi(p) = (x_1, ..., x_n) \in \mathbb{R}^n$ for $p \in U \subset M$. Show that the rank of the Hessian matrix of $f$ at $a$, which is given by
   $$\mathrm{Hess_a}(f) := \left[\frac{\partial^2 (f \circ \varphi^{-1})}{\partial x_i \partial x_j}(\varphi(a))\right]_{i,j=1,\ldots,n},$$
   is independent of local coordinates.

3. [10 pts + 10 pts] Let $V$ be a bounded open set in $\mathbb{R}^3$ with smooth boundary $S$, and let $\mathbf{F}$ be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\iiint_V \mathrm{div}(\mathbf{F})\, d\mathrm{Vol}$ as the surface integral over the boundary $S$.
   (a) State precisely this theorem.
   (b) Show carefully how it can be obtained as a special case of the Stokes theorem for differential forms. (You may use a formula for the element of area $dS$.)

4. [5 pts + 10 pts] Let $N = (0, 0, 1)$, $S = (0, 0, -1)$ be the north pole and the south pole of $\mathbb{S}^2 \subset \mathbb{R}^3$, respectively.
   (a) Define the stereographic projections
   $$\sigma : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2 \quad \text{and} \quad \tilde{\sigma} : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$$
   which give a smooth atlas of $\mathbb{S}^2$.
   (b) Describe explicitly trivializations and transition functions for the vector bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ on this atlas.

5. [10 pts + 10 pts]
   (a) Show that there is no structure of smooth manifold on $[0, \infty)$, which induces the standard Euclidean topology.
   (b) Give three distinguished smooth structures on $\mathbb{R}$.

6. [5 pts + 10 pts + 5 pts] A symplectic form on a 6-dimensional smooth manifold is defined to be a smooth closed 2-form $\omega$ such that $\omega \wedge \omega \wedge \omega$ is a volume form (that is everywhere nonvanishing). Determine which of following manifolds admits symplectic forms: (a) $\mathbb{S}^6$; (b) $\mathbb{S}^2 \times \mathbb{S}^4$; (c) $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$.

**THE END**

# Ph.D. Qualifying Exam: Complex Analysis
## February 2021

Student ID:

Note: Write your solutions in English.

**Problem 1 (20 points = 6 points + 7 points + 7 points)**

Consider a Möbius transformation

$$Tz = \frac{az+b}{cz+d}$$

where

$$ad - bc = 1.$$

(i) Show that $T$ maps the real line to a circle or a straight line.
(ii) When the image of the real line is a circle, show that its radius $r$ satisfies

$$\frac{1}{r} = 2\left|c^2 \mathrm{Im}\left(\frac{d}{c}\right)\right|.$$

(iii) Explain why the right-hand side does not depend on $a$ or $b$.

**Problem 2 (20 points = 10 points + 10 points)**

Let $U$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be an analytic (or, holomorphic) function.
(i) Show that the diameter

$$d = \sup_{z,w\in U} |f(z) - f(w)|$$

of the image $f(U)$ satisfies that

$$2|f'(0)| \le d.$$

(ii) Prove that equality holds if and only if $f$ is linear, i.e.,

$$f(z) = az + b$$

for some constants $a$ and $b$.

**Problem 3 (20 points = 10 points + 10 points)**

Let $P(x, y)$ and $Q(x, y)$ be polynomials of two variables $x$ and $y$. Suppose that

$$Q(x, y) \ne 0$$

for every $(x, y) \in \mathbb{R}^2$ with

$$x^2 + y^2 = 1.$$

(i) Show that

$$\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)} dt = 2\pi i \sum_{w\in U} \mathrm{Res}(f; w),$$

where $U$ is the open unit disk, and $f(z)$ is the rational function defined by

$$f(z) = \frac{1}{iz} \frac{P\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{Q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}.$$

(ii) Let $a \in \mathbb{R}$, $|a| < 1$. Show that

$$\int_0^{2\pi} \frac{1}{1 - 2a\cos t + a^2}\,dt = \frac{2\pi}{1 - a^2}.$$

(Hint: You may use Part (i).)

**Problem 4 (20 points = 7 points + 7 points + 6 points)**

Let $u(x, y)$ be a real-valued harmonic function in $x^2 + y^2 < R^2$.
(i) Show that a complex-valued function $f(z)$ defined by

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0)$$

is analytic in $|z| < R$.
(Hint: We can expand $u(x, y)$ into a double power series in $x$ and $y$, and substitute $x = \frac{z}{2}$ and $y = \frac{z}{2i}$.)
(ii) Find the real part of $f(z)$.
(iii) Consider the real-valued function

$$u(x, y) = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y}.$$

Show that

$$f(z) = \tan z$$

where $f(z)$ is defined in Part (i).

**Problem 5 (20 points = 7 points + 6 points + 7 points)**

In the following problems $u(z)$ is a real-valued nonnegative harmonic function.
(i) Suppose that $u(z) \geq 0$ is harmonic in $|z| < R$. Show that

$$\frac{R - |z|}{R + |z|} u(0) \leq u(z) \leq \frac{R + |z|}{R - |z|} u(0).$$

(ii) Suppose that $u(z) \geq 0$ is harmonic in the disk $D(a, r)$ of center $a$ and radius $r$. Show that

$$\frac{1}{3} u(a) \leq u(z) \leq 3u(a)$$

for all $z$ in the disk $D(a, r/2)$.
(iii) Let $u(z) \geq 0$ be harmonic on a connected open set $D \subset \mathbb{C}$. Let $K \subset D$ be a compact subset. Show that there exists a constant $A > 0$, depending only on $D$ and $K$, such that

$$u(z_1) \leq A u(z_2)$$

for all $z_1, z_2 \in K$.

# Ph.D. Qualifying Exam: Numerical Analysis
## February 2021

1. [7 points each] Consider the fixed point iteration scheme

$$x_{n+1} = g(x_n).$$

   (a) State the conditions for the convergence of such a scheme to fixed point $x = \alpha$ and prove the convergence under those conditions. Make the conditions as general as possible.

   (b) Suppose that $g(x)$ is continuously differentiable near $\alpha$. Given

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha) \quad \text{with } |g'(\alpha)| < 1,$$

   find an upper bound for the absolute error $|\alpha - x_n|$.

   (c) Derive the conditions that the root finding method is $p$th order convergent.

   (d) Consider the following iteration for calculating $\gamma^{1/3}$:

$$x_{n+1} = ax_n + b\frac{\gamma}{x_n^2} + c\frac{\gamma^2}{x_n^5}$$

   Assuming that this iterative scheme converges for $x_0$ sufficiently close to $\gamma^{1/3}$, determine $a, b, c$ such that the method has the highest possible convergence rate.

2. Consider the Hermite problem of constructing a polynomial $p(x)$ of degree $\le 3$ such that

$$p(x_1) = y(x_1), p'(x_1) = y'(x_1), p(x_2) = y(x_2), p'(x_2) = y'(x_2).$$

   (a) [9 points] Derive a Lagrange type formula for $p(x)$.

   (b) [6 points] Derive an error formula.

   (c) [6 points] Prove that the interpolation is unique.

3. [12 points] Find the Gaussian quadrature formula for $n = 2$ on the interval $[0, 1]$ with weight function $w(x) = x$. That is, find the weights and nodes for the formula

$$\int_0^1 f(x)x\,dx \approx \sum_{i=1}^{2} w_i f(x_i)\,.$$

   What is the degree of precision of this formula?

4. [9 points each] Consider the boundary value problem (BVP)

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \quad u(0) = u(1) = 0,$$

where $a(x) > \delta \geq 0$ is a bounded differentiable function in $[0,1]$. We note that the above BVP can be written as

$$-\frac{da}{dx}\frac{du}{dx} - a(x)\frac{d^2u}{dx^2} = f(x), \quad u(0) = u(1) = 0.$$

We assume that, although $a(x)$ is available, an expression for its derivative, $\frac{da}{dx}$, is not available.

(a) Using finite differences and an equally spaced grid in $[0,1]$, $x_l = hl$, $l = 0,\ldots,n$ and $h = 1/n$, we discretize the BVP to obtain a linear system of equations, yielding an $O(h^2)$ approximation of the BVP. After the application of the boundary conditions, the resulting coefficient matrix of the linear system is an $(n-1)\times(n-1)$ tridiagonal matrix. Provide a derivation and write down the resulting linear system (by giving the expressions of the elements).

(b) Utilizing all the information provided, find a disc in $\mathbb{C}$, the smaller the better, that is guaranteed to contain all the eigenvalues of the linear system constructed in part (a).

5. For $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ with rank $r$, let $A = U\Sigma V^T$ be its singular value decomposition. Denote $u_i$ the $i$-th column of $U$ and $v_j$ the $j$-th column of $V$. $\Sigma$ collects all singular values along its diagonal with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$.

(a) [6 points] Show that $\|A\|_2 = \sigma_1$, the largest singular value of $A$.

(b) [6 points] Show that the Frobenius norm $\|A\|_F = \sqrt{\sigma_1^2 + \cdots \sigma_r^2}$, where $\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2$.

(c) [9 points] Show that $A_\nu = \sum_{i=1}^{\nu} \sigma_i u_i v_i^T$ is the best $\nu$-rank approximation to $A$ in $L_2$ norm.

**THE END**

# 2021 QUALIFYING EXAM - REAL ANALYSIS

**Problem 1.** (20 pt) Let $f : [0,1] \to [0, M]$ be a bounded (Lebesgue) measurable function. Show that $f$ is Riemann integrable if and only if $f$ is continuous almost everywhere.

**Problem 2.** (15 pt) Let $\{u_n : \mathbb{R} \to \mathbb{R}\}$ be a sequence of continuous functions on $\mathbb{R}$ that are equicontinuous and satisfy $|u_n(x)| \le \frac{1}{1+|x|^2}$ for all $n$. Show that there is a convergence subsequence in $L^1$-norm. (Hint. You may use Arzela-Ascoli theorem)

**Problem 3.** (15 pt) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For given $\epsilon > 0$, there exists a continuous function $g(x)$ such that $g'(x)$ exists and equals 0 almost everywhere and

$$\sup_{x\in[0,1]} |f(x) - g(x)| \le \epsilon.$$

(Hint. Mimic the Cantor function.)

**Problem 4.** We define the 1d Fourier transform by $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx$.

(1) (10pt) Assume that for each integer $N$, we have a decay $|\widehat{f}(\xi)| \le C_N \frac{1}{1+|\xi|^N}$. Show that $f \in C^\infty \cap L^2$.
(2) (10pt) Show that if we further assume $|\widehat{f}(\xi)| \le Ce^{-\alpha|\xi|}$ for some $\alpha > 0$, then $f(x)$ is real-analytic.

**Problem 5.** Let $\phi(x)$ be a radially decreasing bump function. i.e. $\phi \in C_c^\infty(\mathbb{R}^d)$, supp $\phi \in B_1(0)$, $\int \phi = 1$. We denote a rescaled function $\phi_\delta(x) = \delta^{-d}\phi(\frac{x}{\delta})$ for $\delta > 0$.

(1) (10pt) For each $f \in L^1_{loc}(\mathbb{R}^d)$, show that $f * \phi_\delta$ converges to $f$ almost everywhere as $\delta \to 0$. (Hint. You may use Lebesgue differentiation theorem)
(2) (10pt) Show that if $f$ is continuous and compactly supported, then $f * \phi_\delta$ converges uniformly to $f$.
(3) (10pt) Show that if $f \in L^p(\mathbb{R}^d)$ for $1 \le p < \infty$, then $\lim_{\delta\to 0} \|f * \phi_\delta - f\|_{L^p} = 0$.

# PH.D. QUALIFYING EXAM GUIDE: REAL ANALYSIS

This guide and coverage of the exam is **only for 2021 Feb exam**. The coverage may change in the future. The coverage is selected on the basis of topics taught in MAS 540 in Spring 2019 and 2020.
We also assume examinees are familiar with major undergraduate topics such as Vector calculus, Linear algebra, Analysis, and Topology.

## Scope of the subject

*The exam will cover the following topics*:

- Riemann integrals
- General measure and integral theory
- Product measure and Fubini theorem.
- Differentiation e.g. Hardy-Littlewood maximal inequality, Lebesgue differentiation theorem, Fundamental theorem of calculus, Radon-Nikodym theorem, Lebesgue decomposition.
- Function of bounded variation and differentiability of functions on $\mathbb{R}$.
- $L^p$ spaces and related topics (Folland Ch.6, Lieb-Loss Ch.2,Ch.4)
- Fourier transforms (Folland Ch.8.1–8.3, Lieb-Loss Ch.5)
- Hilbert spaces.
- Bounded linear functionals on Banach spaces and Riesz representation theorems.
- Distributions and Sobolev spaces(Folland Ch.9, Lieb-Loss Ch.6)

*The exam will NOT include the following topics*:

- Hahn-Banach theorem and its applications
- Baire category theorem and its applications
- Locally convex Hausdorff space, Topological vector spaces.
- Duality on Radon measure
- Compact operators, Unbounded operators, Spectral theorems
- Weak topology

## Suggested references

Folland, *Real analysis.*
Lieb and Loss, *Analysis* Jones, *Lebesgue integrals in Euclidean spaces.*
Stein and Shakarchi, *Real analysis.*

Tao, *An epsilon of room.*
Rudin, *Real and complex analysis.*
Bass, *Real analysis.*
Royden, *Real analysis.*
Stein and Shakarchi, *Fourier analysis.*

# Ph.D. Qualifying Exam: Probability
## Spring 2021

Student ID:

Note: use English only for your answers.

1. [20 pts]Suppose $T : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$ is measurable. Suppose $X$ is a random variable on $(\Omega_1, \mathcal{F}_1)$. Show that $X$ is measurable with respect to the $\sigma$-field generated by $T$ iff there is a random variable $Y$ on $(\Omega_2, \mathcal{F}_2)$ such that $X = Y \circ T$.

2. [20 pts]Let $\{X_n\}$ be iid with mean 0 and variance $\sigma^2$. Let $\{R_n\}$ be a sequence of positive random variables such that for some sequence of integers $a_n \to \infty$, $R_n/a_n \to 1$ in distribution. Show that as $n \to \infty$, $\frac{S_{R_n}}{\sigma\sqrt{a_n}}$ converges in distribution to $N(0,1)$. You may assume the standard CLT (be careful on this problem).

3. [20 pts] Prove the Strong Law of Large Numbers: Let $(X_k, k \in \mathbb{N})$ be an i.i.d. sequence such that $EX_1^4 < \infty$ and $EX_1 = 0$. Show that, almost surely,
$$\frac{X_1 + \cdots + X_n}{n} \to 0$$
as $n \to \infty$.

4. [20 pts] If $\{X_n\}$ is a sequence of mean-zero independent random variables with $E(X_n^2) = \sigma_n^2 < \infty$, show that
$$M_n := \left(\sum_{k=1}^{n} X_k\right)^2 - \sum_{k=1}^{n} \sigma_k^2$$
is a martingale with respect to the natural filtration (what is the filtration?).

5. [20 pts] Suppose the ordered pair of random variables $(X, Y)$ is equal in distribution (jointly) to $(Z, Y)$. Show that $\mathbf{E}(X|\sigma(Y))$ is a.s. equal to $\mathbf{E}(Z|\sigma(Y))$ where $\sigma(Y)$ is the $\sigma$-field generated by $Y$.