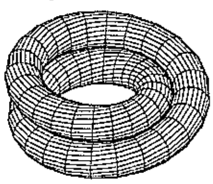
Doctoral Qualifying Exam

Algebraic Topology I **July 31, 2012**

Write your answer as detailed as you can. Problems 1, 2 and 3 are worth 20 points each and the rest 10 points each.

1. Let $X = S^1 \times S^1 - \{p, q, r, s\}$ be the torus with four holes.
   (a) Compute the fundamental group of X.
   (b) Compute the singular homology groups of X.

2. Let $X = S^1 \vee S^1 = S^1 \times \{s_0\} \cup \{s_0\} \times S^1 \subset S^1 \times S^1$ and let $f : X \to X$ be the restriction of the map $(s, t) \mapsto (t, s)$ of $S^1 \times S^1$ to X. Consider the quotient space
$$Y = \frac{X \times [0,1]}{(x, 0) \sim (f(x), 1), \forall x \in X}$$
which can be visualized by the figure below.



   (a) Find the Euler characteristic of Y.
   (b) Compute a presentation for $\pi_1(Y)$.

3. Let A be the rectangular region in the plane $\mathbb{R}^2$ with two square holes as in the figure below.

Use the Seifert-van Kampen theorem to compute the fundamental group of each space.
   (a) $X = \partial(A \times [0,1])$.
   (b) $Y = \mathbb{R}^3 - A \times [0,1]$.

4. Let n be a positive integer and let $P \in \mathbb{R}^n$. Compute $H_k(\mathbb{R}^n, \mathbb{R}^n - P)$ for $k \geq 0$.

5. The following is a diagram of abelian groups and homomorphisms in which the rows are exact and the squares are commutative. Show that if α and γ are isomorphisms then so is β.
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & D & \xrightarrow{h} & E & \xrightarrow{k} & F & \longrightarrow & 0 \end{array}$$

6. The *suspension* of a space X, denoted ΣX, is defined as the quotient space
$$\Sigma X = \frac{X \times [0,1]}{\sim}$$
where $\sim$ indicates the identifications $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. Show that $H_n(\Sigma X) \cong \widetilde{H}_{n-1}X$ for $n \geq 1$.

7. Let A be a closed subset of a space X. Show that $H_k(X, A) \cong \widetilde{H}_k(X/A)$, if A is a deformation retract of one of its neighborhood.

(gtj 2012/7/27 9:23)

Real Analysis

1. **[15]** Let X and Y be nontrivial normed vector spaces such that the set of all bounded linear operators from X^* to Y is a Banach space. Show that Y is complete. You may use Hahn-Banach Extension Theorem.

2. **[15]** Let X and Y be normed vector spaces. Let Γ be a collection of bounded linear operators from X to Y such that the set of all points $x \in X$ with $\sup_{A\in\Gamma} \|Ax\| < \infty$ is of the second category in X. Show that $\sup_{A\in\Gamma} \|A\| < \infty$.

3. **[15]** Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable, real-valued functions on $(X, \mathcal{A})$ with the property that, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\mu(\{x \in X | |f_k(x) - f_l(x)| \geq \epsilon\}) < \epsilon$ for all $k, l \in \mathbb{N}$ with $k, l \geq N$. Show that there is a subsequence of $\{f_n\}$ that converges pointwise a.e. to a function f on X. You may use Borel-Cantelli Lemma.

4. **[15]** Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and ν be a finite measure on $\mathcal{A}$. Show that $\nu \perp \mu$ if and only if there is no nonzero finite measure ρ on $\mathcal{A}$ such that $\rho \ll \mu$ and $\rho(E) \leq \nu(E)$ for all $E \in \mathcal{A}$. You may use Radon-Nikodym Theorem.

5. **[10]** Let $(X, \mathcal{A}, \mu)$ be a measure space and f be an integrable function on $(X, \mathcal{A}, \mu)$. Show that, given $\epsilon > 0$, there is $E \in \mathcal{A}$ with $\mu(E) < \infty$ such that f is bounded on E and $\int_{X\setminus E} |f| d\mu < \epsilon$. You may use Monotone Convergence Theorem.

6. **[15]** Let $(X, \mathcal{A}, \mu)$ be a measure space and Γ be the set of integrable simple functions on $(X, \mathcal{A}, \mu)$. Show that if μ is semifinite, then $\int_X |f| d\mu = \sup_{\varphi\in\Gamma} \int_X (|f| \wedge \varphi) d\mu$ for all measurable functions f on $(X, \mathcal{A})$. Decide whether the converse holds.

7. **[15]** Let $1 < p < \infty$. Let $(X, \mathcal{A}, \mu)$ be a measure space and f be a measurable function on $(X, \mathcal{A})$ such that $|f| > 0$ a.e. and $|\int_X f\varphi d\mu| \leq 1$ for all simple functions φ on $(X, \mathcal{A})$ with $\int_X |\varphi|^p d\mu = 1$. Show that if μ is semifinite, then it is σ-finite.

KAIST DMS
QUALIFYING EXAM IN COMPLEX ANALYSIS
JULY 2012

Student ID: **Name:**

No cell phone or electronic device allowed. One two-sided sheet of notes allowed. Duration: two hours. Please return all material, including this sheet and draft paper used during the test.

If you can not solve a problem, it is better to write clear and relevant ideas than irrelevant computations. Do not claim that you solved a problem unless you think you did. Good luck!

We denote by $\mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \}$ the unit disk in $\mathbb{C}$.

(1) -10 pts- Give the statement (not the proof) of Liouville's theorem on bounded entire functions.

(2) -10 pts- Give the definition of a meromorphic function on $\mathbb{C}$.

(3) -10 pts- Does there exist a holomorphic bijection between $\mathbb{D}$ and $\{ z = x + iy \in \mathbb{C} \mid x > 0,\ y > 0,\ x + y < 1 \}$?
(*Justify your answer in no more than two lines.*)

(4) -10 pts- Does there exist a holomorphic bijection $f : \mathbb{C} \to \mathbb{D}$?
(*Justify your answer in no more than two lines.*)

(5) -10 pts- Let $x \in \mathbb{R}$ with $x > 1$. Set $z = e^{i\theta}$, and use complex integration and the residue theorem to prove

$$\int_0^{2\pi} \frac{d\theta}{x + \cos\theta} = \frac{2\pi}{\sqrt{x^2 - 1}}.$$

(*Hint:* $-x - \sqrt{x^2-1} < -1 < -x + \sqrt{x^2-1} < 0$.)

(6) -10 pts- Let $\mathcal{A}$ be a family of functions holomorphic in $\mathbb{D}$. Denote $\mathcal{A}' = \{ f' \mid f \in \mathcal{A} \}$. Assume $\mathcal{A}'$ is normal and

$$\sup \{ |f(0)| \mid f \in \mathcal{A} \} < \infty.$$

Show that $\mathcal{A}$ is normal.

(7) -10 pts- Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of points in the complex plane converging to 0. Let $S = \{0\} \cup \{ z_n \mid n \in \mathbb{N} \}$, and let f be a holomorphic function on $\mathbb{C} - S$. Show that either f extends to a meromorphic function on some disk centered at 0, or else for any $y \in \mathbb{C}$ there exists a sequence of complex numbers $(w_n)_{n\in\mathbb{N}}$ such that $w_n \to 0$ and $f(w_n) \to y$.

(8) -15 pts- Let Ω be a simply connected nonempty open subset of $\mathbb{C}$ such that $0 \notin \Omega$. Without using the Riemann Mapping Theorem, show that there is a holomorphic bijection $\varphi : \Omega \to \Omega'$ with $\Omega' \subset \mathbb{D}$.

(9) -15 pts- Fix τ in the upper half-plane. We consider elliptic functions with respect to the lattice generated by 1 and τ.
  (a) (i) Describe the poles of the Weierstrass $\wp$ function.
  (ii) Recall that the restriction to a fundamental parallelogram of the function $z \mapsto \wp(z) - \wp(a)$ has exactly one zero at a when $a = 1/2, \tau/2$ or $(1+\tau)/2$, and exactly two zeros at a and $-a$ otherwise. What is the order of these zeros?
  (b) Show that any elliptic function can be written as a rational expression in $\wp$.

QUALIFYING EXAM: COMPLEX ANALYSIS
(3 HOURS)

Problem 1. *[10pt] Let $f(z)$ be a holomorphic function on a domain D. Prove that $\overline{f(\overline{z})}$ is a holomorphic function on $\{\overline{z} : z \in D\}$.*

Problem 2. *[20pt] Evaluate the following integrals*

(1)
$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$$

(2)
$$\int_{0}^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\, d\theta$$

Problem 3. *[10pt] Establish the identity*

$$\int_{-\infty}^{\infty} e^{\alpha x^2} e^{i\beta x}\, dx = \sqrt{\frac{\pi}{\alpha}} e^{-\beta^2/4\alpha}$$

where α, β are real numbers with $\alpha > 0$.

Problem 4. *[20pt] Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n.*

(1) *Prove that for any $M > 0$, there exists a constant $R > 0$ such that $|p(z)| > M$ on $\{z \in \mathbb{C} : |z| > R\}$.*
(2) *Prove that if $|p(z)|$ attains a local minimum at z_0, then $|p(z_0)| = 0$.*
(3) *Deduce the fundamental theorem of algebra, saying that $p(z)$ has at least a zero in $\mathbb{C}$.*

Problem 5. *[15pt] Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function with compact support. Show that if the Fourier transform of f,*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$$

also has a compact support, then $f \equiv 0$. (Hint. Use the identity theorem)

Problem 6. *[15pt] Let $f(z)$ be a nonconstant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.*

Problem 7. *[10pt] Let $f(z)$ be a holomorphic function on $\mathbb{C}$. Show tha if $Re\, f \geq 0$, then f is a constant.*

2012 Algebra Qualifying Exam

(1) Let G be an abelian group of order 72. Classify it in terms of both the elementary divisor decomposition and the invariant factor decomposition.

(2) Let G be a finite group of order 56. By making use of Sylow's theorem determine whether it is a simple group or not.

(3) Let R be a PID. Prove that R is a UFD.

(4) Find the minimal polynomial for $\alpha = \sqrt{3} + \sqrt{5}$ over $\mathbb{Q}$.

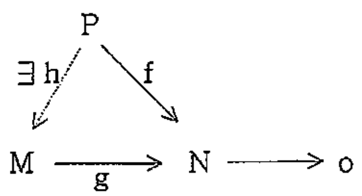
(5) Let $K = \mathbb{Q}(\zeta)$ with ζ a primitive 5-th root of unity, and let $G = \mathrm{Gal}(K/\mathbb{Q})$.

(i) Find the group G.

(ii) Let H be a nontivial proper subgroup of G, and K^H be its fixed field. Find K^H.

(iii) When $K^H = \mathbb{Q}(\alpha)$ in (ii), find the minimal polynomial of α over $\mathbb{Q}$.

(6) Let R be a commutative ring with 1 and P be an R-module. We say "P is projective" if given any diagram of R-linear maps with the bottom row exact, there exists an R-linear map $h: P \to M$ such that $f = g \circ h$.



Prove that every R-module is a homomorphic image of a projective R-module.

Qualifying Exam in Probability Theory (July 2012)

1. (10 pts) Verify that

$$P\{\liminf_{n\to\infty} A_n\} \le \liminf_{n\to\infty} P\{A_n\} \le \limsup_{n\to\infty} P\{A_n\} \le P\{\limsup_{n\to\infty} A_n\}.$$

2. (10 pts) Let $(\Omega, \mathcal{B})$ be a measurable space, and X and Y be two random variables. Define a new random variable $Z := XY$. Consider $\sigma(Z)$ and $\sigma(X,Y)$. Prove or disprove that $\sigma(Z) = \sigma(X,Y)$.

3. (10 pts) Let $\{X_n\}$ be independent random variables and $S_n = X_1 + \ldots + X_n$. Show that the event $\{\omega : \sum X_n(\omega) \text{ converges}\}$ has probability 0 or 1.

4. (10 pts) A sequence $\{X_n\}$ is said to converge completely to a random variable X if $\sum_{n=1}^{\infty} P\{|X_n - X| > \epsilon\} < \infty$ for every $\epsilon > 0$. Show that, if X_n converges to X almost surely, then there exists a subsequence $\{X_{n_j}\}$ such that X_{n_j} converges to X completely.

5. (10 pts) For random variables X_1, X_2 and Y with $Y \in L_1$, suppose that $\sigma(Y, X_1)$ is independent of $\sigma(X_2)$. Show almost surely $E[Y|X_1, X_2] = E[Y|X_1]$.

6. (10 pts) Suppose that $\{(X_n, \mathcal{B}_n), n \ge 0\}$ is a martingale. Show the following: $\{X_n\}$ is L_1-convergent if and only if there exists $X \in L_1$ such that $X_n = E[X|\mathcal{B}_n], n \ge 0$.

7. (10 pts) Suppose X and Y are independent with common distribution function $F(x)$ having mean zero and variance 1, and suppose further that

$$\frac{X+Y}{\sqrt{2}} \stackrel{d}{=} X \stackrel{d}{=} Y.$$

Show that both X and Y have a $\mathcal{N}(0,1)$ distribution.

8. (10 pts) If X_n converges in distribution to X and Y_n converges in distribution to a constant c, show that X_nY_n converges in distribution to cX. (Hint: Prove it first with $c = 0$.)

9. (20 pts) True or false. You don't need to prove! 2 points for correct answers, but -2 points for incorrect answers.

   (a) Suppose X is a non-negative random variable satisfying $P\{0 \le X < \infty\} = 1$. $\lim_{n\to\infty} nE[\frac{1}{X}1_{\{X>n\}}] = 0$.

   (b) If $X_n \to X$ a.s., then $E[X_n] \to E[X]$.

   (c) If $X_n \to X$ in L_p $(p = 1, 2, \cdots)$, then $E[X_n^p] \to E[X^p]$.

   (d) If $X_n \stackrel{L_p}{\to} 0$, $\frac{S_n}{n} \stackrel{L_p}{\to} 0$ for $p \ge 1$.

   (e) Let $\{X_n\}$ be a sequence of random variables with $P\{X_n = \pm\frac{1}{n}\} = \frac{1}{2}$. $X_n \to 0$ a.s..

   (f) If $X_n \stackrel{P}{\to} X$, there exists a subsequence $\{X_{n_k}, k \ge 1\}$ which converges to X a.s.

   (g) For i.i.d. random variables $\{X_n\}$, if X_1 is integrable, then $\lim_{n\to\infty} \frac{|X_n|}{n} = 0$ a.s..

   (h) If X_n converges in distribution to X, then for every continuous function $f : \mathbb{R} \to \mathbb{R}$, $E[f(X_n)]$ converges to $E[f(X)]$.

   (i) Let X be an integrable random variable. If $\sigma(X)$ and a σ-field $\mathcal{F}$ are independent, then $E[X|\mathcal{F}] = E[X]$ almost surely.

   (j) If $X_n \stackrel{L_p}{\to} X$, then $E[X_n|\mathcal{B}] \stackrel{L_p}{\to} E[X|\mathcal{B}]$ for a σ-field $\mathcal{B}$.

Qualifying Exam-2 2012 in Advanced Statistics

1. Let $(Y_1, \cdots, Y_k)$ be a multinomial random vector with cell probabilities, $p_1, \cdots, p_k$, such that $\sum_{i=1}^{k} p_i = 1$, and let $\sum_{i=1}^{k} X_k = n$, where n is the total number of multinomial trials or the sample size.

   (a) [10 pts] For $l < k$, find the distribution of $(Y_1, \cdots, Y_l)$.

   (b) [5 pts] For $l < m \leq k$, find the conditional distribution of $(Y_{l+1}, \cdots, Y_m)$ given that $(Y_1, \cdots, Y_l) = (y_1, \cdots, y_l)$.

2. [15 pts] Let the statistic $T(X_1, \cdots, X_n)$ be a minimal sufficient statistic(MSS), where $X_1, \cdots, X_n$ are iid from a distribution. Is it true then that any complete sufficient statistic is also a MSS? Why?

3. Let $X_1, \cdots, X_n$ be a random sample from the gamma distribution, $gamma(\alpha, \beta)$. Let $\bar{X}$ and S^2 be the sample mean and sample variance, respectively.

   (a) [10 pts] Are $\bar{X}$ and S^2 independent of each other? Why?

   (b) [10 pts] Are $\bar{X}$ and $S^2/\bar{X}^2$ independent of each other? Why?

4. [15 pts] Let $X_1, \cdots, X_n$ be a Normal random sample from $N(\mu, 1)$ and define $Y_n = \max\{X_1, \cdots, X_n\}$. Find a best unbiased estimator of $P(Y_n > a)$ for a real value a.

5. [15 pts] Find the 1-α confidence set for a that is obtained based on a random sample $X_1, \cdots, X_n$ from the normal distribution, $N(\theta, a\theta)$, with θ unknown. [Hint: Think of inverting an LRT.]

6. Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution, $U(\theta, \theta+1)$. For testing $H_0: \ \theta = 0$ vs $H_1: \ \theta > 0$, we use the test

   $$\text{reject } H_0 \text{ if } Y_n \geq 1 \text{ or } Y_1 \geq a,$$

   where a is a constant, $Y_1 = \min\{X_1, \cdots, X_n\}$, and $Y_n = \max\{X_1, \cdots, X_n\}$.

   (a) [8 pts] Find a so that the test is of size α.

   (b) [6 pts] Find an expression for the power function of the test in question (6a).

   (c) [6 pts] Is the test UMP? Why?

THE END

Qualifying Exam-3 2012 in Advanced Statistics

1. [10 pts] Suppose T_1 and T_2 are sufficient and minimally sufficient for θ, respectively, and that U is an unbiased estimator of θ. Let $U_i = E(U|T_i)$, $i = 1, 2$. Compare $Var(U_1)$ and $Var(U_2)$.

2. [20 pts] Given that $N = n$, the conditional distribution of Y is χ^2_{2n}. Suppose that N is a Poisson random variable with parameter θ. Find the limit of the distribution of $(Y - E(Y))/\sqrt{Var(Y)}$ as $\theta \to \infty$.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from the pdf $f(x;\theta)$ and write $\mathbf{X} = (X_1, \cdots, X_n)$. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, using a test with rejection region R that satisfies

$$\begin{array}{l} \mathbf{x} \in R \text{ if } f(\mathbf{x};\theta_1)/f(\mathbf{x};\theta_0) > k \\ \text{and} \\ \mathbf{x} \in R^c \text{ if } f(\mathbf{x};\theta_1)/f(\mathbf{x};\theta_0) < k \end{array} \tag{1}$$

for some $k \geq 0$, and

$$\alpha = P_{\theta_0}(\mathbf{X} \in R). \tag{2}$$

Then show the following statements:

(a) [10 pts] Any test that satisfies (1) and (2) is a UMP level α test.

(b) [10 pts] If there exists a test satisfying (1) and (2), every UMP level α test satisfies (1) and (2) except a null set A such that $P_{\theta_0}(\mathbf{X} \in A) = P_{\theta_1}(\mathbf{X} \in A) = 0$.

4. [15 pts] Let X and Y be independent exponential random variables, with

$$f(x;\lambda) = \frac{1}{\lambda}e^{-x/\lambda},\ x > 0, \quad f(y;\mu) = \frac{1}{\mu}e^{-x/\mu},\ y > 0.$$

Suppose we have a random sample, (Z_i, W_i), $i = 1, 2, \cdots, n$, where

$$Z_i = \min(X_i, Y_i) \quad \text{and} \quad W_i = \begin{cases} 1 & \text{if } Z_i = X_i, \\ 0 & \text{if } Z_i = Y_i. \end{cases}$$

Find the MLEs of λ and μ.

5. [20 pts] Let Y and V be random variables with $Y \sim f_Y$ and $V \sim f_V$, where f_Y and f_V share the same support with

$$M = \sup_y \frac{f_Y(y)}{f_V(y)} < \infty.$$

Suppose that we generate a random number W as follows:
(a). Generate a uniform random number, U, i.e., $U \sim \mathcal{U}(0,1)$, and a random number V from f_V.
(b). If $U < \frac{1}{M} f_Y(V)/f_V(V)$, set $W = V$; otherwise, return to step (a).

Find the distribution of W.

6. [15 pts] Let $X_1, \cdots, X_n$ be a random sample from the normal distribution, $\mathcal{N}(\mu, 1)$ with unknown μ. Find, if it exists, the best unbiased estimator of $\varphi = P(X_1 > c)$ for a constant c.

THE END

Numerical Analysis Qualifying Exam

July 2012

1. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a nonlinear smooth function. To determine a (local) minimum of f one can use a descent method of the form
$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$
where $\alpha_k > 0$ is a suitable parameter and $\mathbf{d}_k$ is a descent direction, i.e., it satisfies
$$f(\mathbf{x}_{k+1}) < f(\mathbf{x}_k)\,. \tag{1}$$
   (a) (7 points) Write the *steepest descent* (or gradient) method and show that there exist $\alpha_k > 0$ such that the resulting method satisfies (1).

   (b) (8 points) Write the *Newton method* and examine whether or not there exist α_k which yield (1). Establish conditions on the Hessian $H(f(x))$ of $f(x)$ which gaurantee the existence of α_k.

   (c) (5 points) If we replace the Hessian by the matrix $H(f(x)) + \gamma_k I$, where $\gamma_k > 0$ and I is the identity matrix, we obtain a *quasi-Newton method*. Find a condition on γ_k which leads to (1).

2. (a) (4 points) Consider the problem of interpolating the following data made up of $n+1$ distinct points:
$$(x_0, f_0), (x_1, f_1), (x_2, f_2), \cdots, (x_n, f_n)\,.$$
   Prove that there exists a unique polynomial of degree at most n that interpolates the above data.

   (b) (6 points) Consider the Chebyshev polynomials:
$$T_0(x) = 1, T_1(x) = x, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad n > 0\,.$$
   Prove all of the following:

   i. $T_n(x)$ is a polynomial of degree exactly n with n distinct real roots between $-1 \le x \le 1$;

   ii. $-1 \le T_n(x) \le 1$ for all $-1 \le x \le 1$;

   iii. The coefficient of x^n in $T_n(x)$ is exactly 2^{n-1} for $n > 0$.

   (c) (8 points) Consider the problem of interpolating the C^{n+1} function $f(x)$ at the $n+1$ distinct points $x_0, x_1, \ldots, x_n$, where $-1 \le x_0 < x_1 < x_2 < \cdots < x_n \le 1$, with the polynomial $p_n(x)$. Show that the max-norm error:
$$\| f(x) - p_n(x) \|_\infty := \max_{-1 \le x \le 1} |f(x) - p_n(x)|$$
   is nearly minimized over all possible choices of the points $x_0, x_1, \ldots, x_n$ if these points are the $n+1$ roots of $T_{n+1}(x)$.

   (d) (2 points) How does $\| f(x) - p_n(x) \|_\infty$ decay with increasing n for the C^∞ function $f(x)$.

3. Consider the Runge-Kutta methods for the ODE $u' = f(u)$:

SCHEME 1:
$$\begin{aligned} f_1 &= f(u^n) \\ f_2 &= f(u^n + b_1 k f_1) \\ u^{n+1} &= u^n + k(c_1 f_1 + c_2 f_2) \end{aligned}$$

SCHEME 2:
$$\begin{aligned} f_1 &= f(u^n) \\ f_2 &= f(u^n + b_1 k f_1) \\ f_3 &= f(u^n + b_2 k f_2) \\ u^{n+1} &= u^n + k(c_1 f_1 + c_2 f_2 + c_3 f_3) \end{aligned}$$

(a) (7 points) Under what conditions is SCHEME 1 second order and SCHEME 2 third order?

(b) (6 points) Are the constants b_i and c_i uniquely determined by the conditions found above?

(c) (7 points) Find the equations defining the stability regions of the above schemes (with $f = \lambda u$). Where do these regions intersect the real and imaginary axis in the complex $k\lambda$ plane?

4. Consider the linear system $Ax = b$ with an $n \times n$ nonsingular matrix A.

   (a) (6 points) Write down the Jacobi iteration for solving $Ax = b$, in the way that it would be programmed on a computer.

   (b) (5 points) Suppose A has m nonzero elements with $m << n^2$. How many operations per iteration does the Jacobi iteration take?

   (c) (9 points) Assume that A is strictly diagonally dominant. i.e., for $i = 1, \cdots, n$,

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| .$$

   Show that the Jacobi iteration converges for any initial guess $x^{(0)}$.

5. Let A be a real square matrix whose eigenvalues are all algebraically simple. We intend to find an eigenvector of A by using the power method. The power method is considered 'convergent' if, for every positive tolerance ϵ, we obtain, after sufficiently many iteration, $x^{(n)}$ so that, for some eigenvector v of A, $\|x^{(n)} - v\| \leq \epsilon\|v\|$.

   (a) (5 points) Explain why the above definition of convergence is preferred over the simpler one. where we merely require that $\|x^{(n)} - v\| \leq \epsilon$.

   (b) (8 points) Is the following statement true? "The power method will converge regardless of the choice of the initial (non-zero) vector, provided that A is symmetric positive definite." Explain your answer.

   (c) (7 points) Will you change your answer to (b) if A is not necessarily positive definite?

Numerical Analysis Qualifying Exam

August 2012

1. Consider the following iteration for calculating $\gamma^{1/3}$:

$$x_{n+1} = ax_n + b\frac{\gamma}{x_n^2} + c\frac{\gamma^2}{x_n^5}$$

   (a) Choose a, b, c so that the iteration will converge to $\gamma^{1/3}$ (for x_0 sufficiently close to $\gamma^{1/3}$ with the highest order possible. Show your work.

   (b) Write an equation that relates the error at step n to the error at step $n+1$. What is the limiting form of this expression?

2. Given the function $f(x) = e^x$,

   (a) Form the divided difference table necessary to construct the cubic Hermite polynomial interpolating $f(x)$ on the nodes $x_0 = 0, x_1 = 1$.

   (b) Use the table to construct the Hermite interpolating polynomial, $p_3(x)$, on this mesh.

   (c) Write the formula for the error, $\mathcal{E}(x) = f(x) - p_3(x)$, involving derivatives of $f(x)$.

   (d) Derive an upper bound for the error: $\max_{x\in[0,1]} |f(x) - p_3(x)|$. This should be a number. Use the approximation $e \leq 3$.

3. Let $p_3(x) = x^3$. Find $q_2^*(x)$, the minimax approximation of $p_3(x)$ on the interval $[-1, 1]$.

4. (a) Derive Simpson's rule from the Trapezoid rule and the Midpoint rule.

   (b) Which quadrature rule for $\int_a^b f(x)\,dx$ do you obtain by the following process? You approximate that integral by $\int_a^b p_2(x)\,dx$, with p_2 the unique polynomial of degree < 3 that agrees with f at the three points a, $(a+b)/2$, and b.

   (c) Discuss how you would go about obtaining, in an efficient way, an accurate value for

$$\int_{-1}^{1} \frac{1 + |x^2 - 1/4|}{\sqrt{1 - x^2}}\,dx$$

   Justify your approach, by citing relevant facts about the quadrature rule(s) you are proposing to use.

5. Let A be real, symmetric, positive definite, and of order n. Consider solving $Ax = b$ using Gaussian elimination without pivoting.

   (a) Show that all of the diagonal elements satisfy $a_{ii} > 0$.

   (b) After elimination of x_1 from equations 2 through n, let the resulting matrix $A^{(2)}$ be written as

$$A^{(2)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & \hat{A}^{(2)} & \\ 0 & & & \end{bmatrix}$$

   Show that $\hat{A}^{(2)}$ is symmetric and positive definite.