2014 대수학I 박사 자격시험

1. (15 points) Let G be a finite group and let $g_1, g_2, \cdots, g_r$ be representatives of the distinct conjugacy classes of G not contained in the center $Z(G)$ of G.
Prove

$$|G| = |Z(G)| + \sum_{i=1}^{r}[G : C_G(g_i)].$$

Here, $C_G(g_i)$ is the centralizer of g_i in G.

2. (20 points) Let p be a prime and G be a group of order p^α for some $\alpha \geq 1$. Prove $Z(G) \neq 1$. Further, if $|G| = p^2$, determine whether G is abelian or not.

3. (20 points) Let A_4 be the alternationg group of degree 4. Determine whether A_4 has a subgroup of order 6.

4. (20 points) Let R be a commutative ring with unity and M be a nontrivial ideal. Show that M is maximal if and only if R/M is a field.

5. (10 points) Find the splitting field E of $x^3 - 2$ over $\mathbb{Q}$.

6. (15 points) Find a QR-decomposition of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Doctoral Qualifying Exam
Differential geometry
5th February 2014

Problem 1
20 points.

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by

$$(x, y, z) \to (r, s) = (xy, z).$$

(1) Find the critical points of F.
(2) Let S^2 be the unit sphere of $\mathbb{R}^3$. Find the critical points of $F_{|S^2}$.
(3) Find the set C of critical values of $F_{|S^2}$.
(4) Determine if C has zero measure.

Problem 2
15 points.

Define a C^∞ non-vanishing vector field on the sphere S^{2n+1}.

Problem 3
15 points.

(1) Consider the differential form

$$\alpha = \frac{1}{2\pi}\frac{xdy - ydx}{x^2 + y^2}$$

on $\mathbb{R}^2 \setminus \{0\}$. Determine if α is closed and/or exact.
(2) Let β denote the restriction of α to the unit circle $S^1 \subset \mathbb{R}^2$. Let $j : S^1 \hookrightarrow \mathbb{R}^2$ the canonical embedding. Determine if $j^*\beta$ is exact.

Problem 4
20 points.

(1) Show that the product of two orientable manifold is orientable.
(2) Show that the total space of the tangent bundle over any manifold is an orientable manifold.

Problem 5
10 points.

Show that the area of the region D bounded by a closed simple curve $[a, b] \to (x(t))$ contained in $\mathbb{R}^2$ is

$$A(D) = \int_D dx\, dy = \frac{1}{2}\int_a^b \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right) dt.$$

Problem 6
20 points.

Show the formula of change of variables for double integrals:

$$\iint_D F(x, y) dx\, dy = \iint_{\phi^{-1}(D)} F(x(u, v), y(u, v))\frac{\partial(x, y)}{\partial(u, v)} du\, dv,$$

corresponding to the coordinate transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $x \circ \phi = x(u, v)$, $y \circ \phi = y(u, v)$.

Algebraic Topology I: Ph.D. Qualifying Exam
February 5, 2014

Instructions (1) Use only ONE-SIDE of each answer sheet. (2) Answers without concrete justification will not be graded. (3) TWENTY points for each.

Problems

1. Let T be a 2-dimensional torus and z_1, z_2 be two distinct points on T. Compute the fundamental group of $T \setminus \{z_1, z_2\}$.
2. Let G be a *topological group*. This means, G is a group and a topological space such that the multiplication map $\mu(x, y) = x \cdot y$ and the inverse map $\nu(x) = x^{-1}$ are both continuous. Prove that $\pi_1(G)$ is abelian.
3. Suppose $f : S^{2n} \to S^{2n}$ is a continuous map without a fixed point. Prove that f maps some point to its antipodal point.
4. Let $T_n = (S^1)^n$ be the n-torus. Suppose $f : T_n \to T_n$ is a continuous map which is homotopic to a constant map. Prove that f has a fixed point.
5. Suppose $f : \mathbb{R}P^3 \to S^2 \times S^1$ is a continuous map. Prove that the induced map $f_* : H_3(\mathbb{R}P^3) \to H_3(S^2 \times S^1)$ is a zero map.

Real Analysis (Winter 2014)

1. (15 points) Give an example of an open set $\mathcal{O}$ with the following property: the boundary of the closure of $\mathcal{O}$ has positive Lebesgue measure.

2. (15 points) Suppose that $a > 0$. Let $f : [0,1] \to [0,\infty]$ be a measurable function satisfying that
$$\int_0^1 f(x)dx = 1.$$
Find the value of the following limit:
$$\lim_{n\to\infty} \int_0^1 n \log\left[1 + \left(\frac{f(x)}{n}\right)^a\right] dx$$

3. (15 points) Let S be a set of all complex, measurable, simple functions on a measure space X with a positive measure μ, satisfying that, for any $f \in S$,
$$\mu(\text{supp}(f)) < \infty.$$
Prove that S is dense in $L^p(X,\mu)$ for any $1 \le p < \infty$.

4. (20 points) Let ν be a signed measure on a measure space X and μ a positive measure on X. Consider the following conditions:
   (a) ν is absolutely continuous with respect to to μ.
   (b) For any $\epsilon > 0$, there exists $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$.

   Prove that (b) implies (a). Prove also that, if $|\nu|$ is a finite measure, then (a) implies (b).

5. (20 points) Suppose that $a, b > 0$. Let
$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}.$$
Prove that f is of bounded variation in $[0,1]$ if and only if $a > b$.

6. (15 points) Prove that, for a bounded linear operator T on a Hilbert space, the following holds:
$$\|TT^*\| = \|T^*T\| = \|T\|^2 = \|T^*\|^2.$$

Ph.D Qualifying Exam
Complex Analysis
Feb 2014
(3 hours)

Problem 1. *Suppose $f(z)$ and $g(z)$ are analytic in domain D and that*

$$\frac{f'(z_n)}{f(z_n)} = \frac{g'(z_n)}{g(z_n)},$$

at a sequence $\{z_n\}$ converging to z_0 in D. Show that $f(z) = Cg(z)$ in D for some constant C.

Problem 2. *Evaluate the following integrals*

(1)

$$p.v. \int_{-\infty}^{\infty} \frac{x^2+1}{x^4+x^2-2}\, dx.$$

(2)

$$\int_0^{\infty} \frac{\sin^3 x}{x}\, dx.$$

Problem 3. *Suppose that $f(z)$ is an entire function such that $f(z)$ is real on the circle $\{|z| = 1\}$. Show that $f(z)$ is constant.*

Problem 4. *If $\alpha > 1$, prove that $f(z) = z + e^{-z}$ takes the value α at exactly one point in the right half-plane.*

Problem 5.

(1) *Suppose $f(z)$ is analytic on $\{|z| < R\}$. Show if $|f(z)| \leq M$ in $\{|z| < R\}$, then $|f(z) - f(0)| \leq \frac{2M|z|}{R}$.*

(2) *Use (1) to give another proof of the Liouville's theorem. (You may use (1) without a proof.)*

Problem 6. *Let $\{f_n(z)\}$ be a sequence of functions analytic in the connected open set D and assume they converge to $f(z)$ uniformly on every compact subset of D.*

(1) *Show that $f(z)$ is analytic in D and*

$$\lim_{n\to\infty} f_n'(z) = f'(z), \qquad for \quad z \in D.$$

(2) *Show that if f_n do not take zero, then either f has no zero, or $f \equiv 0$.*

Qualifying Exam in Probability Theory (February 2014)

1. (10 pts) Let $\{X_n : n \geq 1\}$ be a sequence of independent r.v.s and $\{Y_n : n \geq 1\}$ be a sequence of independent r.v.s. If $P\{\lim_{n\to\infty} X_n \text{ exists}\} > 0$ and $P\{\lim_{n\to\infty} Y_n \text{ exists}\} > 0$, compute $P\{\lim_{n\to\infty} X_n Y_n \text{ exists}\}$.

2. (10 pts) Suppose that $\{X_n : n \geq 1\}$ are random variables on $(\Omega, \mathcal{B}, P)$ and define $S_0 := 0, S_n := \sum_{i=1}^{n} X_i, n \geq 1$. Let $\tau := \inf\{n > 0 : S_n > 0\}$. Assume that $\tau(\omega) < \infty$ for all $\omega \in \Omega$. Show that S_τ is a random variable. Here, S_τ is defined by $S_{\tau(\omega)}(\omega)$ for $\omega \in \Omega$.

3. (10 pts) Let $(\Omega, \mathcal{B}, P)$ be a probability space and $X \in L_1$. For a random variable X', prove the following. $\int_A X\, dP = \int_A X'\, dP$ for any $A \in \mathcal{B}$ if and only if $\int_A X\, dP = \int_A X'\, dP$ for any $A \in \mathcal{P}$ where $\mathcal{P}$ is a π-system generating $\mathcal{B}$ and containing Ω.

4. (20 pts) Let $\{X_n, n \geq 1\}$ be independent and identically distributed (i.i.d.) random variables. Let $S_n = X_1 + X_2 + \cdots + X_n$.

   (a) (10 pts) Assume that $E[|X_1|^p] < \infty$ where $p > 0$. Define $Y_k = X_k I_{\{|X_k| \leq k^{1/p}\}}, k \geq 1$, and $T_n = Y_1 + Y_2 + \cdots + Y_n$. Show that $\frac{S_n}{n^{1/p}} \to 0$ almost surely if and only if $\frac{T_n}{n^{1/p}} \to 0$ almost surely.

   (b) (10 pts) If $\frac{S_n}{n^{1/p}} \to 0$ almost surely for $p > 0$, show that $E[|X_1|^p] < \infty$.

5. (10 pts) Suppose that $\{(X_n, \mathcal{B}_n), n \geq 0\}$ is a martingale. Show the following: $\{X_n\}$ is L_1-convergent if and only if there exists $X \in L_1$ such that $X_n = E[X|\mathcal{B}_n], n \geq 0$.

6. (20 pts) A sequence $\{X_n\}$ is said to converge completely to a random variable X if $\sum_{n=1}^{\infty} P\{|X_n - X| > \epsilon\} < \infty$ for every $\epsilon > 0$. Prove or disprove the following.

   (a) If X_n converges completely to a random variable X, then X_n converges to X almost surely.

   (b) If X_n converges to X almost surely, then X_n converges completely to X.

7. (20 pts) Let $\{\mathcal{B}_n, n \geq 0\}$ be a filtration and $\{(X_n, \mathcal{B}_n), n \geq 0\}$ be a submartingale.

   (a) Show that $E[Y|\mathcal{B}_\tau] = \sum_{n\in\bar{\mathbb{N}}} E[Y|\mathcal{B}_n] I_{\{\tau=n\}}$ for $Y \in L_1$ and a stopping time τ where $\bar{\mathbb{N}} = \{0, 1, \cdots\} \cup \{\infty\}$.

   (b) Show that for every pair of *bounded* stopping times τ_1 and τ_2 such that $\tau_1 \leq \tau_2$, X_{τ_1} and X_{τ_2} are both integrable and $E[X_{\tau_2}|\mathcal{B}_{\tau_1}] \geq X_{\tau_1}$.

Qualifying Exam 2014 in Advanced Statistics

Feburary, 2013

1. (15pt) Let $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from $f_Y(y;\alpha) = \alpha^{-1}e^{-y/\alpha}$, $y > 0$, $\alpha > 0$. It is of interest to make statistical inferences about the unknown parameter $\theta = V(Y)$.

   (a) (7pt) If the observed value of $S = \sum_{i=1}^{n} Y_i$ is $s = 40$ when $n = 50$, compute an appropriate large sample 95% confidence interval for θ.

   (b) (8pt) If $n = 100$ and P(Type I error) $\doteq 0.05$, propose two kinds of appropriate large sample tests for $H_0 : \theta = 2$ vs $H_1 : \theta > 2$. What is the smallest value of $\theta(> 2)$ such that the power of the two tests will be at least 0.95? Based on your results, which test would you prefer?

2. (20pt) Let $(X_1, Y_1), (X_1, Y_1), \ldots, (X_n, Y_n)$ constitute a random sample of size n from

$$f_{X,Y}(x, y; \theta) = 2\theta^{-2}e^{-(x+y)/\theta}, \quad 0 < x < y < \infty$$

   (a) (10pt) Derive an explicit expression for the minimum variance unbiased estimator (MVUE) $\hat{\theta}$ for the unknown parameter θ and check if its variance is equal to the Cramer-Rao lower bound.

   (b) (10pt) Find a reasonable value for the smallest sample size n, say n^*, such that the power of the uniformly most powerful (UMP) test for $H_0 : \theta = 1$ vs $H_1 : \theta > 1$ is at least 0.95 when P(Type 1 error) $\doteq 0.05$ and when the true value of θ is at least equal to 1.20?

3. (30pt) Suppose that $Y_1 \sim Binomial(n_1, \pi_1)$, $Y_2 \sim Binomial(n_2, \pi_2)$ and that Y_1 and Y_2 are independent random variables.

   (a) (15pt) When $n_1 = 1000$ and $Y_1 = 300$, compute an appropriate 95% confidence interval for the unknown parameter $\theta = \pi_1/(1 - \pi_1)$, called an odds.

   (b) (15pt) Let $\hat{\pi}_1 = Y_1/n_1$ and $\hat{\pi}_2 = Y_2/n_2$. Further, let $\theta = ln[\frac{\pi_1/(1-\pi_1)}{\pi_2/(1-\pi_2)}]$ be the log odds ratio, and let $\hat{\theta} = ln[\frac{\hat{\pi}_1/(1-\hat{\pi}_1)}{\hat{\pi}_2/(1-\hat{\pi}_2)}]$ be an estimator for θ. Under the constraint $(n_1 + n_2) = N$, where N is the fixed total sample size, find reasonable expressions for n_1 and n_2 (as functions of N, π_1, and π_2) that minimize $V(\hat{\theta})$. If $N = 100$, $\pi_1 = 0.4$, and $\pi_2 = 0.2$, what are the numerical values of n_1 and n_2? Also, provide a sufficient condition such that $n_1 = n_2$.

4. (10pt) Let $X_1, \ldots, X_n$ be iid observations from a location-scale family. Let $T_1(X_1, \ldots, X_n)$ and $T_2(X_1, \ldots, X_n)$ be two statistics that both satisfy

$$T_i(ax_1 + b, \ldots, ax_n + b) = aT_i(x_1, \ldots, x_n)$$

for all values of $x_1, \ldots, x_n$ and b and for any $a > 0$.

(a) (5pt) Show that T_1/T_2 is an ancillary statistic.

(b) (5pt) Let R be the sample range and S be the sample standard deviation. Verify that R and S satisfy the above condition so that R/S is an ancillary statistic.

5. (10pt) Let $Y_1, \ldots, Y_n$ constitute a random sample from a pdf $f_Y(y) = 5y^4$, $0 < y < 1$. Consider a random variable $U_r = nY_{(1)}^r$ where $Y_{(1)} = min\{Y_1, \ldots, Y_n\}$.

(a) (5pt) For $r = 1$, determine to what random variable U_r converges in distribution as $n \to \infty$.

(b) (5pt) For $r = 5$, determine to what random variable U_r converges in distribution as $n \to \infty$.

6. (15pt) Let $X_1, \ldots, X_n$ constitute an iid random sample from $N(\mu, \sigma^2)$.

(a) (5pt) It is of interest to estimate $\theta = \mu^2$. Consider two estimators; $\hat{\theta}_1 = \bar{X}^2$ and $\hat{\theta}_2$ is an unbiased estimator involving $\bar{X}$ and S^2. Compare the mean squared error (MSE) for the two estimators and decide under what circumstances you would prefer one estimator to another.

(b) (5pt) Now consider $X_1, \ldots, X_n$ are random samples with $E(X_i) = \mu$, $Var(X_i) = \sigma^2$, and $Corr(X_i, X_{i'}) = \rho$. Derive explicitly $E(S^2)$ and comment on using S^2 as an estimator of σ^2 in this correlated data situation.

(c) (5pt) When $\mu = 1$, consider a conjugate prior for σ^2 and propose an appropriate Bayes point and interval estimators for σ^2.

Numerical Analysis Qualifying Exam

February 2014

1. Consider a polynomial $p(x) = x^6 - x - 1$.
   (a) (7 points) Find the number of positive real roots and the number of negative roots of $p(x)$.
   (b) (7 points) Find an upper bound for all of the roots of $p(x)$.
   (c) (6 points) Discuss how to find all real roots of $p(x)$.

2. There exist continuous functions f on the interval $[0, 1]$ whose best uniform approximation from Π_3 is the zero function. Here Π_3 is the space of all polynomials of degree ≤ 3 (in one variable).
   (a) (7 points) Give an example of such a function f.
   (b) (6 points) Would the zero function still be the best approximation to the f you chose in (a) if we replace Π_3 by Π_k for $k = 2, 4$? Explain. (Note that you are asked here two separate questions: one for $k = 2$ and one for $k = 4$).
   (c) (7 points) Would the zero function still be the best approximation to the f you chose in (a) if we replace $[0, 1]$ by $[0, 1/2]$? Explain.

3. Consider the ODE
$$y' = f(t, y)\,.$$
   (a) (7 points) Describe the trapezoidal method for the ODE above. Define the local truncation error and estimate it.
   (b) (7 points) Show that the truncation error for the following multistep method is of the same order as the method used in (a):
$$y_{n+1} = 2y_n - y_{n-1} - hf(t_{n-1}, y_{n-1}) + hf(t_n, y_n)\,.$$
   (c) (6 points) What can be said about the global convergence rate for these two methods? Justify your conclusions for both methods.

4. Consider the real system of linear equations
$$Ax = b \qquad (1)$$
where A is nonsingular and satisfies
$$(v, Av) > 0$$
for all real v, where $(\cdot, \cdot)$ denotes the Euclidean inner product.
   (a) (6 points) Show that $(v, Av) = (v, Mv)$ for all real v, where $M = \frac{1}{2}(A + A^T)$ is the symmetric part of A.

(b) (6 points) Prove that

$$\frac{(v, Av)}{(v, v)} \geq \lambda_{\min}(M) > 0\,,$$

where $\lambda_{\min}(M)$ is the minimum eigenvalue of M.

(c) (8 points) Now consider the iteration for computing a series of approximate solutions to (1),

$$x_{k+1} = x_k + \alpha r_k\,,$$

where $r_k = b - Ax_k$ and α is chosen to minimize $\|r_{k+1}\|_2$ as a function of α. Prove that

$$\frac{\|r_{k+1}\|_2}{\|r_k\|_2} \leq \left(1 - \frac{\lambda_{\min}(M)^2}{\lambda_{\max}(A^T A)}\right)^{1/2}.$$

5. (10 points each)

(a) Using Householder matrices, reduce an $n \times n$ real matrix A to an upper Hessenberg form.

(b) Explain how to find all eigenvalues of a symmetric matrix A using above method.