2015 Spring KAIST Qualifying Exam for Ph. D program
Algebraic Topology I

1. (20 points: 10 points each)

   (a) Show that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n \geq 2$ using covering map theory.

   (b) Show that any continuous map $f : \mathbb{R}P^n \to S^1$ is null homotopic for $n \geq 2$.

2. (20 points) Let $T^2 = S^1 \times S^1$ be the torus, and let $X = S^1 \times \{1\} \cup \{1\} \times S^1 \subset T^2$. Show that there is no retract of T^2 to X.

3. (20 points: (a) 5points, (b) 5 points, (c) 10 points)) Let $X = D^2 \cup_f S^1$ where $f : \partial D^2 = S^1 \to S^1$, $z \mapsto z^6$. Namely, X is the result of attaching a 2-cell D^2 to S^1 by the map $f : S^1 \to S^1$ defined by $f(z) = z^6$. Let $\tilde{X}$ be the universal covering of X, i.e., $\tilde{X}$ is the simply connected covering of X.

   (a) Compute $\pi_1(X)$.

   (b) Find the Euler characteristics $\chi(X)$ and $\chi(\tilde{X})$.

   (c) Compute $H_*(\tilde{X})$.

4. (20 points) Let $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, and let $S^{n-1} = \partial D^n = \{x \in D^n \mid |x| = 1\}$. Let $f : D^n \to D^n$ be a continuous map such that $f|_{S^{n-1}}$ is a homeomorphism from S^{n-1} to S^{n-1}. Then show that f is surjective.

5. (20 points) Let X be a CW complex constructed from S^1 by attaching two 2-cells, one by the map $z \to z^4$ and the other by the map $z \to z^6$. Compute the homology of X.

QUALIFYING EXAM (ALGEBRA)

Each problem has 10 points.

1. (a) Show that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic.
   (b) Exhibit a proper subgroup of the additive group $\mathbb{Q}$ is not cyclic.

2. Let p be a prime and let P be a subgroup S_p of order p. Prove that $|N_{S_p}(P)| = p(p-1)$ and $N_{S_p}(P)/C_{S_p}(P) \cong \mathrm{Aut}(P)$. ($N_{S_p}(P)$ is the normalizer of P in S_p, and $C_{S_p}(P)$ is the centralizer of P in S_p.)

3. Let G be a group of order pq where p and q are primes with $p < q$. Let $P \in Syl_p(G)$, and $Q \in Syl_q(G)$.
   (a) Show that $G \cong Q \rtimes P$ for some group homomorphism $\varphi : P \to \mathrm{Aut}(Q)$.
   (b) Determine all isomorphism classes of G.

4. Let $f_1(x), f_2(x), \ldots, f_k(x)$ be polynomials with integer coefficients of the same degree d. Let $n_1, n_2, \ldots, n_k$ be integers which are relatively pime in pairs (i.e., $(n_i, n_j) = 1$ for all $i \neq j$). Prove that there exists a polynomial $f(x)$ with integer coefficients and of degree d with

$$f(x) \equiv f_1(x) \bmod n_1, \quad f(x) \equiv f_2(x) \bmod n_2, \quad \ldots, \quad f(x) \equiv f_k(x) \bmod n_k$$

   i.e., the coefficients of $f(x)$ agree with the coefficients of $f_i(x)$ mod n_i. Show that if all the $f_i(x)$ are monic, then $f(x)$ may also be chosen monic.

5. Let $\mathcal{U}$ be the subset of all open sets of $\mathbb{C}$ containing 0 and define a partial ordering on $\mathcal{U}$ by $U \leq V$ if and only if $V \subseteq U$. For $U \in \mathcal{U}$ let $\mathcal{O}_U$ be the set of all analytic functions on U. Prove that the ring $\mathcal{O} = \varinjlim \mathcal{O}_U$ is a discrete valuation ring.

6. Let R be a principal ideal domain.
   (a) Show that every nonzero prime ideal of R is a maximal ideal.
   (b) Show that a nonzero element of R is a prime if and only if it is irreducible.

7. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$.
   (a) Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor.
   (b) Show that $\{2, x\}$ is not a basis of I.

8. Let R be a commutative ring with 1. Suppose that M and N are projective R-modules. Prove that $M \otimes_R N$ is a projective R-module.

9. Find all similarity classes of 3×3 matrices A over $\mathbb{Q}$ satisfying $A^6 = I$.

10. Suppose A is an $n \times n$ matrix over $\mathbb{C}$ for which $A^k = I$ for some integer $k \geq 1$. Show that A can be diagonalized. Show that the matrix $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ where α is an element of a field of characteristic p satisfies $A^p = I$ and cannot be diagonalized if $\alpha \neq 0$.

Real Analysis Qualifying Exam

1. **[15]** Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. State each of the following theorems and prove the uniqueness assertion in (a).
   (a) Riesz Representation Theorem
   (b) Jordan Decomposition Theorem

2. **[10]** Let X and Y be normed linear spaces. State each of the following theorems.
   (a) Uniform Boundedness Principle
   (b) Closed Graph Theorem

3. **[35]** Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $\{f_n\}$ be a sequence of measurable functions on X. Let g be a monotone real-valued function on $[0, 1]$.
   (a) Let $\mu(X) < \infty$ and $\sup_{n \in \mathbb{N}} \|f_n\|_\infty < \infty$. Use Egoroff Theorem to show that if $\{f_n\}$ converges pointwise a.e. to a function f on X, then
   $$\lim_{n\to\infty} \int_X f_n = \int_X f.$$
   (b) Use the result in (a) to show that
   $$\int_X \liminf_{n\to\infty} |f_n| \le \liminf_{n\to\infty} \int_X |f_n|.$$
   (c) Let $D = \{x \in [0, 1] | g'(x) \in \mathbb{R}\}$ so that $m(D) = 1$. Use the result in (b) to show that $g' \in L^1[0, 1]$. If $g \in C[0, 1]$, then must $[0, 1] \setminus D$ be countable?

4. **[20]** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in L^1(\mu)$. Let $\epsilon > 0$. You may use Radon-Nikodym Theorem or Fatou Lemma to prove the following.
   (a) There is $E \in \mathcal{A}$ with $\mu(E) < \infty$ such that f is bounded on E and $\int_{X\setminus E} |f| < \epsilon$.
   (b) Let μ be a σ-finite measure and ν be a finite measure on $\mathcal{A}$. Then $\nu \perp \mu$ if and only if there is no nonzero measure ρ on $\mathcal{A}$ such that $\rho \ll \mu$ and $\rho \le \nu$ on $\mathcal{A}$.

5. **[20]** Let X be a normed linear space and S be a linear subspace of X. Let $x \in X$ and $\delta = \inf_{s\in S} \|s - x\| > 0$. You may use Hahn-Banach Theorem to prove the following.
   (a) There is $f \in X^*$ with $\|f\| = 1$ such that $f(x) = \delta$ and $f = 0$ on S.
   (b) If X^* is separable, then X is separable. The converse is not true.

Numerical analysis, Qualifying Exam. 2015

1. (20 pts)

   (a) Define a Lagrange interpolation polynomial with data $\{(x_i, f(x_i))\}_{i=0}^{n}$. x_i all distinct.

   (b) What is the error form in the above ? Derive it.

   (c) Define a Newton's form of interpolation polynomial using the same data.

   (d) Explain what happens if some x_i are repeated in the Newton's form. What is the correct data corresponding to the repeated points?

2. (10 pts) Describe Newton's method to solve a system of nonlinear equations $\mathbf{F}(\mathbf{x}) := A\mathbf{x} + g(\mathbf{x})\mathbf{x} = 0$ starting from some initial points $\mathbf{x}_0$. Here $\mathbf{x} = (x, y)$, $A = (a_{ij})$ is 2×2, nonsingular constant matrix and $g(\mathbf{x})$ is a scalar C^1-function of $\mathbf{x}$.

3. (10 pts) Explain Runge phenomena in approximation and suggest how one can avoid it.

4. (15 pts) Consider the following integral equation:

$$x(t) = \int_a^b K(t,s)x(s)\,ds + g(t)$$

   where $K(t, s) : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions.

   (a) Define a Picard method to show the existence of the solution $x(t)$ and state appropriate condition(s) to guarantee the solution.

   (b) Prove the solution exists in $C([a, b])$, the space of continuous functions under the maximum norm $\|\cdot\|_\infty = \max_{[a,b]} |x(t)|$.

5. (15 pts)

   (a) State a Householder algorithm to reduce an $n \times n$ real matrix A to an upper Hessenberg form

   (b) Explain how to find eigenvalues of a symmetric matrix A using the Householder algorithm.

(Continue on the next page)

6. (10 pts) Describe Euler's method (both explicit and implicit) to solve an ODE. $\dot{x} = f(t, x(t))$, $x(0) = x_0$. Compare these two method each other and discuss the advantages and disadvantages.

7. (20 pts) Assume A is a nonsingular $n \times n$ matrix and $\mathbf{v}^T A^{-1} \mathbf{u} \neq 1$ for some vectors $\mathbf{v}, \mathbf{u}$.

   (a) Show that $A - \mathbf{u}\mathbf{v}^T$ is nonsingular and

   $$(A - \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} + \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{\gamma}. \tag{1}$$

   for some number γ.

   (b) Generalize this formula to the case when U, V are $n \times k$ matrices of rank k, i.e., show that

   $$(A - UV^T)^{-1} = A^{-1} + A^{-1} B A^{-1} \tag{2}$$

   for some $n \times n$ matrix B.

Complex Analysis, Ph.D. Qualifying Exam, February 2015

1. (10 pts) Prove or disprove: Every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$.

2. (20 pts) Evaluate

   (a) $\int_{-\infty}^{\infty} e^{-(x+ia)^2}\,dx$ for a real.

   (b) $p.v. \int_{-i\infty}^{i\infty} \frac{dz}{(z^2-4)\log(z+1)} := \lim_{\epsilon\to 0^+} \int_{i\mathbb{R}\setminus[-i\epsilon,i\epsilon]} \frac{dz}{(z^2-4)\log(z+1)}$.

3. (20 pts)

   (a) Let f be holomorphic in a region containing the annulus $\{z \in \mathbb{C} : r_1 \le |z-z_0| \le r_2\}$ where $0 < r_1 < r_2$. Show that there are constants $a_n \in \mathbb{C}$ such that

   $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n,$$

   where the series converges absolutely in the interior of the annulus.

   (b) Let $f(z)$ be holomorphic in a punctured disc $B_r'(z_0) = \{z \in \mathbb{C} : \ 0 < |z-z_0| < r\}$. Suppose also that there is a $\epsilon > 0$ such that

   $$|f(z)| \le |z-z_0|^{-1+\epsilon} \quad \text{for all } z \text{ near } z_0.$$

   Classify the singularity of f at z_0 with proof.

4. (15 pts) Prove or disprove: If K is a compact set and f is holomorphic in a neighbourhood of K, then f can be approximated uniformly on K by polynomials.

5. (20 pts) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$ where $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. Prove that

   (a) $|f(z)| \le |z|$ for all $z \in \mathbb{D}$.

   (b) If $|f(z_0)| = |z_0|$ for some $z_0 \ne 0$, then f is a rotation.

6. (15 pts) A bijective holomorphic function is called a conformal map. Find all the conformal maps from $\mathbb{D}$ to $\mathbb{D}$ where $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$.
   (Hint: consider Blaschke factors and use the Schwarz lemma which is the problem 5)

Doctoral Qualifying Exam, Differential geometry
4th February 2015

Problem 1 (15 points)

Suppose that A and B are two disjoint closed subsets of a smooth manifold M. Show that there exists a smooth function $f : M \to \mathbb{R}$ such that $f(x) \in [0,1]$ for any $x \in M$, $A = f^{-1}(0)$, and $B = f^{-1}(1)$.

Problem 2 (20 points)

Suppose that M is a smooth manifold without boundary, N is a smooth manifold with boundary and $F : M \to N$ is a smooth map. Show that if at the point $p \in M$ dF_p is non-singular, then $F(p) \in Int(N)$.

Problem 3 (20 points)

Find the tangent plane (at a generic point) to the surface H in $\mathbb{R}^3$ defined by

$$x^2 + y^2 - z^2 + 1 = 0.$$

Problem 4 (20 points)

Let

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$$

be two vector fields on the plane. Compute the flows θ, ψ of X and Y, and verify that the flows do not commute by finding explicit open intervals J and K containing 0 such that $\theta_s \circ \psi_t$ and $\psi_t \circ \theta_s$ are both defined for all $(s,t) \in J \times K$, but they are not equal for some such (s,t).

Problem 5 (25 points)

Let S denote the set of matrices of the form

$$M(u,v,w) := \begin{pmatrix} \cos(w) & \sin(w) & 0 & u \\ -\sin(w) & \cos(w) & 0 & v \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $u, v, w \in \mathbb{R}$.

(1) Show that S is a Lie subgroup of $GL(4,\mathbb{R})$.

(2) Let $\sigma : \mathbb{R}^3 \to GL(4,\mathbb{R})$ the map defined by

$$(u,v,w) \to M(u,v,w).$$

Compute $d\sigma\left(\frac{\partial}{\partial u}\right)$, $d\sigma\left(\frac{\partial}{\partial v}\right)$, $d\sigma\left(\frac{\partial}{\partial w}\right)$ and show that σ is an immersion.

(3) Show that the tangent space to S at the identity element e of S admits the basis

$$\left\{ \left.\frac{\partial}{\partial x_{1,4}}\right|_e, \left.\frac{\partial}{\partial x_{2,4}}\right|_e, \left.\frac{\partial}{\partial x_{1,2}}\right|_e - \left.\frac{\partial}{\partial x_{2,1}}\right|_e + \left.\frac{\partial}{\partial x_{3,4}}\right|_e \right\}.$$

Here $x_{i,j}$ denotes the entry of a matrix in $GL(4,\mathbb{R})$ located on the i-th row and j-th column.

Ph.D Qualifying Exam
Probability
Feb 2015
(3 hours)

Problem 1. *(10pt) Prove directly from definition that if X and Y are independent random variables and f, g are measurable functions then $f(X)$ and $g(Y)$ are independent.*

Problem 2. *(15pt) Suppose $\{X_t, t \geq 0\}$ is a collection of real valued random variables on the probability space $(\Omega, \mathcal{B}, P)$ in which the function $t \to X_t(\omega)$ is continuous almost surely. Let $\tau : \Omega \mapsto [0, \infty)$ be a random variable and define the function $X_\tau : \Omega \mapsto [0.\infty)$ by*

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega), \quad \omega \in \Omega$$

Prove X_τ is a random variable.

Problem 3. *(15pt) Suppose $\{X_n, n \geq 1\}$ are independent random variables with $E(X_n) = 0$ for all n. If*

$$\sum_n E(X_n^2 1_{[|X_n| \leq 1]} + |X_n| 1_{[|X_n| > 1]}) < \infty,$$

then $\sum_n X_n$ converges almost surely.

Problem 4. *(15pt) Let X_n be independent Poisson r.v.s with $EX_n = \lambda_n$, and let $S_n = X_1 + \cdots + X_n$. Show that if $\lambda_n \to \infty$,then $S_n / ES_n \to 1$ a.s.*

Problem 5. *(15pt) Suppose X_n and Y_n are independent for each n and $X_n \Rightarrow X$, $Y_n \Rightarrow Y$. Prove using characteristic functions that*

$$X_n + Y_n \Rightarrow X + Y.$$

Problem 6. *(15pt) Let $\{X_n, n \geq 1\}$ be a sequence of independent r.v.s with $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$. Define $s_n^2 = \sum_{k=1}^n \sigma_k^2$. Show that $S_n^2 - s_n^2$ is a martingale.*

Problem 7. *(15pt) Suppose $\{X_n > 1\}$ are i.i.d. nonnegative r.v. with $EX_n = \mu$ and $Var(X_n) = \sigma^2$. Use the central limit theorem to derive an asymptotic normality result for $N(t) = \sup\{n : S_n \leq t\}$. That is,*

$$\frac{N(t) - t/\mu}{\sigma t^{1/2} \mu^{-3/2}} \Rightarrow N(0, 1).$$