# Ph.D. Qualifying Exam: Algebra I
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. Let $\mathbb{F}_q$ be the finite field of order $q$. Let $G = GL_n(\mathbb{F}_q)$, which is the group of $n \times n$ invertible matrices with the entries in $\mathbb{F}_q$. Compute the order of the group $G$ in two different ways as instructed below.

   (a) **(10 pts)** The first law can be filled up by choosing any members of $\mathbb{F}_q$, except that the row can't be zero. Suppose the first $(k-1)$ rows were chosen. In constructing the $k$-th row, how many candidates are linearly independent from the first $(k-1)$-rows? Using this argument, find $|G|$.

   (b) **(10 pts)** Let $G$ act on $V = \mathbb{F}_q^n$ by left multiplication. Choose any nonzero vector $v_0 \in V$. Using the orbit of $v_0$, find $|G|$.

2. **(10 pts)** Using a group action, prove that $\mathrm{GL}_2(\mathbb{F}_2) \simeq S_3$.

3. Let $F$ be a field. Consider the formal power series ring $F[[t]]$, and its field of fractions $F((t))$. Answer the following questions.

   (a) **(10 pts)** Prove that every element of $F((t))$ has an expansion of the form $\sum_{n \geq N}^{\infty} a_n t^n$ for some $a_n \in F$ and $N \in \mathbb{N}$. Here, the important point is that it has only finitely many terms of negative powers of $t$.

   (b) **(5 pts)** Define $\nu : F((t))^\times \to \mathbb{Z}$ by sending $\sum_{n \geq N} a_n t^n$ to $N$, where $a_N$ is the first nonzero coefficient of the series. Define $\nu(0) := +\infty$ so that $\nu$ extends to all of $F((t))$. Define $|\cdot| : F((t)) \to \mathbb{R}$ by $|f| := e^{-\nu(f)}$, for the transcendental number $e$. Prove that
   $$|fg| = |f| \cdot |g|, \quad |f+g| \leq \max\{|f|, |g|\}$$
   for all $f, g \in F((t))$, and that
   $$d(f,g) := |f - g|$$
   turns $F((t))$ into a metric space.

   (c) **(5 pts)** Prove that $F[[t]] \subset F((t))$ is an open subset, and $F[t] \subset F[[t]]$ is a dense subset.

4. Let $R$ be a principal ideal domain.

   (a) **(5 pts)** Prove that $R$ is a noetherian ring.

   (b) **(5 pts)** Prove that $R$ is a unique factorization domain.

5. **(10 pts)** Let $R$ be a commutative ring with $1 \neq 0$. Let $I \subset R$ be a proper ideal. Using Zorn's lemma, prove that there exists a maximal ideal $M \subset R$ that contains $I$.

6. **(10 pts)** Give an example of a noetherian ring $R$ and an $R$-module $M$ that is not flat. Justify your answer.

7. Let $G$ be a finite group of order $30 = 2 \cdot 3 \cdot 5$.

   (a) **(10 pts)** Prove that there are normal subgroups of order 3 and order 5.

   (b) **(10 pts)** Prove that $G$ has a normal abelian subgroup of index 2.

**THE END**

# Ph.D. Qualifying Exam: Algebra II
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(20 pts)** Prove or disprove the following:
   (a) The rings $\mathbb{Q}[x,y]/(y^2-x^3)$ and $\mathbb{Q}[x,y]/(xy-1)$ are isomorphic.
   (b) $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.
   (c) $\mathbb{Z}/5\mathbb{Z} \otimes \mathbb{Z}/7\mathbb{Z} = 0$
   (d) Let $L$ be a Galois extension of $K$ and $K$ be a Galois extension of $F$. Then, $L$ is a Galois extension of $F$.

2. **(20 pts)**
   (a) Let $(R, m)$ be a commutative local ring and $M$ a finitely generated $R$-module such that $M = mM$. Show that $M = 0$.
   (b) Let $p$ be a prime integer such that $p \equiv 2 \text{ or } 3 \pmod 5$. Prove that the polynomial $x^4 + x^3 + x^2 + x + 1$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$.

3. **(20 pts)** Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers with its quotient field $K = \mathbb{Q}(i)$.
   (a) Show that $\mathbb{Z}[i]$ is integrally closed in its quotient field $\mathbb{Q}(i)$.
   (b) Suppose $\alpha$ is a complex number which is the root of a monic polynomial in $R[x]$. Prove that the minimal monic polynomial of $\alpha$ over $K = \mathbb{Q}(i)$ has all coefficients in $R = \mathbb{Z}[i]$.

4. **(20 pts)**
   (a) Construct a field $F$ of order 27.
   (b) Describe the isomorphism types of $F$ and the multiplicative group $F^\times$ of nonzero elements of $F$ as abelian groups.

5. **(20 pts)**
   (a) Let $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ of degree 5 having exactly two complex and three real zeros in $\mathbb{C}$. Compute the Galois group of $f(x)$ as a subgroup of $S_5$.
   (b) Show that the polynomial $f(x) = 2x^5 - 5x^4 + 5$ is not solvable by radicals over $\mathbb{Q}$.

**THE END**

# Ph.D. Qualifying Exam: Advanced Statistics
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(5 pts)** Three events, A, B, and C satisfy that $P(A \cap B \cap C) = P(A)P(B)P(C)$. Is it true that $P(A \cap B) = P(A)P(B)$? Why?

2. **(10 pts)** Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with pdf $f(x;\theta)$. Let $\hat{\theta}$ be the MLE of $\theta$. Show that, for a function $g$, the MLE of $\eta = g(\theta)$ is given by $g(\hat{\theta})$.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from Poisson distribution with parameter $\lambda$. The Gamma-Poisson equality is given by

$$P(\text{Gamma}(\alpha, \beta) \leq x) = P(\text{Poisson}(x/\beta) \geq \alpha).$$

The pdf of Gamma$(\alpha, \beta)$ is given by

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-x/\beta}.$$

   (a) **(6 pts)** Find a 95% confidence interval of $\lambda$.

   (b) **(9 pts)** For this question, assume a Gamma$(\theta_1, \theta_2)$ prior for $\lambda$. Find the joint probability of the data, $x_1, \cdots, x_n$, and check if $X_n$ is independent of $X_1, \cdots, X_{n-1}$.

   (c) **(10 pts)** Suppose that the sample is from two Poisson distributions with parameters $\lambda_1$ and $\lambda_2$, $\lambda_1 < \lambda_2$, whose values are known. $100\tau$ % of the sample is from Poisson$(\lambda_1)$. Find the MLE of $\tau$.

4. Let $X_1, X_2, \cdots, X_k$ be a random sample of size $n$ from a multinomial distribution with $k$ cells with the cell probabilities, $p_1, \cdots, p_k$, $\sum_{i=1}^k p_i = 1$.

   (a) **(5 pts)** Find the marginal distribution of $X_1, \cdots, X_j$, $j < k$.

   (b) **(10 pts)** Let $X_j = \sum_{l=1}^n X_j^l$ where $X_j^l = 1$ if the $l$-th observation falls in cell $j$ and 0 otherwise. Let $\bar{X}_j = \sum_{l=1}^n X_j^l/n$. Define

$$W_n = \frac{\sum_{l=1}^n (X_1^l - \bar{X}_1)(X_2^l - \bar{X}_2)}{\sqrt{\sum_{l=1}^n (X_1^l - \bar{X}_1)}\sqrt{\sum_{m=1}^n (X_2^m - \bar{X}_2)}}.$$

   Check if $W_n$ converges in probability as $n$ increases.

   (c) **(10 pts)** Find the likelihood ratio test statistic for testing $H_0 : p_1 = \cdots = p_k$ vs. $H_1$ : not $H_0$ and construct a size 0.05 test for the hypotheses.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, \sigma^2)$. Let $\bar{X}$ and $S^2$ be the sample mean and sample variance.

   (a) **(5 pts)** Check if $\bar{X}$ and $S^2$ are independent.

   (b) **(10 pts)** Suppose that $\mu = \mu_0$ is known and an inverse Gamma distribution $IG(\alpha, \beta)$ is assumed as a prior for $\sigma^2$ with $\alpha$, $\beta$ known. If $W$ is a Gamma$(\alpha, \beta)$ random variable, then $1/W$ follows $IG(\alpha, \beta)$.
   Construct a 95% credible interval of $\sigma^2$ using quantiles of a Chi-square distribution.

# Ph.D. Qualifying Exam: Algebraic Topology I
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.
*If you are using certain theorems or facts while you are solving the following problems, you should specify exactly which theorems or facts you are using by either mentioning the name of the theorems or by stating the theorems or the facts.*

1. Prove the following statements.
   (a) **(5 pts)** Let $S_g$ be a closed orientable connected surface of genus $g$. (Equivalently, the connected sum of $g$ copies of 2-tori, where the 2-torus is $S^1 \times S^1$.)
   Show that $S_g$ can be obtained from a $4g$-gon by gluing sides in an appropriate way.
   (b) **(10 pts)** Let $S_{g,1}$ be a surface obtained from $S_g$ by removing one point. Show that $\pi_1(S_{g,1})$ is isomorphic to the free group of rank $2g$.
   (c) **(10 pts)** Show that there exists a non-trivial group homomorphism $f : \mathrm{Mod}(S_{g,1}) \to \mathrm{Out}(F_{2g})$ where $\mathrm{Mod}(S)$ is the group of homotopy classes of orientation-preserving homeomorphisms from $S$ to itself and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G)$ for a group $G$.
2. **(15 pts)** Let $K$ be the image of a topological embedding of $S^1$ into $S^3$. Show that $H_1(S^3 \setminus K; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.
3. **(10 pts)** Show that every finite covering of the 2-torus is homeomorphic to the 2-torus.
   (Here, the 2-torus is $S^1 \times S^1$)
4. **(20 pts)** Using the covering space theory, show that any subgroup of a free group is free.
5. **(15 pts)** Show that a closed connected non-orientable 3-manifold $M$ has infinite fundamental group.
   (Hint. What is the Euler characteristic $\chi(M)$ of $M$?)
6. Solve the following.
   (a) **(5 pts)** For a homeomorphism $f : X \to X$ of a topological space $X$, write the definition of the Lefschetz number $L(f)$ of $f$.
   (b) **(10 pts)**
      i. Show that there exists $f : S_g \to S_g$ such that $L(f) = 0$.
      ii. Show that there exists $f : S_g \to S_g$ such that $L(f) > 0$.
      iii. Show that there exists $f : S_g \to S_g$ such that $L(f) = 2 - 2g$.
   (Verify that each case is actually realized).

**THE END**

# Ph.D. Qualifying Exam: Complex Analysis
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(10 pts)** Let $f$ be holomorphic in $\mathbb{C} \setminus \{0\}$ and $\lim_{|z|\to\infty} f(z) = 0$. Show that
$$\frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)}{\xi - z}\, d\xi = -f(z) \quad \text{for } |z| > 1.$$

2. **(15 pts)** Compute the integral
$$\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx \quad \text{for } a > 0,$$
where the integral is defined in the following sense:
$$\lim_{R\to+\infty} \lim_{\delta\to 0^+} \left( \int_{-R}^{-a-\delta} + \int_{-a+\delta}^{a-\delta} + \int_{a+\delta}^{R} \right).$$

3. **(15 pts)** Suppose that $f$ is holomorphic in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$ and that there exists a sequence of polynomials converging to $f$ uniformly on every compact subset of this annulus. Show that $f$ has an extension which is holomorphic in the disk $\{z \in \mathbb{C} : |z| < 2\}$.

4. **(15 pts)** Let $f$ be a non-constant holomorphic function in the disk $\{z \in \mathbb{C} : |z| < 2\}$ such that $|f(z)| = 1$ for $|z| = 1$. Prove that the image of $f$ contains the open unit disk centered at the origin.

5. **(15 pts)** Let $\Omega$ be an open set which contains $\{z \in \mathbb{C} : |z| \le R\}$ for some $R > 0$. Assume that $g$ is a nowhere vanishing holomorphic function in $\Omega$. Show that
$$\log |g(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\theta})|\, d\theta.$$
(This is a special case of Jensen's formula.)

6. **(15 pts)** Show that if $f$ is an entire function of finite order whose image omits two points, then $f$ is constant.

   (This is a special case of Picard's Little Theorem.)

7. **(15 pts)** Suppose that $f$ is a holomorphic function in the unit disk centered at the origin such that
$$|f(z)| < 1 \quad \text{for all } |z| < 1.$$
If $f(0) = \frac{1}{2}$, how large can $|f'(0)|$ possibly be?

**THE END**

# Ph.D. Qualifying Exam: Numerical Analysis
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(20 pts)**
   (a) Define a Lagrange interpolation polynomial with data $\{(x_i, f(x_i))\}_{i=0}^n$, $x_i$ all distinct.
   (b) What is the error form in the above ? Derive it.
   (c) Define Newton's form of an interpolation polynomial using the same data.
   (d) Explain what happens if some $x_i$ are repeated, and in this case what is the correct data corresponding to the repeated points?

2. **(20 pts)** We would like to solve a system of nonlinear equations $\mathbf{F}(\mathbf{x}) := A(\mathbf{x})\mathbf{x} + \mathbf{b} = 0$ using Newton's method. Here $\mathbf{b} = (b_1, \cdots, b_n)$ is a constant vector, $\mathbf{x} = (x_1, \cdots, x_n)$, $A(\mathbf{x})$ is $n \times n$, nonsingular matrix of $C^2$-variable entries. Answer the following.
   (a) Describe the Newton's method for solving $A(\mathbf{x})\mathbf{x} + \mathbf{b} = 0$ starting from some initial points $\mathbf{x}_0$.
   (b) In (a) compute the entries $DF(\mathbf{x})$.
   (c) State and prove a theorem concerning the convergence of the Newton's method when $n = 1$. (State conditions precisely)
   (d) State a similar theorem for $n > 1$ and sketch the proof.

3. **(20 pts)** Let $A$ be a symmetric positive definite matrix. Prove the following.
   (a) $\max_i a_{ii} = \max_{i,j} |a_{ij}|$.
   (b) The submatrices $(a_{ij}^{(k)})$, $1 \le k \le n$ appearing during the Gaussian eliminations are also symmetric positive definite.
   (c) $a_{ii}^{(k)} \le a_{ii}^{(k-1)}$, for $k \le i \le n$.
   (d) If $A$ is diagonally dominant, then so are the submatrices appearing in the Gaussian elimination.

4. **(20 pts)** Define the Gauss-Seidel iteration to solve the nonsingular linear $n \times n$ system $A\mathbf{x} = \mathbf{b}$. Prove the Gauss-Seidel iteration converges if $A$ is weakly diagonal dominant, i.e.,

$$|a_{ii}| \ge \sum_{j \ne i} |a_{ij}|, \quad i = 1, \ldots, n$$

   and the equality does not hold for at least one $i$.

5. **(20 pts)** State the Givens Householder algorithm to reduce an $n \times n$ real matrix $A$ to an upper Hessenberg form. Include a strategy of choosing some vector to avoid instability.

**THE END**

# Ph.D. Qualifying Exam: Real Analysis
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(15 pts)** Suppose $f \in L^p(\mathbb{R}^d)$. Set a rescaled function $f_\lambda(x) = f(x/\lambda)$ for $\lambda > 0$. Show that
$$\lim_{\lambda \to 1} \|f_\lambda - f\|_{L^p(\mathbb{R}^d, dm)} = 0.$$

2. **(15 pts)** Let $H_1, H_2$ be Hilbert spaces over $\mathbb{R}$. Suppose $T : H_1 \to H_2$ is a bounded linear transform. Show that there exists a unique adjoint (bounded linear) transform $T^* : H_2 \to H_1$ satisfying
$$\langle y, Tx \rangle_2 = \langle T^* y, x \rangle_1$$
for any $y \in H_2$ and $x \in H_1$. (Here, $\langle \cdot, \cdot \rangle_i$ is the inner product of $H_i$ for each $i = 1, 2$.)

3. **(20 pts)** Consider the Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2i\pi x\xi}\, dx$. Show that if $f \in L^1(\mathbb{R})$, then $\widehat{f} \in c_0(\mathbb{R})$. (i.e. $\widehat{f}$ is continuous and $\lim_{|x| \to \infty} f(x) = 0$.) Find an example such that $f \in L^1(\mathbb{R})$ but $\widehat{f} \notin L^2(\mathbb{R}) \cup L^1(\mathbb{R})$.

4. **(15 pts)** Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$. Explain how to define the convolution $f * g$ as an $L^p$ function.

   (Hint: Prove Young's inequality and use it.)

5. **(15 pts)** Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function. Show that $f$ is differentiable a.e. Moreover, show that if we further assume $f'(x) = 0$ a.e., then $f$ is constant.

6. **(20 pts)** Let $H$ be a Hilbert space and $\{u_i\}_{i=1}^\infty \subset H$ be an orthonormal set. Consider the ball $B = \{x \in \operatorname{span}\{u_i\}_{i=1}^\infty : \|x\| \le 1\}$. Show that for any $y \in H$, there exists a unique $z \in \overline{B}$ with $\|y - z\| = \operatorname{dist}(B, y)$. Find an expression of $z$ in terms of $u_i$'s. Is $z \in B$?

   (Here, span $A$ means a vector subspace spanned by $A$ in $H$.)

**THE END**

# Ph.D. Qualifying Exam: Probability
## August 2018

Department of Mathematical Sciences, KAIST

Student ID: Name:

Note: Be sure to use English for your answers.

1. **(18 pts)** Suppose $T : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$ is measurable. Suppose $X$ is a random variable on $(\Omega_1, \mathcal{F}_1)$. Show that $X$ is measurable with respect to the $\sigma$-field generated by $T$ if and only if there is a random variable $Y$ on $(\Omega_2, \mathcal{F}_2)$ such that $X = Y \circ T$.

2. **(18 pts)** Suppose $\{X_n\}$ are iid random variables. Show that
$$\mathbf{P}(\sup_{n\in\mathbb{N}} X_n < \infty) = 1$$
if and only if
$$\sum_{n=1}^{\infty} \mathbf{P}(X_n > M) < \infty, \quad \text{for some } M < \infty.$$

3. **(10 pts)** An urn contains a very large number (think infinite) of same-sized candies of 3 different flavors: Apple, Banana, Chocolate. Suppose a fraction $a$ are apple flavored, a fraction $b$ are banana flavored, and a fraction $c$ are chocolate flavored. Find the expected number of candies you need to randomly pick before you have at least one of each flavor.

4. **(18 pts)** Prove directly the well-known fact that if $X_n \to X$ in probability, then for all bounded and continuous $f : \mathbb{R} \to \mathbb{R}$, we have $\mathbf{E}f(X_n) \to \mathbf{E}f(X)$ as $n \to \infty$. Show, conversely, that if for all bounded and continuous $f : \mathbb{R} \to \mathbb{R}$, $\mathbf{E}f(X_n) \to c$, where $c$ is a constant, then $X_n \to c$ in probability.

5. **(18 pts)** Show that if $X_1 \in L^1$ and $\{X_i\}$ are iid, then the family $\{\frac{S_n}{n}, n \in \mathbb{N}\}$ is uniformly integrable.

6. **(18 pts)** Suppose an urn starts with $r$ red balls and $b$ blue balls. We increase the number of balls as follows: At step $n$ a ball is selected at random from the urn, then replaced by $C_n$ balls of the same color, where $C_n$ is a positive random integer that may depend on the outcomes of the first $n-1$ balls drawn. After completion of the nth step, let $R_n$ denote the number of red balls and $B_n$ the number of blue balls in the urn. Show that $\left(\frac{R_n}{R_n+B_n}\right)_{n\in\mathbb{N}}$ is a martingale with respect to a suitable filtration.

**THE END**

6. Let $X = (X_1, \cdots, X_d)'$ be a Normal random vector of dimension $d$ with distribution $N(\mu, \Sigma)$, where $\mu$ is a $d$-vector and $\Sigma$ a $d \times d$ matrix. The joint pdf of $X$ is given by

$$f(x; \mu, \Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right).$$

   (a) **(7 pts)** Find the MGF(moment generating function) of $X$.

   (b) **(7 pts)** Let $Y_i = X_{i+1} - X_i$. Find the joint distribution of $Y_1, \ldots, Y_{d-1}$.

   (c) **(6 pts)** Find a linear transformation $g$ of $X$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ so that each component of $g(X)$ is independent of the others.

**THE END**