Ph.D. Qualifying Exam: Algebraic Topology 1 (August 2020)

Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. If answers are not well organized, we will take points off. (Total 100 pts.)

1. (20 pts.) Let X and Y be disjoint subspaces in $\mathbf{S}^3$ homeomorphic to a circle. Compute $H_*(\mathbf{S}^3 - X - Y, G)$ for a given finitely generated abelian group G. (Hint: We may not assume that X and Y are contained in disjoint cells respectively.)

2. (25 pts.) Let $\pi : X \to Y$ be a two-sheeted covering map, and let $g : X \to X$ be the unique nontrivial deck transformation. We define a chain map $t : C_*(Y, \mathbf{Z}_2) \to C_*(Y, \mathbf{Z}_2)$ by sending a simplex $\tau : \Delta_p \to Y$ to $\sigma + g \circ \sigma : \Delta_p \to X$ where σ is a lift of τ.
   (1) (7 pts.) Show that t is a well-defined chain map.
   (2) (8 pts.) Show that there is a long exact sequence
   $$\cdots \to H_p(Y, \mathbf{Z}_2) \xrightarrow{t_*} H_p(X, \mathbf{Z}_2) \xrightarrow{\pi_*} H_p(Y, \mathbf{Z}_2) \xrightarrow{\partial_*} H_{p-1}(Y, \mathbf{Z}_2) \to \cdots .$$
   (3) (10 pts.) Do this for $X = \mathbf{S}^n$ and $Y = \mathbf{RP}^n$ and describe the maps and the homology groups using generators.

3. (25 pts.) Let X be the space $\mathbf{S}^1 \times \mathbf{S}^1 - \{p\}$ for a point p.
   (1) (5 pts.) Compute the fundamental group of X.
   (2) (5 pts.) Find two finite covering spaces of X that are not homeomorphic.
   (3) (8 pts) Let $\pi : Y \to X$ be a covering map. There is a homomorphism $\pi_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$. π induces a homology homomorphism $\pi_H : H_1(Y; \mathbf{Z}) \to H_1(X, \mathbf{Z})$. Discuss the relationship between π_* and π_H and prove your discussions.
   (4) (7 pts.) Compute π_* and π_H explicitly for a finite covering and an infinite covering of X.

4. (30 pts.) Let X be the space $\mathbf{S}^n - Y - Z$ where Y and Z are disjoint embedded subspaces homeomorphic to spheres of dimension l and k for $0 < l, k < n$. Let f be the inclusion map $Y \to \mathbf{S}^n - Z$. Let $f_* : \tilde{H}_*(Y, \mathbf{Z}) \to \tilde{H}_*(\mathbf{S}^n - Z, \mathbf{Z})$ denote the induced homomorphism. Here, $\tilde{H}_*$ indicates the reduced homology groups.
   (1) (15 pts.) For each n, $n > 2$, find an example of Y and Z where f_* is not trivial.
   (2) (15 pts.) Let T be an $(n-1)$-sphere embedded as a subcomplex of S^n realized as a cell-complex. Suppose that Y and Z are in different components of $S^n - T$. Show that f_* is trivial. (hint: The closure of each component is also a subcomplex.)

Ph.D. Qualifying Exam: Algebraic Topology II (August 2020)

Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. If answers are not well organized, we will take points off. The meaning of "describe the ring" means that you need to find a basis and write the products of two basis elements in terms of the basis. (Total 100 pts.)

1. (15 pts.) Let X be a union of m mutually disjoint spheres of respective dimensions $i_j, 0 < i_j < n-1$, $j = 1, \ldots, m$, in $\mathbf{S}^n$, $n \geq 3$. Compute $H_*(\mathbf{S}^n - X; \mathbf{Z})$.

2. (20 pts.) Let X and Y be cellular complexes. Let $T : X \times Y \to Y \times X$ be a map given by $(x, y) \mapsto (y, x)$.
   (1) (10 pts.) Prove that $\alpha \times \beta = (-1)^{pq} T^*(\beta \times \alpha)$ for $\alpha \in H^p(X, \mathbf{Z})$ and $\beta \in H^q(Y, \mathbf{Z})$.
   (2) (10 pts.) Show that $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$ for $\alpha \in H^p(X, \mathbf{Z})$ and $\beta \in H^q(X, \mathbf{Z})$.

3. (20 pts.)
   (1) (8 pts.) Describe the cohomology ring $H^*(\mathbf{RP}^n, \mathbf{Z}_2)$ where $n \geq 2$.
   (2) (12 pts.) Describe the cohomology ring $H^*(X, \mathbf{Z}_2)$ where $X := \mathbf{S}^4 \times \mathbf{RP}^4$.

4. (25 pts.) Let $\mathbf{S}^3 \subset \mathbf{C}^2$ be a unit sphere. Let $\omega = e^{2\pi i/p}$ for a prime number p. Let q be an integer relatively prime to p. Let $T_q : \mathbf{S}^3 \to \mathbf{S}^3$ be a map defined by $(u, v) = (\omega u, \omega^q v)$ for $u, v \in \mathbf{C}$. A lens space L is defined as $\mathbf{S}^3 / \langle T_q \rangle$.
   (1) (13 pts.) Compute $H^*(L, \mathbf{Z})$ and describe the ring structure under cup product operations.
   (2) (12 pts.) Compute $H^*(L, \mathbf{Z}_l)$ for any integer l, $l > 1$.

5. (20 pts.) Find examples of compact cell complexes and the coefficient groups denoted by G with the following properties. If it is not possible, prove the nonexistence.
   (1) (5 pts.) $H_*(M, G) \neq H_*(M) \otimes G$.
   (2) (7 pts.) $H^*(M, G) \neq H^*(M) \otimes G$.
   (3) (8 pts.) $H^*(M, \mathbf{Z}_p) = H^*(M) \otimes \mathbf{Z}_p$ for a prime p, but this fails for some other primes.

Ph.D. Qualifying Exam: Algebra I
August 2020

Department of Mathematical Sciences, KAIST

Student ID: Name:

1. (20 pts) Let $n \geq 3$ be an odd integer. Let $G = D_{2n} \times Q_8$, where D_{2n} denotes the dihedral group of order $2n$ and Q_8 denotes the quaternion group of order 8.

   (a) (10 pts) Find all conjugacy classes of G.

   (b) (10 pts) Determine the group of inner automorphisms of G.

2. (20 pts) Let p and q be prime integers with $5 \leq p < q$ and let G be a group of order $4pq$. Determine whether G is simple or not.

3. (20 pts) Let R be a domain such that for all $a \in F$ (the field of fractions of R) either $a \in R$ or $a^{-1} \in R$, i.e., R is a valuation ring. Let I be an ideal of R.

   (a) (10 pts) Show that R is local and integrally closed.

   (b) (10 pts) Show that if R is Noetherian, then $I = (p^n)$ for some prime p and some nonnegative integer n.

4. (20 pts) Let R be the ring of 2×2 upper triangular matrices over a field.

   (a) (10 pts) Determine all two-sided ideals of R.

   (b) (10 pts) Show that the group of units of R is solvable.

5. (20 pts) Let G be a group of order p^4 for some prime integer p. Assume that G has a normal subgroup H of order p^2. Prove that G has an abelian subgroup A with $|A| \geq p^3$.

   (Hint: One may consider a homomorphism $\phi : G \to \mathrm{Aut}(H)$ given by $\phi(g)(h) = ghg^{-1}$)

THE END

Ph.D. Qualifying Exam: Algebra II
August 2020
Department of Mathematical Sciences, KAIST

Student ID: Name:

1. (20 pts) Let p be a prime integer. Determine whether the polynomial $x^{p^n} - x - 1$ is reducible or irreducible over the field of p elements.

2. (15 pts) Find all square free integers $d \geq 2$ such that $\mathbb{Z}[\sqrt{-d}]$ is a unique factorization domain.

3. (20 pts) Compute the degree of the splitting field of $x^6 + 5x^3 - 2$ over $\mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers.

4. (30 pts) Determine the Galois group over K of each of the following polynomials $f(x)$.

   (a) (15 pts) $K = \mathbb{Q}$, $f(x) = x^{15} + 3$.

   (b) (15 pts) $K = \mathbb{Q}(\sqrt{-3})$, $f(x) = x^6 - 3$.

5. (15 pts) Let $K(x)$ be the rational function field over a field K and let $y \in K(x)$. Prove that $K(x) = K(y)$ if and only if $y = \dfrac{ax+b}{cx+d}$ for some $a, b, c, d \in K$ with $ad - bc \neq 0$.

THE END

Ph.D. Qualifying Exam: Differential Geometry
August 2020

Student ID: Name:

Note: Be sure use English for your answers.

1. [15 pts] Prove directly the following special case of Stokes' Theorem. Suppose $P(x,y,z), Q(x,y,z)$, and $R(x,y,z)$ are C^∞ functions on $\mathbb{R}^3$ which vanish identically if $\max\{|x|,|y|,|z|\} \geq 5$. Prove that
$$\int_{-6}^{6}\int_{-6}^{6}\int_{-6}^{6} d(Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy) = 0.$$
2. [15 pts] Show that there is no smooth submersion $F : \mathbb{S}^n \to \mathbb{R}^n$ for any $n > 0$.
3. [15 pts] Let M be a smooth manifold of dimension n. Let $\omega_1, ..., \omega_k$ be smooth 1-forms that $\{\omega_1|_p, ..., \omega_k|_p\}$ is linearly independent for each $p \in M$. Given smooth 1-forms $\alpha_1, ..., \alpha_k$ such that
$$\sum_{i=1}^{k} \omega_i \wedge \alpha_i = 0,$$
show that there exist smooth functions f_{ij} so that
$$\alpha_i = \sum_{i=1}^{k} f_{ij}\omega_j, \quad i = 1, ..., k.$$
4. [15 pts] Let M be a smooth manifold and V, W, X be smooth vector fields on M. Show that
$$\mathcal{L}_{[V,W]}X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X,$$
where $\mathcal{L}_V W$ is the Lie derivative of W with respect to V and $[V, W]$ is the Lie bracket of V, and W.
5. [20 pts] The vector field V on $\mathbb{R}^3$ whose value at $p = (x, y, z) \in \mathbb{R}^3$ is
$$V_p = x\frac{\partial}{\partial x}\bigg|_p + y\frac{\partial}{\partial y}\bigg|_p + z\frac{\partial}{\partial z}\bigg|_p.$$
Let c be real number, and let $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ be a smooth function that is positively homogeneous of degree c, meaning that $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^3 \setminus \{0\}$.
   (a) Show that $Vf = cf$.
   (b) Show that if $g \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $Vg = cg$ for some $c \in \mathbb{R}$, then g is positively homogeneous of degree c.
6. [20 pts] Let $\mathbf{M}(n;\mathbb{R})$ be the space of $n \times n$ real matrices and $\mathbf{M}_k(n;\mathbb{R})$ be the subspace of all matrices of rank k in $\mathbf{M}(n;\mathbb{R})$.
   (a) Show that $\mathbf{M}_1(2;\mathbb{R})$ is a 3-dimensional submanifold of $\mathbf{M}(2;\mathbb{R}) \equiv \mathbb{R}^4$.
   (b) Show that $\mathbf{M}_k(n;\mathbb{R})$ is a smooth submanifold of $\mathbf{M}(n;\mathbb{R}) \equiv \mathbb{R}^{n^2}$ of codimension $(n-k)^2$.

THE END

Ph.D. Qualifying Exam: Complex Analysis
August 2020

Student ID:

Note: use English only for your answers.

Problem 1 (5+8 pts) Let $G \subset \mathbb{C}$ be a connected open set and $f : G \to \mathbb{C}$ analytic.
a) Show that $g(z) = \overline{f(\overline{z})}$ is analytic in $\overline{G} = \{\overline{z} : z \in G\}$.
b) Show that $h(z) = f(\overline{z})$ is *not* analytic in $\overline{G}$, unless f is constant.

Problem 2 (11 pts) Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and suppose that there is a constant M and an integer n such that

$$|f(z)| \leq M(1 + |z|^n)$$

for all $z \in \mathbb{C}$. Show that f is a polynomial of degree $\leq n$.

Problem 3 (11 pts) Let f be a non-constant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 4 (11 pts) $\mathbb{D}$ is the open unit disc centered at the origin. Let $f : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function and assume that f extends continously to $\partial\mathbb{D} \setminus \{1\}$. Show that $|f| \leq 1$ in $\mathbb{D}$ if $|f| \leq 1$ on $\partial\mathbb{D} \setminus \{1\}$.
Hint: find a function related to f that is subharmonic and use the maximum principle.

Problem 5 (4 pts each) Let $0 < |a| < |b| < \infty$ and consider the function $f(z) = \frac{1}{(z-a)(z-b)}$. Find the Laurent series of f in the three domains $\{|z| < |a|\}$, $\{|a| < |z| < |b|\}$ and $\{|b| < |z|\}$.

Problem 4 (7 pts each) Calculate

$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

and

$$\int_0^\infty \frac{dx}{x^3+1}.$$

Problem 7 (4+11 pts) State the Schwarz Lemma. Use this lemma to show that f is an analytic bijection of $\mathbb{D}$ to itself if and only if

$$f(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}$$

for some $a \in \mathbb{D}$.

Problem 8 (13 pts)
Give an example of a domain and a bounded harmonic function u whose harmonic conjugate v (i.e., a function v such that $f = u + iv$ is analytic) is not bounded.

Ph.D. Qualifying Exam: Numerical Analysis

August 2020

1. [5 points each] Let $p(x)$ be a continuously differentiable function from $\mathbb{R}$ into $\mathbb{R}$ with $p(x) > 0$ and $\int_{-\infty}^{\infty} p(x)\,dx = 1$. Define $F(x) = \int_{-\infty}^{x} p(t)\,dt$.

   (a) Show that $F(x)$ is invertible.

   (b) Let $x \in F^{-1}(y)$. Write Newton's method to solve for x given $y \in (0,1)$ using only evaluations of the functions $p(x)$ and $F(x)$.

   (c) Explain why the method is locally at least quadratically convergent for every $y \in (0,1)$.

2. [5 points each]

   (a) Determine the values of a, b, c so that the following is a cubic spline with knots at $0, 1, 2$:

   $$s(x) = \begin{cases} 3 - 2x + 2x^3 & \text{for } x \in [0,1] \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{for } x \in [1,2] \end{cases}$$

   (b) A spline is called natural if the second derivatives are zero at two end points. Is there a value for d that makes $s(x)$ a natural cubic spline? Explain your answer.

3. Consider the integral $\int_0^{\infty} f(x)\,dx$ where f is continuous, $f'(0) \neq 0$, and $f(x)$ decays like $x^{-1-\alpha}$ with $\alpha > 0$ in the limit $x \to \infty$.

   (a) [5 points] Suppose you apply the equispaced composite trapezoidal rule with n subintervals to approximate $\int_0^{L} f(x)\,dx$. What is the asymptotic error formula for the error in the limit $n \to \infty$ with L fixed?

   (b) [10 points] Suppose you consider the quadrature from (a) to be an approximation to the full integral from 0 to ∞. How should L increase with n to optimize the asymptotic rate of total error decay? What is the rate of error decrease with this choice of L?

   (c) [10 points] Make the following change of variable $x = L(1+y)/(1-y)$, $y = (x-L)/(x+L)$ in the original integral to obtain $\int_{-1}^{1} F_L(y)\,dy$. Suppose you apply the equispaced composite trapezoidal rule; what is the asymptotic error formula for fixed L?

   (d) [5 points] Depending on α, which method - domain truncation or change-of-variable - is preferable?

4. [5 points each] Consider the 1D linear ODE

$$u'(t) = \lambda u \,,$$

where $\lambda < 0$ is a constant.

(a) Write down the forward Euler, backward Euler and trapezoidal schemes. What is the order of accuracy of each of these methods (in terms of the local truncation error)?

(b) What are the advantages of forward Euler and backward Euler schemes, respectively? Write down a few sentences to explain the reasons.

(c) Starting with the same initial value and using the same time step, is the trapezoidal scheme always more accurate than the backward Euler? Show your proof or give a counterexample.

(d) Write down a two-stage explicit and second order accurate Runge-Kutta scheme. Using the local truncation error, prove that it is second order accurate. State the meaning of each of the two stages and explain intuitively that this method is second order accurate.

5. (a) [5 points] Let A be an $N \times N$ symmetric positive definite matrix and $f \in \mathbb{R}^N$. Prove that

$$\min_{u \in \mathbb{R}^N} \frac{1}{2} u^T A u - f^T u$$

is equivalent to

$$Au = f \,.$$

(b) [10 points] Write down the steps of the gradient descent method with fixed time step for computing the solution to $Au = f$. Show when the method is guaranteed to converge.

(c) [5 points] Write down the power method.

(d) [5 points] Why must a method for determining the eigenvalue of a matrix A (size bigger than 5) generally be iterative instead of direct?

THE END

2020 QUALIFYING EXAM - REAL ANALYSIS

1.(20pt) Find a sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ on $[0,1]$ such that $\{\varphi_n\}$ is a dense subset of $L^p(\Omega)$ for any $p \in [1,\infty)$.

2.(20pt) Prove that for any $f \in L^1(\mathbb{R})$, its Fourier transform $\widehat{f}$ is continuous and $\lim_{|x|\to\infty} \widehat{f}(x) = 0$, that is, $\widehat{f} \in C_0(\mathbb{R})$.

3.(20pt) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^p([0,1])$ for $p \in (1,\infty)$. Suppose that there exists a $f \in L^p([0,1])$ satisfying $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx$ for any $g \in L^q([0.1])$ with $\frac{1}{p}+\frac{1}{q} = 1$. Prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

4.(20pt) Let $f \in L^1([0,1])$ and $F(x) = \int_0^x f(t)dt$. Show that F is absolutely continuous, and that F is differentiable a.e. in $[0,1]$.

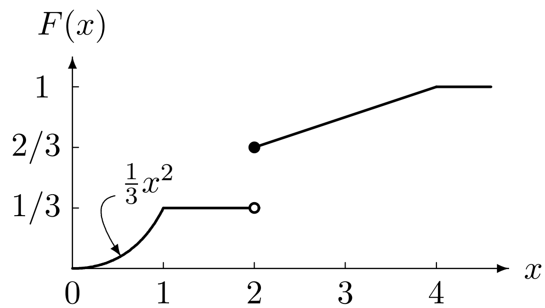
5.(20pt) Prove or disprove that if a function $f \in L^1([0,1])$ is differentiable a.e. in $[0,1]$ and $f' = 0$ a.e. in $[0,1]$, f is identically constant in $[0,1]$.

Qualifying Exam in Probability Theory, August 2020

Student ID: Name:

You have to explain your solutions as well as your answers. No points will be considered for answers without detailed explanations.

1. (20 pts) Let X be a random variable with the distribution function $F(x)$ shown in the following figure. Compute $\mathbb{E}[X^3]$.

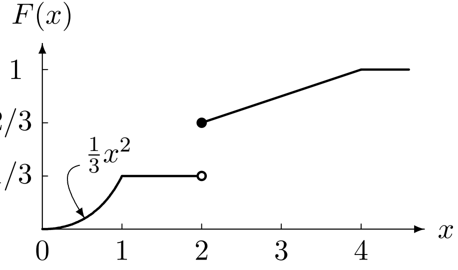


2. (20 pts) Let $X_1, X_2, \dots$ be iid with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$. Characterize the limiting distribution of
$$\lim_{n\to\infty} \frac{\sum_{k=1}^n X_k}{\sqrt{\sum_{k=1}^n X_k^2}}\,.$$

3. (20 pts) Compute the value of
$$\lim_{n\to\infty} 2^n \int_0^\infty \cdots \int_0^\infty \cos\left(\frac{x_1 + \cdots + x_n}{n}\right) e^{-2(x_1+\cdots+x_n)} dx_1 \cdots dx_n\,.$$

4. (20 pts) Suppose that $X_1, \dots, X_n$ are independent, but not necessarily identical, random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}[X_i^2] < \infty$. Set $S_n = X_1 + \ldots + X_n$. Show that
$$\mathbb{P}\Big(\max_{1\le k\le n} |S_k| \ge \lambda\Big) \le \lambda^{-2}\mathrm{Var}(S_n).$$

5. (20 pts) Let X_i be iid with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$ and $S_n = X_1 + \ldots + X_n$. Suppose that $S_0 = x$ for some integer x and $T = \min\{n : S_n \ne (a, b)\}$ for some integers a, b with $a < x < b$. Compute $\mathbb{E}[T]$.



$F(x)$
1
2/3
1/3
$\frac{1}{3}x^2$
0
1
2
3
4
x