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A TWO-LEVEL NONOVERLAPPING SCHWARZ ALGORITHM FOR THE STOKES PROBLEM WITHOUT PRIMAL PRESSURE UNKNOWNS *

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Abstract. A two-level nonoverlapping Schwarz algorithm is developed for the Stokes problem. It is the primal counter part of the FETI-DP algorithm for the Stokes problem without primal pressure components. A preconditioner for the primal form is designed by using close connection of the primal and dual forms and it gives the resulting matrix of the primal form with all its eigenvalues as positive real numbers. Convergence of the method is analyzed and numerical results are included.

Key words. Two-level Schwarz algorithm, FETI-DP, Stokes problem, preconditioner

AMS subject classifications. 65N30, 65N55, 76D07

1. Introduction. FETI-DP algorithms belong to a family of dual iterative substructuring methods, which are known to be one of the most scalable domain decomposition methods for solving numerical partial differential equations, see [3, 4]. A pair of inf-sup stable velocity and pressure finite element spaces is introduced to a given triangulation in the domain and the continuity of the finite element spaces are relaxed by decomposition of the given domain into subdomains. Among the degrees of freedom on subdomain interfaces, some are selected as primal unknowns. A strong continuity is enforced to the primal unknowns and at the remaining part of unknowns on the interface the continuity is imposed weakly by using Lagrange multipliers. After elimination of the unknowns other than the Lagrange multipliers, a system on the dual unknowns, i.e., the Lagrange multipliers, is solved iteratively with a preconditioner. The preconditioner accelerates the convergence of the iteration.

The FETI-DP algorithms have been successfully developed for the elliptic problems and elasticity problems [6, 9, 16, 17]. As a primal counter part of FETI-DP algorithms, BDDC (balancing domain decomposition by constraints) algorithms were introduced by Dohrmann [2] and further analyzed by Mandel and Dohrmann [23]. Their close connection to the FETI-DP algorithms was studied in [1, 7, 21, 24]. Recently, extensions to irregular subdomains and to inexact subdomain solvers have been done in [14, 15, 22]. Both algorithms were also extended to the Stokes problem [8, 10, 11, 18, 19, 20].

In the authors' previous works for the Stokes problem [8, 11], no primal pressure un-

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knowns are introduced differently to other algorithms [18, 19, 20, 26]. Only some velocity unknowns are selected as the primal unknowns in the FETI-DP formulation, which produces a symmetric and positive definite coarse problem matrix. No primal pressure unknowns result in a more practical FETI-DP algorithm, which also allows the use of a quite cheap lumped preconditioner.

In this work, we develop a primal counter part of the FETI-DP algorithm for solving the Stokes problem. The primal algorithm is derived similarly to the work in [21] by using the connection between the FETI-DP and the BDDC algorithms. Velocity values at subdomain corners are selected as the primal unknowns in the two dimensional case and in addition to them averages of the velocity over common faces are selected as primal unknowns in the three dimensional case. In [8, 11] it was proved that such selection of the primal unknowns gives the condition number bound, $C(H/h)(1+\log(H/h))$ in $2D$, and $C(H/h)$ in $3D$ for the FETI-DP algorithm with the lumped preconditioner.

In the primal formulation, by using its close connection to the dual form of the Stokes problem, a preconditioner for the primal form is designed so that the resulting preconditioned linear system has all its eigenvalues as positive real numbers. The primal counter part of the FETI-DP algorithm turns out to be a two-level nonoverlapping Schwarz algorithm. In the previous studies of the Stokes problem [12, 13], two-level overlapping Schwarz preconditioners were employed and numerical results presented a good scalability. However, the analysis for the methods was left as a difficult task. The good scalability was explained due to the distribution of eigenvalues, which are all located at the right half of the complex plane.

In our approach, using that the resulting algebraic system has the same spectra as the algebraic system of the FETI-DP formulation, we are able to provide analysis for the convergence of the GMRES (Generalized Minimal Residual) method applied to the primal form with the preconditioner. The suggested preconditioner consists of solving independent local Stokes problems and solving one global coarse problem related to certain coarse velocity functions, which are related to primal velocity unknowns in the FETI-DP algorithm.

It is known that when applied to parallel processors, Chebyshev iteration outperforms Krylov subspace methods, such as CG (Conjugate Gradient) method and GMRES method, which require to compute the inner product of vectors in contrast to the Chebyshev iteration. The choice of parameters for the Chebyshev iteration can be done by estimating the extreme eigenvalues of the preconditioned primal form. Convergence of the Chebyshev iteration applied to the primal form is also provided.

In Section 2, we recall the FETI-DP algorithm developed in [8, 11] and in Section 3 we develop a primal counter part of the FETI-DP algorithm and investigate the relation between the two algorithms. Convergence analysis is carried out in Section 4 and numerical results are presented in Section 5. Throughout this paper, C denotes a generic positive constant which does not depend on any mesh parameters and the number of subdomains.

2. A FETI-DP algorithm for the Stokes problem without primal pressure unknowns. We introduce the FETI-DP algorithm developed in [8, 11]. We consider the Stokes problem,

$$\begin{aligned} -\triangle \boldsymbol{u} + \nabla p &= \boldsymbol{f} \ \text{ in } \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \ \text{ in } \Omega, \\ \boldsymbol{u} &= 0 \ \text{ on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a bounded polygonal (polyhedral) domain in $\mathbb{R}^2$ ($\mathbb{R}^3$) and $\boldsymbol{f} \in [L^2(\Omega)]^2$ ($[L^2(\Omega)]^3$). A triangulation is provided for the domain Ω and a pair of velocity and pressure space $(\widehat{X}, P)$ is equipped for the triangulation. Functions in the velocity space $\widehat{X}$ are continuous across the elements with zero trace on the boundary of Ω and those in the pressure space P are discontinuous. We enforce the average zero condition on the pressure space and denote the resulting pressure space by $\overline{P}$, i.e.,

$$\overline{P} = P \bigcap L_0^2(\Omega),$$

where $L_0^2(\Omega)$ is the space of square integrable functions that have zero average in Ω. We assume that the pair $(\widehat{X}, \overline{P})$ is inf-sup stable and obtain a discrete problem for (2.1):

Find $(\widehat{\boldsymbol{u}}, \overline{p}) \in (\widehat{X}, \overline{P})$ satisfying

$$\begin{aligned} &\int_\Omega \nabla \widehat{\boldsymbol{u}} \cdot \nabla \boldsymbol{v}\, dx - \int_\Omega \overline{p}\, \nabla \cdot \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx, \ \forall \boldsymbol{v} \in \widehat{X}, \\ &- \int_\Omega \nabla \cdot \widehat{\boldsymbol{u}}\, q\, dx = 0, \ \forall q \in \overline{P}. \end{aligned} \tag{2.2}$$

We now decompose Ω into a non-overlapping subdomain partition $\{\Omega_i\}_{i=1}^N$ in such a way that the subdomain boundaries align to the given triangulation in Ω. We introduce local finite element spaces,

$$X^{(i)} = \widehat{X}|_{\Omega_i} \text{ and } P^{(i)} = P|_{\Omega_i}.$$

In the product spaces X and P defined as

$$X = \prod_{i=1}^N X^{(i)} \text{ and } P = \prod_{i=1}^N P^{(i)},$$

the functions can be discontinuous across the subdomain boundaries. Among those unknowns in X, we select some unknowns on the subdomain interface as primal unknowns and enforce strong continuity to obtain $\widetilde{X}$, where functions can be discontinuous at the remaining part of the interface unknowns. We call the remaining part of unknowns the dual unknowns. The notations, $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)}$ are used to denote the unknowns at the interior part of $\Omega^{(i)}$, the dual unknowns on $\partial\Omega^{(i)}$, and the primal unknowns, respectively. The spaces $X_I^{(i)}$, $X_\Delta^{(i)}$, and $X_\Pi^{(i)}$ consist of the corresponding velocity unknowns, $\boldsymbol{u}_I^{(i)}$, $\boldsymbol{u}_\Delta^{(i)}$, and $\boldsymbol{u}_\Pi^{(i)}$, respectively. In this work, we select the primal unknowns which are the velocity values at the subdomain

corners in $2D$ and the velocity values at the subdomain corners and averages of the velocity over common faces in $3D$.

By enforcing the continuity at the dual unknowns with Lagrange multipliers $\boldsymbol{\lambda} \in \Lambda$, we obtain an equivalent discrete problem to (2.2):

Find $((\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi), \overline{p}, \boldsymbol{\lambda}) \in (\widetilde{X}, \overline{P}, \Lambda)$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & \overline{B}_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & \overline{B}_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & \overline{B}_\Pi^T & 0 \\ \overline{B}_I & \overline{B}_\Delta & \overline{B}_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \widehat{\boldsymbol{u}}_\Pi \\ \overline{p} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}, \tag{2.3}$$

where $\overline{B}_I$, $\overline{B}_\Delta$, and $\overline{B}_\Pi$ are matrices which arise from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx = 0, \quad \forall q \in \overline{P},$$

J_Δ is a Boolean matrix that represents jumps of dual velocity unknowns across the subdomain interface Γ_{ij},

$$J_\Delta \boldsymbol{u}_\Delta|_{\Gamma_{ij}} = \boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)},$$

and the other blocks, K_{II}, $K_{I\Delta}$, $K_{I\Pi}$, $K_{\Delta\Delta}$, $K_{\Delta\Pi}$, and $K_{\Pi\Pi}$ are from

$$\sum_i \int_{\Omega_i} \nabla \widetilde{\boldsymbol{u}} \cdot \nabla \widetilde{\boldsymbol{v}}\, dx.$$

To remove all the pressure unknowns by solving the independent local Stokes problems the pressure space $\overline{P}$ is replaced with P. Recall that P contains a constant pressure component. It gives an additional condition on $\widetilde{\boldsymbol{u}}$,

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx = 0, \quad q = c, \tag{2.4}$$

which is equivalent to

$$\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, c\, dx = c \sum_{ij} \int_{\Gamma_{ij}} (\boldsymbol{u}_\Delta^{(i)} - \boldsymbol{u}_\Delta^{(j)}) \cdot \boldsymbol{n}_{ij}\, ds = 0.$$

Here, Γ_{ij} is the common part of Ω_i and Ω_j, which is a common edge in $2D$ and a common face in $3D$, respectively. In the above identity, the following property of the primal unknowns is used for $\widetilde{\boldsymbol{u}}$,

$$\int_{\Gamma_{ij}} (\boldsymbol{u}_\Pi^{(i)} - \boldsymbol{u}_\Pi^{(j)}) \cdot \boldsymbol{n}_{ij}\, ds = 0, \quad \forall \Gamma_{ij}.$$

The additional condition (2.4) is in fact a linear sum of $J_\Delta \boldsymbol{u}_\Delta = 0$. By using the pressure space P instead of $\overline{P}$, we still obtain an equivalent algebraic system to (2.3):

Find $((\boldsymbol{u}_I, \boldsymbol{u}_\Delta, \widehat{\boldsymbol{u}}_\Pi), p, \boldsymbol{\lambda}) \in (\widetilde{X}, P, \Lambda)$ such that

$$\begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} & B_I^T & 0 \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} & B_\Delta^T & J_\Delta^T \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} & B_\Pi^T & 0 \\ B_I & B_\Delta & B_\Pi & 0 & 0 \\ 0 & J_\Delta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \widehat{\boldsymbol{u}}_\Pi \\ p \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ \boldsymbol{f}_\Pi \\ 0 \\ 0 \end{pmatrix}. \tag{2.5}$$

Here B_I, B_Δ, and B_Π are from

$$-\sum_i \int_{\Omega_i} \nabla \cdot \widetilde{\boldsymbol{u}}\, q\, dx = 0, \quad \forall q \in P,$$

and the other terms are the same as those in (2.3).

The unknowns $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ can then be eliminated by solving independent local Stokes problems,

$$\begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ p \end{pmatrix} = S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} \widehat{\boldsymbol{u}}_\Pi - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} \right), \tag{2.6}$$

where S is given by

$$S = \begin{pmatrix} K_{II} & K_{I\Delta} & B_I^T \\ K_{I\Delta}^T & K_{\Delta\Delta} & B_\Delta^T \\ B_I & B_\Delta & 0 \end{pmatrix}. \tag{2.7}$$

We note that S can be represented as diagonal blocks of local Stokes problems after appropriate ordering.

Substituting $(\boldsymbol{u}_I, \boldsymbol{u}_\Delta, p)$ into (2.5) and then solving for $\widehat{\boldsymbol{u}}_\Pi$

$$S_{\Pi\Pi} \widehat{\boldsymbol{u}}_\Pi = \boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} \right), \tag{2.8}$$

where

$$S_{\Pi\Pi} = K_{\Pi\Pi} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}, \tag{2.9}$$

we obtain the resulting algebraic system for $\boldsymbol{\lambda}$,

$$F_{DP} \boldsymbol{\lambda} = d, \tag{2.10}$$

where

(2.11)

$$F_{DP} = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}$$

and

$$d = \begin{pmatrix} 0 \\ J_\Delta^T \\ 0 \end{pmatrix}^T S^{-1} \left(\begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix} S_{\Pi\Pi}^{-1} \left(\boldsymbol{f}_\Pi - \begin{pmatrix} K_{I\Pi} \\ K_{\Delta\Pi} \\ B_\Pi \end{pmatrix}^T S^{-1} \begin{pmatrix} \boldsymbol{f}_I \\ \boldsymbol{f}_\Delta \\ 0 \end{pmatrix} \right) \right).$$

The resulting system on $\boldsymbol{\lambda}$ is symmetric and positive semidefinite. More precisely it has one null space component $\boldsymbol{\mu}_0$ which is given by

$$\boldsymbol{\mu}_0|_{\Gamma_{ij}} = \zeta_{ij} \boldsymbol{n}_{ij}, \ \forall \Gamma_{ij}. \tag{2.12}$$

Here, ζ_{ij} is

$$\zeta_{ij}(x_l) = \int_{\Gamma_{ij}} \phi_l(x(s), y(s), z(s))\, ds, \tag{2.13}$$

where ϕ_l is the nodal basis function related to the node x_l. For details, we refer [11, Section 2.2].

We now introduce a subspace of Λ, which is orthogonal to the null space of F_{DP},

$$\Lambda_c = \left\{ \boldsymbol{\mu} \in \text{Range}(J_\Delta) \subset \Lambda \ : \ \sum_{ij} \boldsymbol{\mu}|_{\Gamma_{ij}} \cdot \boldsymbol{\mu}_0|_{\Gamma_{ij}} = 0 \right\}.$$

Then F_{DP} is positive definite on Λ_c and the system in (2.10) is solved by the conjugate gradient method with a lumped preconditioner of the form,

$$M^{-1} = J_\Delta D_\Delta K_{\Delta\Delta} D_\Delta J_\Delta^T, \tag{2.14}$$

where D_Δ is a diagonal matrix with certain weight factors.

In [11], we proved the condition number bound,

$$\kappa(M^{-1} F_{DP})) \leq C \frac{H}{h} \left(1 + \log \frac{H}{h} \right),$$

which determines the convergence of the conjugate gradient iteration. The same bound was proved to be optimal for the FETI-DP algorithm of the elliptic problems with a lumped preconditioner, see [22]. In more detail, the minimum eigenvalue of $M^{-1} F_{DP}$ is bounded below by some positive number which is independent of any mesh parameters and the maximum eigenvalue follows the growth of $(H/h)(1 + \log(H/h))$. We also note that a bound for

the maximum eigenvalue, $C(H/h)$, is obtained for the three dimensional case by employing additional primal velocity unknowns which are averages of each velocity component on common faces, see [8].

In the FETI-DP algorithm, fully redundant Lagrange multipliers are commonly used for treating the continuity of the dual unknowns across the subdomain interfaces in three dimensional problems. The implementation of the continuity can be done by considering only two subdomains so that the fully redundant Lagrange multipliers make the implementation simpler and easier. For the three dimensional Stokes problem, two types of null space components appear in our approach. One is by introducing the fully redundant Lagrange multipliers and the other, which is given in (2.12), is by adding the constant pressure component.

The elimination of these null space components requires to find additional null space components and it makes the algorithm less practical for the three dimensional problem. This becomes a motivation of developing the primal counter part of the FETI-DP algorithm. As we will see, without the use of Lagrange multipliers the primal form provides a much simpler algorithm with as a good scalability as the FETI-DP algorithm.

3. The primal counterpart to the FETI-DP algorithm. In this section, we will derive the primal counterpart of the FETI-DP algorithm. We recall (2.5) and let

$$\widetilde{K} = \begin{pmatrix} K_{II} & K_{I\Delta} & K_{I\Pi} \\ K_{I\Delta}^T & K_{\Delta\Delta} & K_{\Delta\Pi} \\ K_{I\Pi}^T & K_{\Delta\Pi}^T & K_{\Pi\Pi} \end{pmatrix}, \quad \widetilde{\boldsymbol{u}} = \begin{pmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_\Delta \\ \widehat{\boldsymbol{u}}_\Pi \end{pmatrix},$$

$$\widetilde{B} = \begin{pmatrix} B_I & B_\Delta & B_\Pi \end{pmatrix}, \quad \widetilde{J}_\Delta = \begin{pmatrix} 0 & J_\Delta & 0 \end{pmatrix}.$$

We then rewrite (2.5) into

$$\begin{pmatrix} \widetilde{K} & \widetilde{B}^T & \widetilde{J}_\Delta^T \\ \widetilde{B} & 0 & 0 \\ \widetilde{J}_\Delta & 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{u}} \\ p \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widetilde{\boldsymbol{f}} \\ 0 \\ 0 \end{pmatrix}.$$

By eliminating the unknowns $(\widetilde{\boldsymbol{u}}, p)$ in the above, we obtain the system on the dual unknowns $\boldsymbol{\lambda}$,

$$\begin{pmatrix} \widetilde{J}_\Delta & 0 \end{pmatrix} \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{J}_\Delta^T \\ 0 \end{pmatrix} \boldsymbol{\lambda} = d.$$

In fact, we have the identity

$$F_{DP} = \begin{pmatrix} \widetilde{J}_\Delta & 0 \end{pmatrix} \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{J}_\Delta^T \\ 0 \end{pmatrix}.$$

We note that the lumped preconditioner can be rewritten as

$$M^{-1} = J_\Delta D_\Delta K_{\Delta\Delta} D_\Delta J_\Delta^T = \begin{pmatrix} \widetilde{J}_\Delta D & 0 \end{pmatrix} \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix} \begin{pmatrix} D\widetilde{J}_\Delta^T \\ 0 \end{pmatrix}, \tag{3.1}$$

where the diagonal matrix D has its value as D_Δ at the location of the dual unknowns and as one at the other part of the unknowns. The values of D_Δ are given by

$$D_\Delta(x) = \frac{1}{\nu(x)},$$

where x is a node related to dual velocity unknowns and $\nu(x)$ is the number of subdomains sharing the node x.

We now develop a primal form of the Stokes problem which is closely related to the FETI-DP algorithm in the previous section. Let $\widetilde{R}$ be an extension from $\widehat{X}$ to $\widetilde{X}$, i.e.,

$$\widetilde{R} \ : \ \widehat{X} \to \widetilde{X}.$$

The Stokes problem discretized by the pair $(\widehat{X}, P)$ then becomes

$$\begin{pmatrix} \widehat{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\boldsymbol{u}} \\ p \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{f}} \\ 0 \end{pmatrix}, \tag{3.2}$$

where

$$\begin{pmatrix} \widehat{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{R}^T & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{R} & 0 \\ 0 & I \end{pmatrix}.$$

Let

$$\widehat{A} = \begin{pmatrix} \widehat{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{pmatrix} \ \text{ and } \ \widetilde{A} = \begin{pmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{pmatrix}.$$

We note that since $P(= \prod_{i=1}^N P^{(i)})$ has a constant pressure component, $\widehat{A}$ has one null space component, which is given by

$$\text{Null}(\widehat{A}) = \left\{ (\boldsymbol{v}, q) \in (\widehat{X}, P) \ : \ \boldsymbol{v} = 0 \text{ and } q = c \text{ for any constant } c \right\}.$$

We propose a preconditioner M_p^{-1} for the primal form $\widehat{A}$,

$$M_p^{-1} = \begin{pmatrix} \widetilde{R}^T D & 0 \\ 0 & I \end{pmatrix} \widetilde{A}^{-1} \begin{pmatrix} D\widetilde{R} & 0 \\ 0 & I \end{pmatrix},$$

where the matrix D is the one in (3.1). The preconditioner consists of solving independent local Stokes problems and solving one global coarse problem, i.e.,

$$M_p^{-1} = \sum_i (R_\Delta^{(i)})^T \widetilde{D}_\Delta^{(i)} (S_{\Delta\Delta}^{(i)})^{-1} \widetilde{D}_\Delta^{(i)} R_\Delta^{(i)} + R_0^T S_{\Pi\Pi}^{-1} R_0.$$

Here $R_\Delta^{(i)}$ is the restriction from $\widehat{X} \times P$ to $X_\Delta^{(i)} \times P_i$ and R_0 is the matrix with its columns as coefficient vectors of coarse basis functions. The matrix $S_{\Delta\Delta}^{(i)}$ is the algebraic system of the local Stokes problem in Ω_i and $\widetilde{D}_\Delta^{(i)}$ is a diagonal matrix with the weight factors equal to D_Δ at the location of dual velocity unknowns and equal to one at the other part. The explicit form of the coarse problem $S_{\Pi\Pi}$ is given in (2.9) and only the primal velocity unknowns are selected for the construction of the coarse problem. The preconditioner M_p^{-1}, applied to the primal algebraic system in (3.2), results in a two-level nonoverlapping Schwarz algorithm [29, Chapter 2].

We now investigate connection between the two algorithms. We have that

$$M^{-1}F_{DP} = \begin{pmatrix} \widetilde{J}_\Delta D & 0 \end{pmatrix} \widetilde{A} \begin{pmatrix} D\widetilde{J}_\Delta^T \\ 0 \end{pmatrix} \begin{pmatrix} \widetilde{J}_\Delta & 0 \end{pmatrix} \widetilde{A}^{-1} \begin{pmatrix} \widetilde{J}_\Delta^T \\ 0 \end{pmatrix} \tag{3.3}$$

and

$$M_p^{-1}\widehat{A} = \begin{pmatrix} \widetilde{R}^T D & 0 \\ 0 & I \end{pmatrix} \widetilde{A}^{-1} \begin{pmatrix} D\widetilde{R} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{R}^T & 0 \\ 0 & I \end{pmatrix} \widetilde{A} \begin{pmatrix} \widetilde{R} & 0 \\ 0 & I \end{pmatrix}.$$

We introduce

$$G_d = \begin{pmatrix} \widetilde{J}_\Delta^T \\ 0 \end{pmatrix} \begin{pmatrix} \widetilde{J}_\Delta D & 0 \end{pmatrix} \widetilde{A} \begin{pmatrix} D\widetilde{J}_\Delta^T \\ 0 \end{pmatrix} \begin{pmatrix} \widetilde{J}_\Delta & 0 \end{pmatrix} \widetilde{A}^{-1}$$

and

$$G_p = \begin{pmatrix} \widetilde{R} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{R}^T D & 0 \\ 0 & I \end{pmatrix} \widetilde{A}^{-1} \begin{pmatrix} D\widetilde{R} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{R}^T & 0 \\ 0 & I \end{pmatrix} \widetilde{A}.$$

Let ϕ be an eigenvector of the matrix $M^{-1}F_{DP}$ with the nonzero eigenvalue μ, i.e.,

$$M^{-1}F_{DP}\phi = \mu\phi.$$

By multiplying the both sides of the above matrix equation with $\begin{pmatrix} \widetilde{J}_\Delta & 0 \end{pmatrix}^T$, we can see that the two matrices G_d and $M^{-1}F_{DP}$ have the same eigenvalues except the eigenvalue zero. The same result holds for the matrices G_p and $M_p^{-1}\widehat{A}$. For details, we refer [21] where similar calculation was carried out.

LEMMA 3.1. *The operators G_d and $M^{-1}F_{DP}$ have the same nonzero eigenvalues. The same holds for the operators G_p and $M_p^{-1}\widehat{A}$.*

In the following, we will show that G_d and G_p have the same eigenvalues except the eigenvalues zero and one. As a result, $\widehat{M}^{-1}F_{DP}$ and $\widehat{M}_p^{-1}\widehat{A}$ have the same eigenvalues except the eigenvalues zero and one.

LEMMA 3.2. *The operators G_d and G_p have the same eigenvalues except the eigenvalues zero and one.*

Proof. We introduce two operators defined on $\widetilde{X}$,

$$P_D = D\widetilde{J}_\Delta^T \widetilde{J}_\Delta \text{ and } E_D = \widetilde{R}\widetilde{R}^T D.$$

Let $\nu(x)$ be the number of subdomains which contain the node x. Since $D(x) = 1/\nu(x)$ for the nodes x associated with the dual unknowns and $D(x) = 1$ at the other nodes x, we have that

$$P_D + E_D = I. \tag{3.4}$$

Moreover P_D and E_D are projections which satisfy

$$P_D^2 = P_D,\ E_D^2 = E_D, \text{ and } E_D P_D = P_D E_D = 0. \tag{3.5}$$

By using P_D and E_D, we define the two projections defined on $\widetilde{X} \times P$,

$$\widetilde{P}_D = \begin{pmatrix} P_D & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{E}_D = \begin{pmatrix} E_D & 0 \\ 0 & I \end{pmatrix}.$$

It can be seen easily that $\widetilde{P}_D$ and $\widetilde{E}_D$ also have those properties of P_D and E_D in (3.4) and (3.5). We then rewrite G_d and G_p into

$$G_d = \widetilde{P}_D^T \widetilde{A} \widetilde{P}_D \widetilde{A}^{-1} \text{ and } G_p = \widetilde{E}_D \widetilde{A}^{-1} \widetilde{E}_D^T \widetilde{A}.$$

For the matrix G_d, let ϕ be an eigenvector with a nonzero eigenvalue μ, i.e.,

$$\widetilde{P}_D^T \widetilde{A} \widetilde{P}_D \widetilde{A}^{-1} \phi = \mu\phi. \tag{3.6}$$

Let

$$\psi = \widetilde{E}_D \widetilde{A}^{-1} \phi.$$

When $\psi \neq 0$, by using the properties of $\widetilde{P}_D$ and $\widetilde{E}_D$ we can prove that ψ is an eigenvector of G_p with the eigenvalue μ, i.e.,

$$G_p \psi = \mu\psi.$$

When $\psi = 0$, from $\widetilde{E}_D = I - \widetilde{P}_D$ we have that

$$\psi = \widetilde{E}_D \widetilde{A}^{-1} \phi = (I - \widetilde{P}_D) \widetilde{A}^{-1} \phi = 0.$$

We then obtain

$$\widetilde{P}_D \widetilde{A}^{-1} \phi = \widetilde{A}^{-1} \phi.$$

From this identity it follows that

$$\begin{aligned} G_d\phi &= \widetilde{P}_D^T \widetilde{A}(\widetilde{P}_D \widetilde{A}^{-1}\phi) \\ &= \widetilde{P}_D^T \widetilde{A}(\widetilde{A}^{-1}\phi) \\ &= \widetilde{P}_D^T \phi. \end{aligned}$$

Since ϕ satisfies (3.6), ϕ is in the range space of $\widetilde{P}_D^T$. We thus obtain

$$G_d\phi = \phi,$$

which shows that the eigenvalue μ in (3.6) is equal to one.

The same argument holds for G_p by selecting ϕ as an eigenvector with the nonzero eigenvalue μ for the matrix G_p,

$$G_p\phi = \mu\phi. \tag{3.7}$$

In this case, we select $\psi = \widetilde{P}_D^T \widetilde{A}\phi$ and proceed in the same manner as before to show that when ψ is nonzero it is also an eigenvector of G_d with the eigenvalue μ. When the vector ψ becomes zero, the eigenvalue μ in (3.7) is one.

As a result, we proved that the two operators G_d and G_p have the same eigenvalues except the values zero and one. □

From Lemmas 3.1 and 3.2 we obtain the following result:

THEOREM 3.3. *The two operators $M^{-1}F_{DP}$ and $M_p^{-1}\widehat{A}$ have the same eigenvalues except the eigenvalues zero and one.*

Let λ_{min} and λ_{max} be the minimum and the maximum eigenvalues of a given matrix A with all its eigenvalues as real numbers,

$$Ax = \lambda x.$$

For such a matrix A, even when A is not symmetric, we will use the notation

$$c \leq A \leq C,$$

to denote that all eigenvalues of the matrix A are located in the interval $[c, C]$.

It has been shown that $M^{-1}F_{DP}$ is positive semidefinite. When the operator is restricted to the subspace Λ_c, the minimum eigenvalue is determined by a constant independent of any mesh parameters and the maximum eigenvalue follows the growth of $(H/h)(1 + \log(H/h))$ in two dimensions and H/h in three dimensions; see [8, 11].

LEMMA 3.4. *The FETI-DP algorithm for the two-dimensional Stokes problem with velocity unknowns at subdomain corners as primal unknowns gives*

$$c \leq M^{-1}F_{DP} \leq C\frac{H}{h}\left(1 + \log\frac{H}{h}\right),$$

and the same algorithm for the three-dimensional problem with velocity unknowns at subdomain corners and velocity averages over common faces as primal unknowns gives

$$c \leq M^{-1}F_{DP} \leq C\frac{H}{h},$$

when the operator $M^{-1}F_{DP}$ is restricted to the subspace Λ_c.

By Theorem 3.3 the same bound holds for $M_p^{-1}\widehat{A}$ on the subspace of $\widehat{X} \times P$, where $M_p^{-1}\widehat{A}$ is nonsingular. In fact, the operator $M_p^{-1}\widehat{A}$ is nonsingular on $\widehat{X} \times \overline{P}$, where

$$\overline{P} = \left\{ q \in P \,:\, \int_\Omega q\, dx = 0 \right\}.$$

THEOREM 3.5. *The primal algorithm with the preconditioner M_p^{-1} satisfies*

$$c \leq M_p^{-1}\widehat{A} \leq C\frac{H}{h}\left(1 + \log\frac{H}{h}\right)$$

for the two-dimensional problem and

$$c \leq M_p^{-1}\widehat{A} \leq C\frac{H}{h}$$

for the three-dimensional problem when the operator is restricted to the subspace $\widehat{X} \times \overline{P}$.

REMARK 3.6. *The resulting primal algorithm can be seen as a two-level nonoverlapping Schwarz algorithm for the Stokes problem discretized with the finite element space $(\widehat{X}, P)$. The preconditioner consists of local Stokes problems obtained from each subdomain finite element spaces $(X_{I,\Delta}^{(i)}, P^{(i)})$ and one global coarse problem related to the primal velocity unknowns. The null space component of the discrete problem $\widehat{A}$ can be removed by projecting the pressure unknowns onto the subspace $\overline{P}$. The elimination of the null space component in the primal form is much simpler than the FETI-DP algorithm in the three dimensions.*

4. Convergence of the primal algorithm. Since both M_p^{-1} and $\widehat{A}$ are indefinite, we can not apply the conjugate gradient method to the resulting primal algebraic system. We will solve the system with the GMRES (Generalized Minimal Residual) method and provide its convergence rate by using the bound of eigenvalues of the matrix $M_p^{-1}\widehat{A}$.

For the sake of simplicity, we let

$$S = M_p^{-1}\widehat{A}.$$

The matrix S is then nonsymmetric and has all its eigenvalues as real numbers with the bound

$$\alpha \leq S \leq \beta,$$

where α is a constant independent of any mesh parameters and β is bounded by $C(H/h)(1 + \log(H/h))$ or $C(H/h)$. To solve the primal form of the Stokes problem

$$\widehat{A}x = b,$$

we apply the GMRES method to

$$Sx = M_p^{-1} b. \tag{4.1}$$

We introduce $\|v\|_2$ to denote the l_2-norm of a vector v.

Algorithm *(GMRES iteration in [27])*

Step 1. Let x_0 be the initial guess and TOL be given.

$r_0 = M_p^{-1}(b - \widehat{A}x_0)$

$v_1 = r_0/\|r_0\|_2$

Step 2. Iterate:

$k = 0$

while ($\|r_k\|_2/\|r_0\|_2 >$ *TOL)*

$k = k + 1$

$h_{jk} = \langle M_p^{-1}\widehat{A}v_k, v_j\rangle,\ j = 1, \cdots, k.$

$\widehat{v}_{k+1} = M_p^{-1}\widehat{A}v_k - \sum_{j=1}^{k} h_{jk}v_j$

$h_{k+1,k} = \|\widehat{v}_{k+1}\|_2$

$v_{k+1} = \widehat{v}_{k+1}/h_{k+1,k}$

Compute $\|r_k\|_2 := \|M_p^{-1}(b - \widehat{A}x_k)\|_2$,

where

$x_k = x_0 + V_k y_k$ *with* $y_k = H_k^{-1}\|r_0\|_2 e_1$.

end

Here, V_k is a matrix with columns $v_1, \cdots, v_k$, H_k is a matrix with entries $(H_k)_{ij} = h_{ij}$ with $1 \le i, j \le k$, and e_1 is the vector in $\mathbb{R}^k$ such that $e_1 = (1, 0, \cdots, 0)^T$. During the GMRES iteration, it is not necessary to solve for y_k to compute the current residual norm $\|r_k\|_2$. We can compute the residual norm $\|r_k\|_2$ by multiplying certain rotation matrices to the matrix $\widetilde{H}_k((\widetilde{H}_k)_{ij} := h_{ij}) \in \mathbb{R}^{(k+1)\times k}$, see [28, Section 3.2] for more detail.

To analyze the convergence of the GMRES method applied to our problem, we refer some important results provided in [27, Chapter 6]. Let x_k be the k-th iterate of the GMRES method. The k-th residual is then defined by

$$r_k = M_p^{-1} b - S x_k.$$

We introduce $C_k(t)$, the Chebyshev polynomial of degree k,

$$C_k(t) = \cosh(k \cosh^{-1}(t)),$$

and $\mathbb{P}_k$, the set of polynomials of degree at most k.

LEMMA 4.1. *Let $[\alpha, \beta]$ be a nonempty interval in $\mathbb{R}$ and let γ be any real scalar outside the interval $[\alpha, \beta]$. Then the minimum*

$$\min_{p\in\mathbb{P}_k,\ p(\gamma)=1} \max_{t\in[\alpha,\beta]} |p(t)|$$

is achieved by the polynomial

$$\widehat{C}_k(t) = \frac{C_k(1+2\frac{\alpha-t}{\beta-\alpha})}{C_k(1+2\frac{\alpha-\gamma}{\beta-\alpha})}.$$

For a matrix X, let $\|X\|_2$ denote l^2-norm of the matrix X.

LEMMA 4.2. *Let the Jordan form of S be given by*

$$S = XJX^{-1}.$$

The residual norm of the GMRES method applied to (4.1) *is bounded by*

$$\|r_k\|_2 \leq \kappa_2(X) \min_{p\in\mathbb{P}_k,\, p(0)=1} \|p(J)\|_2 \|r_0\|_2,$$

where

$$\kappa_2(X) = \|X\|_2\|X^{-1}\|_2.$$

When S is diagonalizable, the matrix J is diagonal with its diagonal entries as the eigenvalues of S. For this case we have the following convergence result by using Lemmas 4.1 and 4.2. We include the proof for its later use in convergence result for a more general S.

THEOREM 4.3. *Assume that S is diagonalizable and α and β be the minimum and the maximum eigenvalues of S. The convergence is then determined by*

$$\|r_k\|_2 \leq 2\epsilon^k \kappa_2(X)\|r_0\|_2,$$

where

$$\epsilon = \frac{\sqrt{\beta/\alpha}-1}{\sqrt{\beta/\alpha}+1}.$$

Proof. Since S is diagonalizable, i.e., J is a diagonal matrix,

$$\|p(J)\|_2 \leq \max_{t\in[\alpha,\beta]} |p(t)|.$$

From the above bound and Lemmas 4.1 and 4.2, we obtain

$$\|r_k\|_2 \leq \|X\|_2\|X^{-1}\|_2\|r_0\|_2 \max_{t\in[\alpha,\beta]} \widehat{C}_k(t),$$

where

$$\widehat{C}_k(t) = \frac{C_k(1+2\frac{\alpha-t}{\beta-\alpha})}{C_k(1+2\frac{\alpha-\gamma}{\beta-\alpha})} \text{ with } \gamma = 0. \tag{4.2}$$

Since the maximum value of $C_k(1+2\frac{\alpha-t}{\beta-\alpha})$ is one for $t \in [\alpha,\beta]$,

$$\max_{t\in[\alpha,\beta]} \widehat{C}_k(t) = \frac{1}{C_k(1+2\frac{\alpha}{\beta-\alpha})}.$$

From

$$C_k(t) \geq \frac{1}{2}\left(t+\sqrt{t^2-1}\right)^k, \quad |t| \geq 1, \tag{4.3}$$

we obtain

$$\max_{t\in[\alpha,\beta]} \widehat{C}_k(t) \leq 2\left(\eta+\sqrt{\eta^2-1}\right)^{-k}, \quad \eta = 1+2\frac{\alpha}{\beta-\alpha}. \tag{4.4}$$

The right hand side of the above bound is further reduced to

$$2\left(\frac{\sqrt{\beta/\alpha}-1}{\sqrt{\beta/\alpha}+1}\right)^k$$

which provides the convergence rate of the GMRES method. □

REMARK 4.4. *For a general case, the matrix J is in the form*

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix},$$

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}_{d_i\times d_i}$$

is the Jordan block associated to the eigenvalue λ_i with its invariant subspace of dimension d_i. In the case,

$$p(J) = \begin{pmatrix} p(J_1) & & \\ & \ddots & \\ & & p(J_k) \end{pmatrix}$$

with

$$p(J_i) = \begin{pmatrix} p(\lambda_i) & p'(\lambda_i) & \frac{1}{2!}p''(\lambda_i) & \cdots & \frac{1}{(d_i-1)!}p^{(d_i-1)}(\lambda_i) \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{1}{2!}p''(\lambda_i) \\ & & & \ddots & p'(\lambda_i) \\ & & & & p(\lambda_i) \end{pmatrix}.$$

The following results are provided in [25, Theorem 2.2]:

LEMMA 4.5. *If λ_i is an eigenvalue of S with its invariant subspace of dimension d_i, then $\|p_k(S)\|_2 \to 0$ as $k \to \infty$ if and only if $p_k^{(j)}(\lambda_i) \to 0$ as $k \to \infty$ for every $j < d_i$, for each eigenvalue λ_i, where k is the degree of the polynomial $p_k(x)$.*

We note that $\{\widehat{C}_k(t)\}_k$ in (4.2) is a sequence of polynomials which converge to zero uniformly on $[\alpha, \beta]$, see (4.4). Therefore its derivatives also converge to zero and in more detail

$$|\widehat{C}_k^{(j)}(t)| \le K(t,j)k^j\epsilon^k \text{ for large } k, \tag{4.5}$$

where $K(t,j)$ is a constant depending only on t and j. This bound can be proved by the following result in [25, Section 2.4]:

$$|\overline{C}_k^{(j)}(z)| \le K(z,j)k^j r(z)^k, \tag{4.6}$$

where

$$r(z) = \lim_{k\to\infty} |\overline{C}_k(z)^{1/k}|$$

and

$$\overline{C}_k(z) = \frac{C_k((c-z)/d)}{C_k(c/d)} \text{ with } c > d > 0.$$

Here the Chebyshev polynomial C_k is extended to the complex variable z. We note that

$$\max_{z\in E(c,d,a)} \overline{C}_k(z) = \frac{C_k(a/d)}{C_k(c/d)},$$

where $E(c,d,a)$, $c > a > d$, is an ellipse centered at c with foci at $c-d$ and $c+d$ and with semimajor axis a. Since all the eigenvalues of S is located in $[\alpha,\beta]$, by selecting

$$c = \frac{\alpha+\beta}{2}, a = \frac{\beta-\alpha}{2},$$

the interval $[\alpha,\beta]$ is contained in the ellipse $E(c,d,a)$ for any $d < a$. Letting $d \to a$, we can obtain the bound

$$\lim_{d\to a} \max_{z\in E(c,d,a)} \overline{C}_k(z) = \lim_{d\to a} \frac{C_k(a/d)}{C_k(c/d)} \doteq \lim_{d\to a} \frac{a+\sqrt{a^2-d^2}}{c+\sqrt{c^2-d^2}} = \epsilon^k$$

for sufficiently large k, and

$$\widehat{C}_k^{(j)}(t) = \lim_{d\to a} \overline{C}_k^{(j)}(t),\ t \in [\alpha,\beta].$$

The inequality in (4.6) combined with the above two identities gives (4.5).

THEOREM 4.6. *Assume that S is not diagonalizable. The GMRES method converges with its convergence determined by $\epsilon = \frac{\sqrt{\beta/\alpha}-1}{\sqrt{\beta/\alpha}+1}$ and the dimension d_i of the invariant subspace of each eigenvalue λ_i.*

Proof. Since $\widehat{C}_k(t) \leq 2\epsilon^k$ for all $t \in [\alpha, \beta]$ with $\epsilon = \frac{\sqrt{\beta/\alpha}-1}{\sqrt{\beta/\alpha}+1}$, by Lemma 4.5 and (4.5) the GMRES method converges for the general case of S. The convergence is determined by ϵ and d_i. □

REMARK 4.7. *In our numerical experiments, the convergence rate of GMRES method is observed to follow ϵ, see Section 6, even though the preconditioned system $M_p^{-1}\widehat{A}_p$ is not known to be diagonalizable.*

5. Chebyshev iteration. Since all the eigenvalues of S are real, when the extreme eigenvalues are estimated, we suggest Chebyshev iteration for solving

$$M_p^{-1}\widehat{A}x = M_p^{-1}b. \tag{5.1}$$

The Chebyshev iteration provides an algorithm which is more suitable for parallel processors, since it does not require to compute inner product of vectors. As we will see in the following, convergence of the Chebyshev iteration with the optimal choice of parameters is as competitive as that of the GMRES iteration.

Let α and β be the estimates for the smallest and the largest eigenvalues of $M_p^{-1}\widehat{A}$, respectively. Let $\gamma = \frac{2}{\beta+\alpha}$ and $\mu = \frac{\beta+\alpha}{\beta-\alpha}$. Let constant c_k be the value of the k-th Chebyshev polynomial evaluated at μ. Then we have the following recurrence relation

$$c_{k+1} = 2\mu c_k - c_{k-1},\ k \geq 1,$$

with $c_0 = 1$ and $c_1 = \mu$. In the following we refer the Chebyshev iteration introduced in [5, 25].

Algorithm *(Chebyshev iteration with γ, μ, $\{c_k\}_k$)*

Step 1. Let x_0 be the initial guess and TOL be given.

$r_0 = b - \widehat{A}x_0,\ z_0 = M_p^{-1}r_0$

$x_1 = x_0 + \gamma z_0$

Step 2. Iterate:

$k = 0$

while ($\|r_k\|_2/\|r_0\|_2 >$ TOL)

$k = k + 1$

$r_k = b - \widehat{A}x_k,\ z_k = M_p^{-1}r_k$

$\omega_{k+1} = 2\mu\frac{c_k}{c_{k+1}}$

$x_{k+1} = x_{k-1} + \omega_{k+1}(\gamma z_k + x_k - x_{k-1})$

end

We now provide convergence of the Chebyshev iteration. Let

$$e_k = x - x_k \text{ and } S = M_p^{-1}\widehat{A}.$$

Using the Jordan form of $S = XJX^{-1}$, we obtain the error formula, see [5],

$$e_k = X\frac{C_k(\mu J)}{c_k}X^{-1}e_0. \tag{5.2}$$

The error formula combined with $\alpha \leq S \leq \beta$ gives the convergence of the Chebyshev iteration, of which proof is similar to that of the GMRES iteration in the previous section:

THEOREM 5.1. *Let the matrix $S(= M_p^{-1}\widehat{A}_p)$ in* (5.1) *satisfy*

$$\alpha \leq S \leq \beta,$$

and the parameters of the Chebyshev iteration be given by

$$\mu = \frac{\beta+\alpha}{\beta-\alpha} \text{ and } \gamma = \frac{2}{\beta+\alpha}.$$

When S is diagonalizable, the Chebyshev iteration applied to (5.1) *with the parameters μ and γ converges with the error*

$$\|e_k\|_2 \leq C\kappa_2(X)\epsilon^k\|e_0\|_2,$$

where the rate ϵ is given by

$$\epsilon = \frac{\sqrt{\beta/\alpha}-1}{\sqrt{\beta/\alpha}+1},$$

and X is the matrix which appears in the Jordan form of S,

$$S = XJX^{-1}.$$

For a general case of S, the Chebyshev iteration converges and the convergence is determined by ϵ and the dimension of invariant subspaces in the Jordan form of S.

6. Numerical results. We consider the Stokes problem in $\Omega = [0,1]^3$ with the exact solution

$$\boldsymbol{u} = \mathbf{0}, \quad p = xyz - \frac{1}{8}.$$

The domain is uniformly partitioned into cubic subdomains. The number of subdomains $N_d = 3^3$ means that the domain is divided into three subdomains in each x, y, and z-directions. For a given mesh size h, we partition each subdomain into small cubes of its side length h and then divide each cubes into two prism elements. We denote by H the diameter of each subdomain and by H/h the number of cubes in each edges of the subdomain. We introduce a pair of velocity and pressure finite element spaces associated to the prism elements. Velocity basis functions are piecewise linear in each prism element and pressure basis functions are constant in each cube consisting of two prism elements. This pair of velocity and pressure finite element spaces was shown to be inf-sup stable, see [18]. We solve the

	Primal form				FETI-DP form			
H/h	Iter	$\frac{-6\log 10}{\log \epsilon}$	$\lambda_{\min}$	$\lambda_{\max}$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$
6	18	18	0.252	2.002	17	8.428	0.237	2.002
8	20	21	0.246	2.593	19	10.225	0.253	2.592
10	22	23	0.267	3.250	21	12.158	0.267	3.250
12	25	26	0.258	3.985	24	14.935	0.266	3.985
14	27	28	0.262	4.743	25	17.697	0.268	4.743
16	29	31	0.260	5.519	27	20.216	0.273	5.519

TABLE 1

Performance as increase of the local problem size H/h in a fixed subdomain partition with $N_d = 3^3$. Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, and λ_{max}: the maximum eigenvalue.

Stokes problem employing the proposed preconditioner M_p^{-1} with iterative methods. The iteration is performed up to the relative residual norm reduced by a factor of 10^6.

In Table 1, performances of two algorithms are compared. The number of iterations and estimated extreme eigenvalues are presented with respect to the local problem size H/h in a fixed subdomain partition $N_d = 3^3$. The extreme eigenvalues are approximated by computing those of the upper Hessenberg matrix obtained from the GMRES iteration. Even though $S(= M_p^{-1}\widehat{A})$ is not known to be diagonalizable, we compute the rate of convergence, $\epsilon = \frac{\sqrt{\beta/\alpha}-1}{\sqrt{\beta/\alpha}+1}$ by using the estimated extreme eigenvalues, $\alpha = \lambda_{min}$ and $\beta = \lambda_{max}$. The computed rate of convergence of the primal form is used to predict the number of iterations, which is required to reduce the relative residual norm by a factor of 10^6,i.e.,

$$\frac{-6\log 10}{\log \epsilon}.$$

The predicted number of iterations agrees quite well to the true ones. The extreme eigenvalues and iteration count from the FETI-DP algorithm, which is described in Section 2, are also presented. They are almost identical to those from the primal form. The FETI-DP algorithm results in a symmetric and positive definite system which is solved by the preconditioned conjugate gradient method with the lumped preconditioner.

In Table 2, scalability of the two methods is presented regarding to the number of subdomains. We observe that primal form gives as scalable results as the FETI-DP form and the approximated rate of convergence in the GMRES iteration fits well to the required number of iterations.

In Tables 3 and 4, Chebyshev iteration is applied to solving the primal form. With the optimal choice of parameters, i.e., (α, β) as the estimated extreme eigenvalues, the Chebyshev iteration provides almost identical number of iterations to the GMRES iteration in Tables 1 and 2. With less optimal choice of parameters $(\alpha, \beta) = (0.2, 4.0)$ or $(0.2, 6.0)$, which is

	Primal form				FETI-DP form			
N_d	Iter	$\frac{-6\log 10}{\log \epsilon}$	$\lambda_{\min}$	$\lambda_{\max}$	Iter	κ	$\lambda_{\min}$	$\lambda_{\max}$
3^3	20	21	0.246	2.593	19	10.225	0.253	2.592
4^3	21	23	0.242	2.850	20	11.230	0.253	2.850
5^3	22	23	0.240	2.987	21	11.844	0.252	2.987
6^3	23	24	0.238	3.067	21	12.290	0.249	3.067
7^3	23	24	0.238	3.117	21	12.487	0.249	3.116
8^3	23	24	0.238	3.148	22	12.598	0.250	3.149
9^3	23	24	0.238	3.171	22	12.786	0.248	3.171
10^3	23	24	0.237	3.186	22	12.906	0.247	3.188

TABLE 2

Performance as increase of the number of subdomains N_d with a local problem size $H/h = 8$. Iter: the number of iterations, κ: the condition number, λ_{min}: the minimum eigenvalue, and λ_{max}: the maximum eigenvalue.

	$(\alpha,\beta) = (\lambda_{min}, \lambda_{max})$	$(\alpha,\beta) = (0.2, 4.0)$	$(\alpha,\beta) = (0.2, 6.0)$
H/h	Iter	Iter	Iter
6	19	31	38
8	22	31	38
10	24	31	38
12	25	31	38
14	27	–	38
16	29	–	38

TABLE 3

Performance of the Chebyshev iteration depending on the choice of parameters α and β as increase of the local problem size H/h in a fixed subdomain partition with $N_d = 3^3$. Iter: the number of iterations.

chosen so that all the eigenvalues of S are contained in the interval $[\alpha, \beta]$, we still observe relatively good convergence as well as a good scalability.

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	$(\alpha,\beta)=(\lambda_{min},\lambda_{max})$	$(\alpha,\beta)=(0.2,4.0)$
N_d	Iter	Iter
3^3	22	31
4^3	23	30
5^3	24	30
6^3	24	30
7^3	24	30
8^3	24	30

TABLE 4

Performance of the Chebyshev iteration depending on the choice of parameters α and β as increase of the number of subdomains N_d with the local problem size $H/h = 8$. Iter: the number of iterations.

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