PhD Qualifying Exam: Advanced Statistics
February, 2020

1. [10 pts] Let $X_1, \ldots, X_n$ be independently and identically distributed as Uniform(θ, θ+1) where $\theta > 0$ is an unknown parameter.

   (a) [5 pts] Is the maximum likelihood estimate (MLE) for θ unique?

   (b) [5 pts] If the answer in part (a) is YES, find the MLE. If not, find all possible values of θ for which the likelihood attains its maximum.

2. [15 pts] Let $X_1, \ldots, X_n$ be independently and identically distributed as N(μ, σ^2) where μ is unknown and $\sigma^2 > 0$ is knwon.

   (a) [10 pts] For fixed and known $t \neq 0$, find the best unbiased estimator for $e^{t\mu}$.

   (b) [5 pts] Show that the variance of the best unbiased estimator obtained in part (a) does not achieve the Cramer-Rao lower bound, but it is asymptotically efficient.

3. [15 pts] Let $X_1, \ldots, X_n$ be independently and identically distributed as Beta(1, β) whose pdf is given by

$$f(x|\beta) = \beta(1-x)^{\beta-1}, \;\; 0 < x < 1$$

   where $\beta > 0$ is an unknown parameter.

   (a) [5 pts] Show that $\sum_{i=1}^{n} log(1 - X_i)$ is a complete and sufficient statistic for β.

   (b) [5 pts] Derive a method of moments estimator for β as a function of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Is it unbiased?

   (c) [5 pts] Suppose, for some function g, you found that $S^* = g(\bar{X})$ is an unbiased estimator for β. Will S^* be the best unbiased estimator for β?

4. [20 pts] Let $X_1, \ldots, X_n$ be independently and identically distributed as Uniform(0, θ) where $\theta > 0$.

   (a) [5 pts] Develop what you consider to be the best $100(1-\alpha)$% confidence interval only with a lower limit for the unknown parameter θ. In other words, find an explicit expression for a random variable $L(\mathbf{X})$ such that $P(\theta > L(\mathbf{X})) = 1 - \alpha$.

   (b) [5 pts] Let $T = X_{(n)}$ be the largest observation. Show that for testing hypothesis $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ for $\theta_1 > \theta_0$, the likelihood ratio for T is a non-decreasing function.

   (c) [5 pts] Find the most powerful level $\alpha = 0.05$ test for $H_0 : \theta \leq 2$ vs $H_1 : \theta > 2$. You need to explicitly find the cut-off value for the rejection region.

(d) [5 pts] Compute and graph the power function for the test in part (c).

5. [40 pts] Let $X_1, \ldots, X_n$ be independently and identically distributed as Laplace(σ) whose pdf is given by

$$f_X(x;\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma}, \; -\infty < x < \infty, \; \sigma > 0$$

(a) [5 pts] Prove that the distribution of $Y_i = |X_i|$ has the pdf as

$$f_Y(y;\sigma) = \frac{1}{\sigma} e^{-y/\sigma}, \; y > 0$$

(b) [5 pts] Find the Cramer-Rao lower bound (CRLB) for the variance of any unbiased estimator of σ based on $X_1, \ldots, X_n$.

(c) [5 pts] Find an explicit expression for $\hat{\sigma}_{ML}$, the maximum likelihood estimator (MLE) of σ. Find $E(\hat{\sigma}_{ML})$ and $Var(\hat{\sigma}_{ML})$. Is $\hat{\sigma}_{ML}$ the minimum variance unbiased estimator (MVUE) of σ?

(d) [10 pts] Find an explicit expression for $\hat{\sigma}_B$, the Bayes estimator for σ. Assume that the prior distribution is $\tau = 1/\sigma \sim Gamma(a, b)$ and consider the posterior mean as the Bayes estimator. The pdf of $Gamma(a, b)$ is

$$f(\tau) = \frac{1}{\Gamma(a) b^a} \tau^{a-1} e^{-\tau/b}, \; \tau > 0, \; a > 0, \, b > 0$$

(e) [5 pts] Compare the the two estimators $\hat{\sigma}_{ML}$ and $\hat{\sigma}_B$ with respect to the mean squared error.

(f) [5 pts] Prove that a uniformly most powerful (UMP) test for $H_0 : \sigma = \sigma_0$ vs $H_1 : \sigma > \sigma_0$ has a rejection region of size α as $\mathcal{R} = \{S : S \geq c_\alpha\}$ where $S = \sum_{i=1}^n |X_i|$ and c_α is chosen such that $\mathrm{P}(S \geq c_\alpha | H_0 : \sigma = \sigma_0) = \alpha$.

(g) [5 pts] Assuming that n is large, obtain a large-sample normal approximation for c_α in part (f) as a function of σ_0, n, and Z_α where $P(Z > Z_\alpha) = \alpha$ when $Z \sim N(0, 1)$.

Ph.D. Qualifying Exam: Algebraic Topology (Februray 2020)

Justify your answers fully. You should state the theorems and the results that you are using exactly. One can obtain partial points for rough ideas but not the full scores. Also, one must write in good English and in a well-organized way for better grades. You must also write the answers in order and not mix up the answers here and there. (Total 100 pts.)

1. (20 pts.) Let $X := \mathbf{RP}^3$ be a projective 3-space.

   (1) Compute $H_n(X, \mathbf{Z}/p\mathbf{Z})$ for each number $p = 2, 3, 4, 5, 6$ and all integer n.

   (2) There is a map $f : X \to \mathbf{S}^3$ that is not homotopic to a constant map. Prove or disprove.

2. (25 pts.) Let Σ be a closed surface of genus 2 that is the boundary of the union of two distinct solid tori T_1 and T_2 symmmetric under the rotation of angle π about the x-axis and $T_1 \cap T_2$ is homeomorphic to a 3-ball. Let $\phi : \Sigma \to \Sigma$ be a homeomorphism given by a rotation of angle π by the x-axis. Describe $\phi_* : H_i(\Sigma, \mathbf{Z}) \to H_i(\Sigma, \mathbf{Z})$ for $i = 0, 1, 2$, i.e., with respect to some basis of $H_i(\Sigma, \mathbf{Z})$ for each i. (Hint 1: use ϕ-invariant cell decompositions.) (Hint 2: A solid torus is a space homeomorphic to $D^2 \times \mathbf{S}^1$.)

3. (25 pts.) Let $p : \mathbf{R}^2 \to \mathbf{S}^1 \times \mathbf{S}^1$ be given by $p(x, y) = (\exp(2\pi xi), \exp(2\pi yi))$. Let $G_1 := \pi_1(\mathbf{S}^1 \times \mathbf{S}^1, (1, 1))$ be the fundamental group based on loops. Let $G_2 := \mathrm{Aut}(\mathbf{R}^2, p)$ denote the group of homeomorphisms of $\mathbf{R}^2$ satisfying $p \circ f = p$. Define a homomorphism $f : G_1 \to G_2$ given by sending each homotopy class $[\gamma]$ based at $(1, 1)$ to a deck transformation τ_γ sending $(0, 0)$ to $\tilde{\gamma}(1)$ where $\tilde{\gamma}(0) = (0, 0)$ and $p \circ \tilde{\gamma} = \gamma$. (Hint: $\mathbf{S}^1$ is the unit circle in the complex plane.)

   (1) Prove that f is well-defined indepedent of choice of γ in $[\gamma]$.

   (2) Show that f is an isomorphism.

   (3) Describe G_1 and G_2, and write f explicitly.

4. (30 pts.) Let $\phi : \mathbf{S}^2 \to \mathbf{S}^2$ be the antipodal map $x \mapsto -x$. Define

$$\Phi : (x, y) \to (\phi(x), \phi(y)).$$

Let X be the quotient space of $\mathbf{S}^2 \times \mathbf{S}^2$ by indentifying $(x, y), \Phi(x, y)$ for all $x, y \in \mathbf{S}^2$. Compute $\pi_1(X, *)$ and $H_n(X, \mathbf{Z})$ for all n.

Ph.D. Qualifying Exam: Algebra I
Feb 2020

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let G be the quaternion group of order 8 and let $H = \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ denotes the automorphism group of G.
   (a) (15 pts) Show that H is isomorphic to a semidirect product of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by S_3, i.e., $H \simeq (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes S_3$, where S_3 denotes the symmetric group on 3 letters.
   (b) (10 pts) Determine the automorphism group $\mathrm{Aut}(H)$ of H.

2. Let M be a nonzero module over a commutative ring R with 1. Let n be an integer greater than 1.
   (a) (15 pts) Prove that if M is finitely generated, then the n-fold tensor product $M^{\otimes n}$ over R is nonzero.
   (b) (10 pts) Give an example of M with $M^{\otimes n} = 0$.

3. (10 pts) Let G be a nonabelian group of order 343. Find the number of conjugacy classes of G.

4. (10 pts) Let G be a nontrivial finite group. A maximal subgroup of G is a proper subgroup not contained in any other proper subgroup of G. Prove that if every maximal subgroup of G is normal, then every Sylow subgroup of G is normal.

5. (10 pts) Let F be a field. Show that every nonzero ideal of the formal power series ring $F[[x]]$ is generated by a single element.

6. Let $R = \mathbb{Q}[x, y]$ be the polynomial ring in two variables x and y over the rational numbers $\mathbb{Q}$.
   (a) (10 pts) Determine the intersection of all maximal ideals of R.
   (b) (10 pts) Find a non-noetherian subring of R (we require that a subring contains the unity).

THE END

1. (20pts) Show that the only field automorphism of the field $\mathbb{R}$ of real numbers is identity.

2. (20 pts) Let K be a finite extension field of a field k and $Aut_k(K)$ be the set of all the automorphisms of K fixing k. Show that K is a Galois extension of k if and only if k is the fixed field of $Aut_k(K)$.

3. Let R be a commutative ring with identity. We say that two ideals I and J of R are said to be *co-maximal* if $I + J = R$. Let $I_1, I_2, \ldots, I_m$ be pairwise co-maximal ideals of R.

i) (10 pts) Show that

$$I_1 I_2 \cdots I_m = I_1 \cap I_2 \cap \cdots \cap I_m.$$

ii) (10 pts) Show that the map

$$\phi : R \longrightarrow R/I_1 \times R/I_2 \times \cdots \times R/I_m$$

defined by $\phi(a) = (a + I_1, a + I_2, \ldots, a + I_m)$ is a surjective homomorphism with kernel $I_1 I_2 \cdots I_m$.

4. (20 pts) Let R be a commutative ring with identity. Show that if $a_0 + a_1 X + \cdots + a_n X^n$ in $R[X]$ is invertible, then a_0 is a unit and a_i with $i > 0$ is nilpotent,i.e., $a_i^m = 0$ for some positive integer m.

5. (20 pts) Show that the ring

$$\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right] := \left\{ m + n\frac{1+\sqrt{-7}}{2} : m, n \in \mathbb{Z} \right\}$$

is a Euclidean domain.

Ph.D. Qualifying Exam: Differential Geometry
February 2020

Student ID: Name:

Note: Be sure use English for your answers.

Here $\mathbb{R}^k, \mathbb{S}^k$ for $k \geq 1$ are the Euclidean space and the unit sphere with their standard smooth structures and the standard orientations, respectively.

1. [20 pts] For a positive integer n, the n-torus is the product space
$$\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n \text{ times}}.$$
   (a) Show that $\mathbb{T}^n$ is a topological manifold of dimension n.
   (b) Construct explicitly a smooth structure on $\mathbb{T}^n$.

2. [25 pts] Let $\mathrm{Mat}(n, \mathbb{R})$ with $n \geq 1$ be the set of $n \times n$ matrices with real entries. The set of invertible matrices is denoted by $\mathrm{GL}(n, \mathbb{R})$.
   (a) Show that $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold, and the tangent space of $\mathrm{GL}(n, \mathbb{R})$ at the identity matrix I_n can be identified with $\mathrm{Mat}(n, \mathbb{R})$.
   (b) Let $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ denote the determinant function. Show that for any $A \in \mathrm{Mat}(n, \mathbb{R})$
$$\left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA) = \mathrm{tr}A,$$
   where $\mathrm{tr}(A^i_j) = \sum_i A^i_i$ is the trace of A.
   (c) Given $X \in \mathrm{GL}(n, \mathbb{R})$ and $B \in T_X\mathrm{GL}(n, \mathbb{R})$ the tangent space at X, compute the differential of the determinant map at X applied to B, i.e., $d(\det)_X(B)$.

3. [20 pts] Let $\mathrm{GL}(\mathrm{n}, \mathbb{R})$ and $\mathrm{Mat}(n, \mathbb{R})$ be as in Problem 2. Define a map $\Phi : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Mat}(n, \mathbb{R})$ by $\Phi(A) = A^T A$, where A^T represents the transpose of A.
   (a) Show that Φ is a smooth map of constant rank.
   (b) Let $SO(n) = \{A \in \mathrm{Mat}(n, \mathbb{R}) : \det A = 1\}$. Show that $SO(n)$ is a smooth embedded submanifold of dimension $n(n-1)/2$ in $\mathrm{GL}(n, \mathbb{R})$.

4. [20 pts] Define vector fields X and Y on the plane $\mathbb{R}^2$ by
$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$
   (a) Compute $[X, Y]$, and compute the flows θ_s of X and ψ_t of Y.
   (b) Find open intervals J and K containing 0 such that both $\theta_s \circ \psi_t$ and $\psi_t \circ \theta_s$ are defined for all $(s, t) \in J \times K$, but they are unequal for some (s, t).

5. [15 pts] Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the sets of points (x, y, z, w) such that $x^2 + y^2 = z^2 + w^2 = 1$, with the product orientation determined by the standard orientation on $\mathbb{S}^1$. Compute
$$\int_{\mathbb{T}^2} yzw \, dx \wedge dz.$$

THE END

Ph.D Qualifying Examination, February, 2020
Complex Analysis

Each Problem is worth **10** points.

Problem 1
(i) Find all values of i^{-i}.
(ii) Solve $\log z = -1$, $0 \neq z \in \mathbb{C}$.

Problem 2
Let $f(z) = u(x,y) + iv(x,y)$ be an entire function where $z = x+iy$ and u, v are real-valued functions. Suppose that $u_y - v_x = 1$ for all $z \in \mathbb{C}$. Find all such $f(z)$ if there exists such a function.

Problem 3
Evaluate

$$\int_{|z|=1} \frac{e^z}{z(z+2)}\, dz.$$

Problem 4
Assume that $f(z)$ is analytic for some $r > 0$ and f satisfies the equation $f(2z) = f(z)^2$ for all z sufficiently close to 0. Show that $f(z)$ can be extended to an entire function. Find all such entire function $f(z)$ explicitly.

Problem 5
Find all the entire functions $f(z)$ such that

$$\lim_{|z|\to\infty} \frac{f(z)}{z^2} = 0.$$

Problem 6
Consider the function $f(z) = e^{-z}$ on the unit disk $|z| \leq 1$. (a) What is the maximum of $|f(z)|$ on the unit disk? At which points does the function have the maximum? (b) What is the minimum of $|f(z)|$ on the unit disk? At which points does the function have the minimum?

Problem 7

Using the contour integral in complex analysis, evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+1}.$$

Problem 8

Let Γ be the circle $|z| = \frac{1}{2}$. Evaluate the integral

$$\int_{\Gamma} \sqrt{z^2+1}\,\mathrm{d}z.$$

Problem 9

Suppose that for some constant $r > 0$ a bilinear transformation $w = g(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$, maps $\{z \in \mathbb{C} : r < |z| < 1\}$ onto the domain bounded by $|z - \frac{1}{4}| = \frac{1}{4}$ and $|z| = 1$. Find r.

Problem 10

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be an analytic function in an open domain containing $D = \{z \in \mathbb{C} : |z| \leq 1\}$. Suppose that the mapping $f : D \to \mathbb{C}$ is one-to-one. Compute the area of $f(D)$.

Qualifying Exam. Numerical analysis, 2020 Feb.

For problems with some ambiguous expression such as 'discuss', 'explain', 'state', give as much detail as possible, but not exceeding half page.

1. (10pts) Explain how a typical 32 bit computer stores numbers in memory. Give a separate discussion for real and integers. Include the concept of precision, the range of numbers that can be represented, etc.

2. (10pts) State three different methods to find the root of $f(x) = 0$, where $f \in C^1(I)$, $I = [A, B]$. Discuss the advantages(disadvantages) of each method. Also explain how to find the roots of a system of equations $\mathbf{f}(\mathbf{x}) = 0$ in $\mathbb{R}^n$. Here $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), \cdots, f_n(\mathbf{x}))^T$.

3. (15pts) Explain the Runge phenomena for approximate functions on some interval $[a, b]$ and suggest a method to avoid it.

4. (10pts) Given data $(x_i, f(x_i))_{i=0}^n$, $a = x_0 < \cdots < x_n = b$, $f \in C^2[a, b]$, $h = \max_i |x_i - x_{i-1}|$, answer the following ; Define the natural spline approximation $s(x)$ of f (You do not need to show how to construct it).

5. (10pts) State Gaussian quadrature using n points to approximate $\int_a^b f(x)dx$ and prove it is exact for polynomials of degree $2n - 1$.

6. (15pts) Given a matrix $A = \{a_{ij}\}_{i,j=1}^n$, do the followings:

   (a) Define the matrix norm of A subordinate to $\|\cdot\|$ for any norm $\|\cdot\|$ on $\mathbb{R}^n$.

   (b) Derive a formula for $\|A\|_1$.

   (c) Derive a formula for $\|A\|_\infty$.

   Here $\|\mathbf{u}\|_1 = \sum_i |u_i|$ and $\|\mathbf{u}\|_\infty = \max_i |u_i|$.

7. (10 pts) Explain the Givens rotation and Givens algorithm to tridiagonalize a symmetric matrix. What is main usage or purpose of Givens algorithm?

8. (20pts) State QR algorithm to find all the eigenvalues of A. Discuss detailed algorithm including the QR decompostion, computational complexity(i.e, the number of operations), effective implementation, etc.

PH.D QUALIFYING EXAM: February 2020
REAL ANALYSIS
3 HOURS

Problem 1. (20 pt) Let $1 \le p, q \le \infty$. Consider measure spaces $(\mathbb{R}^d, dm)$, $(\mathbb{Z}^d, dc)$, and (B, dm) for a bounded ball $B = B(0,1) \subset \mathbb{R}^d$, where dm is the Lebesgue measure and dc is the counting measure.

(1) (10 pt) Find the necessary and suffient condition on exponents for which the inequality holds in cases $X = \mathbb{R}^d, B$, or $\mathbb{Z}^d$.

$$\int_X fg \le C\|f\|_{L^p(X)}\|g\|_{L^q(X)},$$

for $f \in L^p, g \in L^q$.

(2) (10 pt) Find a necessary condition on exponents for which the inequality holds in cases $X = \mathbb{R}^d$ or $X = B$,

$$\|f\|_{L^p(X)} \le C\|\nabla f\|_{L^2(X)},$$

for $f \in C_c^\infty(X)$.

Problem 2. (20 pt) Let $f \in L^2(\mathbb{R}^d)$. Explain how to define the Fourier transform of f. Prove that $\lim_{R\to\infty} S_R f = f$ in L^2-norm where $S_R f(x) = \int_{|\xi|<R} \widehat{f}(\xi) e^{i2\pi\xi\cdot x}\, d\xi$.

Problem 3. (20 pt) Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz class on $\mathbb{R}^d$.

(1) (10pt) Show that if $f, g \in \mathcal{S}(\mathbb{R}^d)$, then $f * g \in \mathcal{S}(\mathbb{R}^d)$.
(2) (10pt) Let $\phi \in \mathcal{D}'(\mathbb{R}^d)$ and $f \in C_c^\infty(\mathbb{R}^d)$. Explain how to define $\phi * f$. Moreover, show that $\phi * f \in C^\infty$.

Problem 4. (20 pt) Let μ and ν be finite unsigned measures on X. Suppose $\nu \ll \mu$.

(1) (5 pt) Use Riesz representation theorem for $L^2(X, d(\mu+\nu))$ to find $g \in L^2(X, d(\mu+\nu))$ such that

$$\int_X f(1-g)d\mu = \int_X fg d\nu,$$

for μ-integrable function f.
(2) (7 pt) Show that $0 < g \le 1$ $\mu - a.e.$
(3) (8 pt) Conclude the Radon-Nikodym theorem.

Problem 5. (20 pt) Prove the rearrangement theorem. i.e.

Assume $a_n \ge 0$ for all $n \in \mathbb{Z}_+$. Let $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ be a 1-1 correspondence. Show that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}.$$

Show the rearrangement theorem fails, if one drops the condition $a_n \ge 0$.

Date: Feb 2020.

Ph.D. Qualifying Exam: Probability
Spring 2020

Student ID:

Note: use English only for your answers.

1. [10 pts] Suppose that $\{A_n\}$ is a sequence of events. Show that

$$P(\limsup A_n) = P(A_n \text{infinitely often}) \geq \limsup P(A_n).$$

2. [20 pts] (a) Let $T_k = X_1 + \cdots + X_k$, be the "kth arrival" of a Poisson process with rate λ. Show that the number of arrivals before time t has a Poisson distribution with parameter λt.

   (b) Condition on the event that there are exactly k arrivals of the Poisson process with rate λ before time one. In other words,

$$0 < T_1 < \cdots < T_k < 1$$

   while $T_{k+1} > 1$. Under this condition, show that $(T_1, \cdots, T_k)$ is uniformly distributed on the simplex in $\mathbb{R}^k$,

$$\{(x_1, \ldots, x_k) : 0 < x_1 < \cdots < x_k < 1\}.$$

3. [16 pts] $\{X_n\}$ is said to *converge completely* to X if $\sum_n P(|X_n - X| > \epsilon) < \infty$ for every $\epsilon > 0$. Prove or disprove:

   (a) If $\{X_n\}$ converges completely to X then it converges a.s. to X.

   (b) If $\{X_n\}$ converges a.s. to X then it converges completely to X.

4. [20 pts] The *Stieltjes transform* is defined, for any probability measure μ on $\mathbb{R}$, as

$$S_\mu(z) := \int \frac{1}{z - x}\, \mu(dx), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

   Suppose μ has no atoms, i.e., $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Show the following inversion formula for the Stieltjes transform. For any open interval $I \subset \mathbb{R}$,

$$\mu(I) = \lim_{y \to 0} \int \frac{S_\mu(x + iy) - S_\mu(x - iy)}{2\pi i}\, dx.$$

   Hint: If X has distribution μ consider $X + C_\sigma$ where C_σ is an independent Cauchy(σ) random variable with density $\frac{\sigma dx}{\pi(x^2+\sigma^2)}$ (consider *small* values of σ).

5. [16 pts] If X is equal in distribution to $E(X|\mathcal{F})$ then, in fact, they are equal a.s.

   (for partial credit, you may assume $E(X^2) < \infty$)

6. [18 pts] If $\{X_n\}$ is a sequence of mean-zero independent random variables with $E(X_n^2) = \sigma_n^2 < \infty$, show that

$$M_n := \left(\sum_{k=1}^{n} X_k \right)^2 - \sum_{k=1}^{n} \sigma_k^2$$

is a martingale with respect to the natural filtration (what is the filtration?).