Gabor windows supported on [-1, 1] and dual windows with small support

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BK21 Research Report

10 - 05

March 31, 2010

DEPARTMENT OF MATHEMATICAL SCIENCES

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March 15, 2010

Abstract

Consider a continuous function $g \in L^2(\mathbb{R})$ that is supported on $[-1, 1]$ and generates a Gabor frame with translation parameter 1 and modulation parameter $0 < b < \frac{2N}{2N+1}$ for some $N \in \mathbb{N}$. Under an extra condition on the zeroset of the window g we show that there exists a continuous dual window supported on $[-N, N]$. We also show that this result is optimal: indeed, if $b > \frac{2N}{2N+1}$ then a dual window supported on $[-N, N]$ does not exist. In the limit case $b = \frac{2N}{2N+1}$ a dual window supported on $[-N, N]$ might exist, but cannot be continuous.

1 Introduction

Given a Gabor frame with compactly supported wondow, it is natural to ask whether a compactly supported dual window exists. Various results about this can be found in the literature. For example, in the case of rational oversampling a characterization of the cases where a compactly supported dual exists can be found in the paper [1] by Bölsckei and Janssen. For a window supported on $[-1, 1]$ the authors showed in [4] that a compactly supported dual window always exists.

If a compactly supported dual window exists, the size of the support is clearly important for practical applications. The purpose of this paper is to provide a detailed analysis of the necessary size of the support for the dual window. In particular, we will see that the length often can be shortened by a factor of two compared to previously known results.

We will consider a continuous function $g \in L^2(\mathbb{R})$ that is supported on $[-1, 1]$ and generates a Gabor frame with translation parameter 1 and modulation parameter $0 < b < \frac{2N}{2N+1}$ for some $N \in \mathbb{N}$. Under an extra condition on the zeroset of the window g we show that there exists a continuous dual window supported on $[-N, N]$. We also show that this relationship between the parameter b and the size of the support is the best one can hope for: indeed, if $b > \frac{2N}{2N+1}$ then a dual window supported on $[-N, N]$ does not exist. In the limit case $b = \frac{2N}{2N+1}$ a dual window supported on $[-N, N]$ might exist, but cannot be continuous. The proofs of these results are quite technical, so we provide small examples to illustrate the main ideas.

In the rest of the introduction we state a few well known definitions and some of the needed results from the literature. The new results are stated in Section 2, and all the proofs are in Section 3.

Let $g \in L^2(\mathbb{R})$ and consider the Gabor system $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ given by

$$E_{mb}T_n g(x) := e^{2\pi i m b x} g(x - n), \ x \in \mathbb{R}.$$

Recall that $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, E_{mb}T_n g\rangle|^2 \leq B\,||f||^2, \ \forall f \in L^2(\mathbb{R}).$$

If at least the upper frame condition is satisfied, $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a *Bessel sequence.* Given a frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$, a Bessel sequence $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$ is a *dual frame* if

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_n h\rangle E_{mb}T_n g, \quad \forall f \in L^2(\mathbb{R}).$$

The function g generating the frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is called the *window* and h is called a *dual window.* For more information about Gabor frames and their role in time-frequency analysis we refer to, e.g., [6] and [2].

A characterization of all pairs of dual Gabor frames was provided by Ron & Shen [10, 11] and Janssen [8]. We will only consider windows g and dual windows h having compact support. Specifying the size of the support of the function g and h leads to a characterization of the duality in terms of a finite collection of equations. Our starting point is the following result, which is a slightly reformulated version of Corollary 1.2 in [4]:

Proposition 1.1 *Let $b \in]0,1[$ and $N \in \mathbb{N}$. Assume that g and h are bounded and real-valued functions with* $\text{supp } g \subseteq [-1,1]$ *and* $\text{supp } h \subseteq [-N,N]$*, and that*

$$\sum_{k\in\mathbb{Z}} g(x+k)h(x+k) = b, \ \textit{a.e. } x \in [0,1]. \tag{1.1}$$

Then the conditions (i) – (ii) below are equivalent:

(i) $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_nh\}_{m,n\in\mathbb{Z}}$ *form dual frames for* $L^2(\mathbb{R})$*;*

(ii) *For* $n = \pm 1, \pm 2, \cdots, \pm N$,

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0, \ \textit{a.e. } x \in [\frac{n}{b} - 1, \frac{n}{b}]. \tag{1.2}$$

In this article we will consider windows g belonging to the following subset of $L^2(\mathbb{R})$:

$$V := \{f \in C(\mathbb{R}) \mid \text{supp } f = [-1,1], \ f \text{ has a finite number of zeros on } [-1,1]\}. \tag{1.3}$$

In Theorem 2.3 in [4] it is shown that if a function $g \in V$ generates a Gabor frame for some $b < 1$, then there exists a continuous dual window h with compact support. Furthermore, the size of the support of a possible choice of h can be estimated in terms of the size of b:

Proposition 1.2 *Let $N \in \mathbb{N} \setminus \{1\}$. Suppose that a function $g \in V$ generates a Gabor frame $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$. If $b < \frac{2N}{2N+1}$, then there exists a continuous dual window h with* $\text{supp } h \subseteq [-2N, 2N]$*.*

Note that Theorem 2.3 in [4] also contains a characterization of the functions $g \in V$ for which $\{E_{mb}T_ng\}_{m,n\in\mathbb{Z}}$ is a Gabor frame for a given $b < 1$.

Note that in principle one can consider the characterization of dual frames for windows g supported on a larger interval than $[-1,1]$. However, the number of equations to consider might be very large, and technically it is very difficult to deal with this. The long-time goal is to extend the calculations presented here to arbitrary compactly supported windows g.

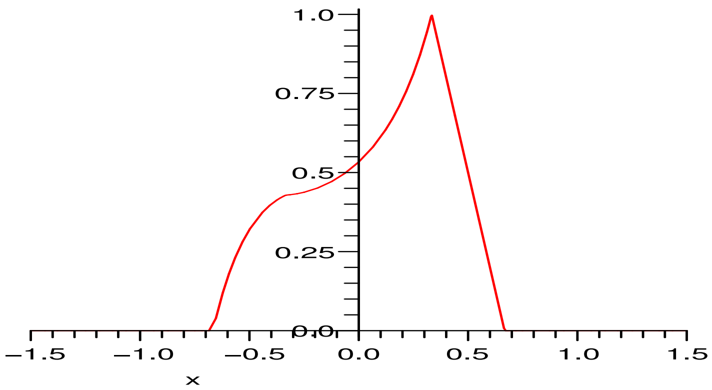

Figure 1: The function h in (2.2)

2 The main results

In this section we present the results. The proofs are quite technical (see Section 3), so we will provide small examples to illustrate the main ideas.

With Proposition 1.1 as starting point we will provide a closer analysis of the relationship between the value of the modulation parameter b and the necessary size of the support of the dual window. The following example motivates the analysis:

Example 2.1 Let $b := \frac{3}{5}$ and

$$g(x) := \frac{27}{20}(x+1)(x-\frac{5}{6})(x-1)\chi_{[-1,1]}(x).$$

By the results in [4] the function g generates a Gabor frame for $a = 1, b = 3/5$, and by Proposition 1.2 we can choose a dual window supported on $[-2, 2]$. We will now show that it actually is possible to find a dual window supported on $[-1, 1]$. In order to do so we check the conditions in Proposition 1.1 with $N = 1$. We consider (1.2) for $n = 1$ and $n = -1$, that is,

$$g(x-\frac{5}{3})h(x) + g(x-\frac{2}{3})h(x+1) = 0, \ x \in [\frac{2}{3}, \frac{5}{3}],$$
$$g(x+\frac{5}{3})h(x) + g(x+\frac{8}{3})h(x+1) = 0, \ x \in [-\frac{8}{3}, -\frac{5}{3}].$$

We see that these equations are satisfied if $h(x) = 0$ for $|x| \geq \frac{2}{3}$. Now, the equation (1.1) means that

$$g(x)h(x) + g(x+1)h(x+1) = \frac{3}{5}, \ x \in [-1, 0]. \tag{2.1}$$

A direct calculation shows that (2.1) is satisfied if we define h by

$$h(x) = \begin{cases} \frac{(x+\frac{2}{3})(18x^3+33x^2-3x+4)}{(x-1)(x+1)(6x-5)}, & x \in [-\frac{2}{3}, -\frac{1}{3}]; \\ \frac{4}{9(x+1)(x-\frac{5}{6})(x-1)}, & x \in [-\frac{1}{3}, \frac{1}{3}]; \\ -3(x-\frac{2}{3}), & x \in [\frac{1}{3}, \frac{2}{3}]. \end{cases} \tag{2.2}$$

Thus, h is a continuous dual window supported on $[-1, 1]$. □

The above considerations can be extended to a general result as follows.

Theorem 2.2 *Let $N \in \mathbb{N}$ and $b \in [\frac{N}{N+1}, \frac{2N}{2N+1}[$. Assume that a function $g \in V$ generates a Gabor frame $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$. Then there exists a continuous dual window h with* supp $h \subseteq [-N, N]$ *if and only if*

$$g(x) \neq 0, \ x \in [\frac{N}{b} - N - 1, -\frac{N}{b} + N + 1].$$

By Theorem 2.2 careful choices of the window g makes it possible to find a dual window h supported on $[-N, N]$ if $b < \frac{2N}{2N+1}$. This turns out to be almost optimal. In fact, we will show that if there exists a dual window supported on $[-N, N]$, then necessarily $b \leq \frac{2N}{2N+1}$:

Theorem 2.3 *Assume that g is a bounded function with* supp $g = [-1, 1]$. *If there exists a dual window h with* supp $h \subseteq [-N, N]$, *then* $0 < b \leq \frac{2N}{2N+1}$.

The following example illustrates Theorem 2.3.

Example 2.4 Let $b := \frac{3}{4}$ and

$$g(x) := (x+1)(x+\frac{3}{4})(x-\frac{3}{4})(x-1)\chi_{[-1,1]}(x). \tag{2.3}$$

By the results in [4] the system $\{E_{3m/4}T_n g\}_{m,n \in \mathbb{Z}}$ is a Gabor frame. Assume that there exists a dual window h supported on $[-1, 1]$. Consider (1.2) for

$n = 1$ and $n = -1$, that is,

$$g(x - \frac{4}{3})h(x) + g(x - \frac{1}{3})h(x+1) = 0,\ x \in [\frac{1}{3}, \frac{4}{3}], \tag{2.4}$$

$$g(x + \frac{4}{3})h(x) + g(x + \frac{7}{3})h(x+1) = 0,\ x \in [-\frac{7}{3}, -\frac{4}{3}]. \tag{2.5}$$

Note that $h(x+1) = 0,\ x \in [\frac{1}{3}, 1]$ and that $h(x) = 0,\ x \in [-2, -\frac{4}{3}]$. These together with (2.4) and (2.5) imply

$$h(x) = 0,\ x \in [-1, -\frac{1}{3}] \bigcup [\frac{1}{3}, 1].$$

Then we have

$$g(x)h(x) + g(x+1)h(x+1) = 0,\ x \in [-\frac{2}{3}, -\frac{1}{3}].$$

But this is a contradiction to (1.1), i.e., the Gabor frame $\{E_{3m/4}T_n g\}_{m,n\in\mathbb{Z}}$ does not have a dual window h supported on $[-1, 1]$. This is in accordance with the general result in Theorem 2.3 for $N = 1$. □

Note that compared with Theorem 2.2, Theorem 2.3 also deals with the limit option $b = \frac{2N}{2N+1}$. In case we want the dual window to be continuous this option is not available:

Theorem 2.5 *Let* $b = \frac{2N}{2N+1}$. *Assume that* $g \in V$, *defined in* (1.3). *Then there does not exist a continuous function* h *with* supp $h \subseteq [-N, N]$ *such that*

$$\sum_{k\in\mathbb{Z}} g(x - n/b + k)h(x+k) = \delta_{0,n},\ x \in [0, 1],\ n \in \mathbb{Z}.$$

The full proof of Theorem 2.2 is technical (see Section 3), so we illustrate the basic idea by an example:

Example 2.6 Let $b := \frac{2}{3}$ and

$$g(x) := (x+1)(x + \frac{3}{4})(x - \frac{3}{4})(x-1)\chi_{[-1,1]}(x). \tag{2.6}$$

Assume that h is a dual window with *supp* $h \subseteq [-1, 1]$. Consider (1.2) for $n = 1$ and $n = -1$, that is,

$$g(x - \frac{3}{2})h(x) + g(x - \frac{1}{2})h(x+1) = 0,\ x \in [\frac{1}{2}, \frac{3}{2}], \tag{2.7}$$

$$g(x + \frac{3}{2})h(x) + g(x + \frac{5}{3})h(x+1) = 0,\ x \in [-\frac{5}{2}, -\frac{3}{2}]. \tag{2.8}$$

Note that $h(x+1) = 0,\ x \in [\frac{1}{2}, 1]$ and that $h(x) = 0,\ x \in [-2, -\frac{3}{2}]$. Together with (2.7) and (2.8) this implies that

$$h(x) = 0,\ x \in [-1, -\frac{1}{2}] \bigcup [\frac{1}{2}, 1].$$

The duality condition with $n = 0$, *i.e.*,

$$g(x)h(x) + g(x+1)h(x+1) = \frac{2}{3},\ a.e.\ x \in [-1, 0]$$

implies that

$$h(x) = \frac{2}{3g(x)} = \frac{2}{(x+1)(x+\frac{3}{4})(x-\frac{3}{4})(x-1)},\ x \in [-\frac{1}{2}, \frac{1}{2}].$$

Hence h is not continuous at $x = \pm\frac{1}{2}$. □

3 Proofs

In this section we provide all the proofs. Note that we do not follow the order in which the theorems are stated in Section 2.

3.1 Proofs of Theorems 2.3 and 2.5

Lemma 3.1 *Let $N \in \mathbb{N}$ and $b \in [\frac{N}{N+1}, 1[$. Assume that g is a bounded function on $\mathbb{R}$ and that* supp $g = [-1, 1]$. *Assume further that h is supported in $[-N, N]$, and that for all $n = \pm 1, \pm 2, \cdots, \pm N$,*

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0,\ a.e.\ x \in [\frac{n}{b} - 1, \frac{n}{b}]. \tag{3.1}$$

Then

$$h(x)=0,\ a.e.\ x\in\left\{\begin{array}{l}\bigcup_{k=1}^{N}\left\{[\frac{N}{b}-N+k-1,k]\bigcup[k,\frac{k}{b}]\right\}\\ \bigcup_{k=1}^{N}\left\{[-\frac{k}{b},-k]\bigcup[-k,-\frac{N}{b}+N-k+1]\right\}\end{array}\right. \tag{3.2}$$

In particular,

$$h(x)=0,\ a.e.\ x\in[-1,-\frac{N}{b}+N]\cup[\frac{N}{b}-N,1]. \tag{3.3}$$

Proof. Note that $b\in[\frac{N}{N+1},1[$ implies that for $n=1,2,\dots,N$, $b\geq\frac{n}{n+1}$; thus,

$$\frac{n}{b}-1\leq n<\frac{n}{b}, \tag{3.4}$$

which will be used at the several instances in the proof.

We first show that $h(x)=0$ *a.e.* on $[\frac{N}{b}-1,N]\cup[N,\frac{N}{b}]$ and use induction on $[\frac{N}{b}-N+k-1]\cup[k,\frac{k}{b}]$ for $k=1,2,\cdots,N-1$ in reverse order.

We consider (3.1) for $n=N$, and split into two cases:

(1) For *a.e.* $x\in[N,\frac{N}{b}]$, $h(x)=0$, due to the support assumption on h.

(2) For *a.e.* $x\in[\frac{N}{b}-1,N]$, which by (3.4) is a subinterval of $[N-1,N]$, we see $h(x+1)=0$, due to the support assumption on h. If we note that, by (3.4) with $n=N$,

$$[\frac{N}{b}-1,N]\subset[\frac{N}{b}-1,\frac{N}{b}+1]=\text{supp } g(\cdot-\frac{N}{b}),$$

then $g(x-\frac{N}{b})\neq 0$ for *a.e.* $x\in[\frac{N}{b}-1,N]$. This together with (3.1) implies that

$$h(x)=0,\ a.e.\ x\in[\frac{N}{b}-1,N].$$

Assuming

$$h(x)=0,\ a.e.\ x\in[\frac{N}{b}-N+n_0-1,n_0]\bigcup[n_0,\frac{n_0}{b}] \tag{3.5}$$

for some $n_0\in\{2,3,\cdots,N\}$, we will show that

$$h(x)=0,\ a.e.\ x\in[\frac{N}{b}-N+n_0-2,n_0-1]\cup[n_0-1,\frac{n_0-1}{b}].$$

An application of (3.4) shows that

$$[\frac{N}{b} - N + n_0 - 2, n_0 - 1] \subset [\frac{n_0 - 1}{b} - 1, n_0 - 1] \bigcap \text{supp } g(\cdot - \frac{n_0 - 1}{b})]$$

and

$$[n_0 - 1, \frac{n_0 - 1}{b}] \subset [n_0 - 1, \frac{n_0}{b} - 1] \bigcap \text{supp } g(\cdot - \frac{n_0 - 1}{b}).$$

Then we have $g(x - \frac{n_0-1}{b}) \neq 0$ for $a.e.$ $x \in [\frac{N}{b} - N + n_0 - 2, n_0 - 1] \cup [n_0 - 1, \frac{n_0-1}{b}]$ and $h(x+1) = 0$ for $a.e.$ $x \in [\frac{N}{b} - N + n_0 - 2, n_0 - 1] \cup [n_0 - 1, \frac{n_0-1}{b}]$ by assumption. Considering (3.1) for $n = n_0 - 1$ leads to

$$h(x) = 0, \; a.e. \; x \in [\frac{N}{b} - N + n_0 - 2, n_0 - 1] \bigcup [n_0 - 1, \frac{n_0 - 1}{b}].$$

This completes our induction and so

$$h(x) = 0, \; a.e. \; x \in [\frac{N}{b} - N + k - 1, k] \bigcup [k, \frac{k}{b}].$$

By symmetry, considering (3.1) for $n = -1, -2, \cdots, -N$ leads to

$$h(x) = 0, \; a.e. \; x \in \bigcup_{k=1}^{N} [-\frac{k}{b}, -k] \bigcup [-k, -\frac{N}{b} + N - k + 1].$$

□

Even though Lemma 3.1 apparently only requires that $b \in [\frac{N}{N+1}, 1[$, the duality condition with $n = 0$, i.e.,

$$g(x)h(x) + g(x+1)h(x+1) = b, \; a.e \; x \in [-1, 0] \tag{3.6}$$

forces an upper bound of b in terms of N as well.

Proof of Theorem 2.3: By (3.3), h at most can be nonzero on the interval $[-\frac{N}{b} + N, \frac{N}{b} - N]$. In order for the duality condition to hold, this interval must have length at least 1; that is, we need to consider b such that $2(\frac{N}{b} - N) \geq 1$, i.e., $b \leq \frac{2N}{2N+1}$.

□

In case g is continuous and we insist on the dual window h being continuous, already $b = \frac{2N}{2N+1}$ has to be excluded:

Proof of Theorem 2.5: Suppose that there exists such a continuous function h. Then by (3.3),

$$h(x) = 0,\ x \in [-1, -\frac{1}{2}] \bigcup [\frac{1}{2}, 1].$$

Thus $h(-\frac{1}{2}) = h(\frac{1}{2}) = 0$. But this is a contradiction to (3.6) for continuous g and h. □

3.2 Proof of Theorem 2.2

Fix $b \in [\frac{N}{N+1}, \frac{2N}{2N+1}[$ for some $N \in \mathbb{N}$. Let $n_+ \in \{1, 2, \cdots, N-1\}$, and define the function R_{n_+} on (a subset of) $[0, (N - n_+)(\frac{1}{b} - 1)]$ by

$$R_{n_+}(y) := \begin{cases} \frac{1}{g(y)}, & \text{if } n_+ = 1; \\ \frac{\prod_{n=1}^{n_+-1} g(y+\frac{n}{b}-n-1)}{\prod_{n=0}^{n_+-1} g(y+\frac{n}{b}-n)}, & \text{if } n_+ = 2, \cdots, N-1. \end{cases}$$

Note that for $n = 0, 1, \ldots, n_+ - 1$,

$$\begin{aligned} y \in [0, (N-n_+)(\frac{1}{b} - 1)] \Rightarrow \frac{n}{b} - n &\leq y + \frac{n}{b} - n \leq (N - n_+)(\frac{1}{b} - 1) + \frac{n}{b} - n \\ &= (N - n_+ + n)(\frac{1}{b} - 1) \\ &\leq (N - n_+ + (n_+ - 1))(\frac{1}{b} - 1) \\ &= (N-1)(\frac{1}{b} - 1) \\ &\leq (N-1)(\frac{N+1}{N} - 1) = \frac{N-1}{N} < 1. \end{aligned}$$

This implies that R_{n_+} is defined on $[0, (N - n_+)(\frac{1}{b} - 1)]$, except maybe on a finite set of points.

Similarly, for $n_- \in \{1, 2, \cdots, N-1\}$, we define the function $L_{n_-}(y)$ on (a subset of) $[-(N-n_-)(\frac{1}{b}-1), 0]$ by

$$L_{n_-}(y) := \begin{cases} \frac{1}{g(y)}, & \text{if } n_- = 1; \\ \frac{\prod_{n=1}^{n_--1} g(y-\frac{n}{b}+n+1)}{\prod_{n=0}^{n_--1} g(y-\frac{n}{b}+n)}, & \text{if } n_- = 2, \cdots, N-1. \end{cases}$$

Lemma 3.2 *Let $N \in \mathbb{N}$ and $b \in [\frac{N}{N+1}, \frac{2N}{2N+1}[$. Assume that $g \in V$, defined in* (1.3)*. Assume that $h(x)$ is continuously chosen for $x \in [-1, 1]$ so that the following four conditions hold:*

(1) $h(x) = 0,\ x \in [-1, -\frac{N}{b} + N] \cup [\frac{N}{b} - N, 1]$;

(2) $g(x)h(x) + g(x+1)h(x+1) = b, x \in [-1, 0]$;

(3) *If there exist $n_+ \in \{1, 2, \cdots, N-1\}$ and $y_+ \in [0, (N-n_+)(\frac{1}{b}-1)]$ such that $g(y_+) = 0$, then the limit*

$$\lim_{y \to y_+} \left\{ h(y + \frac{n_+}{b} - n_+) R_{n_+}(y) \right\} \tag{3.7}$$

exists;

(4) *If there exist $n_- \in \{1, 2, \cdots, N\}$ and $y_- \in [-(N-n_-)(\frac{1}{b}-1), 0]$ such that $g(y_-) = 0$, then the limit*

$$\lim_{y \to y_-} \left\{ h(y - \frac{n_-}{b} + n_-) L_{n_-}(y) \right\}$$

exists.

Then the equations, for $n = \pm 1, \pm 2, \cdots, \pm N$,

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0,\ x \in [\frac{n}{b} - 1, \frac{n}{b}] \tag{3.8}$$

determine $h(x)$ continuously for $x \in \bigcup_{k=1}^{N+1} \left\{ [-k, -\frac{k-1}{b}] \bigcup [\frac{k-1}{b}, k] \right\}$. Moreover,

$$h(x) = 0,\ x \in \bigcup_{k=1}^{N+1} \left\{ [-k, -\frac{N}{b} + N - k + 1] \bigcup [\frac{N}{b} - N + k - 1, k] \right\} \tag{3.9}$$

and

$$\lim_{x \to (\frac{n}{b})^+} h(x) = \lim_{x \to (-\frac{n}{b})^-} h(x) = 0,\ n = 1, \cdots, N. \tag{3.10}$$

Proof. We use induction to show that (3.8) determine $h(x)$ continuously for $x \in \bigcup_{k=1}^{N+1} \{[-k, -\frac{k-1}{b}] \bigcup [\frac{k-1}{b}, k]\}$ and satisfy (3.9). First, by assumption, $h(x)$ is continuously chosen for $x \in [0,1]$ and

$$h(x) = 0,\ x \in [\frac{N}{b} - N, 1]$$

by the condition (1). With the purpose to perform an induction argument, we now assume that, for some $1 \le n_0 \le N$, the function h is known to be continuous on $\bigcup_{n=1}^{n_0} [\frac{n-1}{b}, n]$ and

$$h(x) = 0,\ x \in [\frac{N}{b} - N + n_0 - 1, n_0] \ \ (\subset [\frac{n_0 - 1}{b}, n_0]). \tag{3.11}$$

We consider (3.8) for $n = n_0$, i.e.,

$$g(x - \frac{n_0}{b})h(x) + g(x - \frac{n_0}{b} + 1)h(x+1) = 0,\ x \in [\frac{n_0}{b} - 1, \frac{n_0}{b}]. \tag{3.12}$$

We will use (3.12) for x_0 in the subinterval $[\frac{n_0}{b} - 1, n_0]$. We split the argument into two cases:

(a) We first assume that $g(x_0 - \frac{n_0}{b} + 1) \neq 0$. Then (3.12) implies

$$h(x_0 + 1) = -\frac{g(x_0 - \frac{n_0}{b})h(x_0)}{g(x_0 - \frac{n_0}{b} + 1)}. \tag{3.13}$$

In particular, this and (3.11) imply

$$h(x_0 + 1) = 0,\ \text{if } x_0 \in [\frac{N}{b} - N + n_0 - 1, n_0]. \tag{3.14}$$

(b) We now assume $g(x_0 - \frac{n_0}{b} + 1) = 0$. Take $y := x - \frac{n_0}{b} + 1$ in the condition (3.7). Note that, for $n = 1, \cdots, n_0 - 1$,

$$[\frac{n+1}{b} - 1, n+1] \subset [\frac{n}{b}, n+1]. \tag{3.15}$$

Combining with (3.8) for $n = n_0 - 1$ implies that

$$\frac{h(x)}{g(x - \frac{n_0}{b} + 1)} = -\lim_{x \to x_0} \frac{g(x - \frac{n_0 - 1}{b} - 1)h(x-1)}{g(x - \frac{n_0}{b} + 1)g(x - \frac{n_0 - 1}{b})},\ x \in [\frac{n_0}{b} - 1, n_0],$$

which is well-defined except for a finite number of x-values. Applying (3.8) and (3.15) repeatedly for $n = 1, 2, \cdots, n_0 - 2$ in reverse order implies that

$$\begin{aligned}\frac{h(x)}{g(x - \frac{n_0}{b} + 1)} &= (-1)^{n_0-1}\frac{g(x - \frac{n_0-1}{b} - 1)\cdots g(x - \frac{1}{b} - n_0 + 1)h(x - n_0 + 1)}{g(x - \frac{n_0}{b} + 1)\cdots g(x - \frac{1}{b} - n_0 + 2)} \\ &= (-1)^{n_0-1}\left(h(x - n_0 + 1)R_{n_0}(x - \frac{n_0}{b} + 1)\right).\end{aligned}$$

If $x_0 \in]\frac{N}{b} - N + n_0 - 1, n_0]$ then

$$\lim_{x\to x_0} \frac{h(x)}{g(x - \frac{n_0}{b} + 1)} = 0,$$

by (3.11); if $x_0 \in [\frac{n_0}{b} - 1, \frac{N}{b} - N + n_0 - 1[$, then the limit

$$\lim_{x\to x_0} \frac{h(x)}{g(x - \frac{n_0}{b} + 1)} = (-1)^{n_0-1}\lim_{x\to x_0}\left(h(x - n_0 + 1)R_{n_0}(x - \frac{n_0}{b} + 1)\right)$$

exists by (3.7); if $x_0 = \frac{N}{b} - N + n_0 - 1$, then

$$\lim_{x\to x_0} \frac{h(x)}{g(x - \frac{n_0}{b} + 1)} = (-1)^{n_0-1}\lim_{x\to x_0}\left(h(x - n_0 + 1)R_{n_0}(x - \frac{n_0}{b} + 1)\right) = 0,$$

by (3.7) and (3.11). Note that if $n_0 = N$ and $x_0 \in [\frac{n_0}{b} - 1, \frac{N}{b} - N + n_0 - 1]$, *i.e.*, $x_0 = \frac{N}{b} - 1$, then $g(x_0 - \frac{n_0}{b} + 1) = g(0) \neq 0$. Thus we can define

$$h(x_0+1) = \begin{cases} -\lim_{x\to x_0}\left(\frac{h(x)}{g(x-\frac{n_0}{b}+1)}\right)g(x_0 - \frac{n_0}{b}), & \text{if } x_0 \in [\frac{n_0}{b} - 1, \frac{N}{b} - N + n_0 - 1[; \\ 0, & \text{if } x_0 \in [\frac{N}{b} - N + n_0 - 1, n_0]. \end{cases} \tag{3.16}$$

Note that $g(x - \frac{n_0}{b}), g(x - \frac{n_0}{b} - 1)$ and $h(x - 1)$ are continuous for $x \in [\frac{n_0}{b}, n_0 + 1] \subset [\frac{n_0-1}{b} + 1, n_0 + 1]$. Hence $h(x)$ is determined and continuous for $x \in [\frac{n_0}{b}, n_0 + 1]$ by (3.13) and (3.16), and $h(x) = 0$ for $x \in [\frac{N}{b} - N + n_0, n_0 + 1] (\subset [\frac{n_0}{b}, n_0 + 1])$ by (3.14) and (3.16). By induction, $h(x)$ is continuous for $x \in \bigcup_{k=1}^{N+1}[\frac{k-1}{b}, k]$, and $h(x) = 0$ for $x \in \bigcup_{k=1}^{N+1}[\frac{N}{b} - N + k - 1, k]$.

On the other hand, for $x \in [n, \frac{n}{b}]$, $n = 1, 2, \cdots, N$, the equation

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x + 1) = 0$$

only involves $x \in [n, \frac{n}{b}]$ and $x+1 \in [n+1, \frac{n}{b}+1]$ for h, and

$$\left([n, \frac{n}{b}] \cup [n+1, \frac{n}{b}+1]\right) \cap \left(\bigcup_{k=1}^{N+1} [\frac{k-1}{b}, k]\right) = \emptyset, \ n = 1, 2, \cdots, N.$$

By symmetry, considering (3.8) for $n = -1, -2. \cdots, -N$ determines $h(x)$ continuously for $x \in \bigcup_{k=1}^{N+1} [-k, -\frac{k-1}{b}]$, and $h(x) = 0$ for $x \in \bigcup_{k=1}^{N+1} [-k, -\frac{N}{b} + N - k + 1]$. This proves that $h(x)$ is continuously determined for

$$x \in \bigcup_{k=1}^{N+1} \left\{[-k, -\frac{k-1}{b}] \bigcup [\frac{k-1}{b}, k]\right\}$$

and satisfies (3.9).

For (3.10), the condition (2) and $g(-1) = 0$ imply that $g(0) \neq 0$. So (3.8) implies

$$\lim_{x \to (\frac{n}{b})^+} h(x) = -\lim_{x \to (\frac{n}{b})^+} \frac{g(x - \frac{n}{b} - 1) h(x-1)}{g(x - \frac{n}{b})} = \frac{g(-1) h(n/b - 1)}{g(0)} = 0,$$

for $n = 1, \cdots, N$. Similarly, $\lim_{x \to (-\frac{n}{b})^-} h(x) = 0$ for $n = 1, \cdots, N$.

□

Proposition 3.3 *Under the assumptions in Lemma 3.2, there exists a unique extension of* h *to a function with* supp$h \subseteq [-N, N]$ *so that for* $n = \pm 1, \pm 2, \cdots, \pm N$,

$$g(x - \frac{n}{b}) h(x) + g(x - \frac{n}{b} + 1) h(x+1) = 0, \ x \in [\frac{n}{b}, \frac{n}{b} + 1]. \tag{3.17}$$

This function h *is continuous.*

Proof. We define $h(x)$ for $x \in \bigcup_{k=1}^{N} \left\{[-k, -\frac{k-1}{b}] \bigcup [\frac{k-1}{b}, k]\right\}$ as in the proof in Lemma 3.2 and

$$h(x) = 0, \ x \notin \bigcup_{k=1}^{N} \left\{[-k, -\frac{k-1}{b}] \bigcup [\frac{k-1}{b}, k]\right\}. \tag{3.18}$$

From Lemma 3.2, $h(x)$ is a continuous function with supp $h \subseteq [-N, N]$ satisfying (3.17) for $n = \pm 1, \pm 2, \cdots, \pm N$. □

Theorem 3.4 *Let $N \in \mathbb{N}$ and $b \in [\frac{N}{N+1}, \frac{2N}{2N+1}[$. Assume that $g \in V$. Then the following assertions are equivalent:*

(1) *There exists a dual window h with* $\operatorname{supp} h \subseteq [-N, N]$*;*

(2) *There exists a continuous dual window h with* $\operatorname{supp} h \subseteq [-N, N]$*;*

(3) *The following five conditions are satisfied:*

(i) $|g(x)| + |g(x+1)| > 0, \ x \in [-1, 0]$*;*

(ii) $g(x) \neq 0, \ x \in [\frac{N}{b} - N - 1, -\frac{N}{b} + N + 1]$;

(iii) *If there exist $n_+ \in \{1, 2, \cdots, N-1\}$ and $y_+ \in [0, (N - n_+)(\frac{1}{b} - 1)]$ such that $g(y_+) = 0$ and $\lim_{y \to y_+} |R_{n_+}(y)| = \infty$, then*

$$g(y_+ + \frac{n_+}{b} - n_+ - 1) \neq 0;$$

(iv) *If there exist $n_- \in \{1, 2, \cdots, N-1\}$ and $y_- \in [-(N - n_-)(\frac{1}{b} - 1), 0]$ such that $g(y_-) = 0$ and $\lim_{y \to y_-} |L_{n_-}(y)| = \infty$, then*

$$g(y_- - \frac{n_-}{b} + n_- + 1) \neq 0;$$

(v) *For y_+, y_-, n_+, n_- as in* (iii) *and* (iv),

$$y_+ + \frac{n_+}{b} - n_+ \neq y_- - \frac{n_-}{b} + n_- + 1, \qquad (3.19)$$

Proof. Let $h \in L^2(\mathbb{R})$ be a dual window of g with *supp* $h \subseteq [-N, N]$. Note that such a function h is essentially bounded due to the frame assumption. By Proposition 1.1, for $n = \pm 1, \pm 2 \cdots, \pm N$, we have that

$$g(x - \frac{n}{b})h(x) + g(x - \frac{n}{b} + 1)h(x+1) = 0, \ a.e. \ x \in [\frac{n}{b} - 1, \frac{n}{b}]; \qquad (3.20)$$

further, by a shift of the equation in (1.1) with $n = 0$,

$$g(x)h(x) + g(x+1)h(x+1) = b, \ a.e. \ x \in [-1, 0]. \qquad (3.21)$$

We now verify that the conditions (i)-(v) of (3) in Theorem 3.4 are satisfied.

(i): Since g is continuous and $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ is a frame with lower bound A,

$$\sum_{m\in\mathbb{Z}} |g(x-m)| \geq bA$$

for $x \in \mathbb{R}$; since $\operatorname{supp} g \subseteq [-1,1]$, this leads to (i).
(ii): Note that

$$h(x) = 0, \ a.e. \ x \in [-1, -\frac{N}{b} + N] \bigcup [\frac{N}{b} - N, 1], \tag{3.22}$$

by (3.3). This together with (3.21) implies

$$\begin{aligned} g(x)h(x) = b, &\qquad a.e. \ x \in [\frac{N}{b} - N - 1, 0], \\ g(x+1)h(x+1) = b, &\qquad a.e. \ x \in [-1, -\frac{N}{b} + N], \end{aligned}$$

i.e.,

$$g(x)h(x) = b, \ a.e. \ x \in [\frac{N}{b} - N - 1, -\frac{N}{b} + N + 1];$$

since $g(x)$ is continuous, (ii) holds.
(iii): Suppose n_+ and y_+ satisfy the assumption of (iii). Via (3.4),

$$y_+ \in [0, (N - n_+)(\frac{1}{b} - 1)] \subset [0, 1[.$$

Let

$$x_+ := y_+ + \frac{n_+}{b} - 1 \in [\frac{n_+}{b} - 1, \frac{N}{b} - N + n_+ - 1] \subset [\frac{n_+}{b} - 1, n_+].$$

Consider (3.20) with $n = n_+$, i.e.,

$$g(x - \frac{n_+}{b})h(x) + g(x - \frac{n_+}{b} + 1)h(x+1) = 0, \ a.e. \ x \in [\frac{n_+}{b} - 1, \frac{N}{b} - N + n_+ - 1].$$

Since g have a finite number of zeros in $[-1, 1]$, it follows that

$$\frac{h(x)}{g(x - \frac{n_+}{b} + 1)} = -\frac{h(x+1)}{g(x - \frac{n_+}{b})}, \ a.e. \ x \in [\frac{n_+}{b} - 1, \frac{N}{b} - N + n_+ - 1];$$

since $g(x_+ - \frac{n_+}{b}) = g(y_+ - 1) \neq 0$ by (i) and h is essentially bounded, it follows that

$$\limsup_{L_h \ni x \to x_+} \left| \frac{h(x)}{g(x - \frac{n_+}{b} + 1)} \right| =: M < \infty,$$

where L_h is the set of Lebesgue points of h. As in the proof of Lemma 3.2, we have

$$\limsup_{L_h \ni x \to x_+} \left| \frac{h(x)}{g(x - \frac{n_+}{b} + 1)} \right| = \limsup_{L_h \ni x \to x_+} \left| h(x - n_+ + 1) R_{n_+}(x - \frac{n_+}{b} + 1) \right|.$$

Since $\lim_{x \to x_+} \left| R_{n_+}(x - \frac{n_+}{b} + 1) \right| = \infty$, we conclude that

$$\lim_{L_h \ni x \to x_+} h(x - n_+ + 1) = 0,$$

i.e.,

$$\lim_{L_h \ni y \to y_+} h(y + \frac{n_+}{b} - n_+) = 0. \tag{3.23}$$

By (3.21) and (3.23),

$$\begin{aligned} b &= \lim_{L_h \ni x \to x_+} \{ g(x - n_+) h(x - n_+) + g(x - n_+ + 1) h(x - n_+ + 1) \} \\ &= \lim_{L_h \ni x \to x_+} g(x - n_+) h(x - n_+). \end{aligned}$$

Since $h(x)$ is essentially bounded and $g(x)$ is continuous, we have

$$g(x_+ - n_+) \neq 0,$$

i.e.,

$$g(y_+ + \frac{n_+}{b} - n_+ - 1) \neq 0.$$

This proves that (iii) holds.

(iv): This is similar to the proof of (iii) by symmetry, so we skip it. But we note for use in the proof of (v) that the result corresponding to (3.23) is

$$\lim_{L_h \ni y \to y_-} h(y - \frac{n_-}{b} + n_-) = 0. \tag{3.24}$$

(v): Suppose that y_+, n_+ and y_-, n_- are as in (iii) and (iv), respectively. Then the results in (3.23) and (3.24) holds, i.e,

$$\lim_{L_h \ni y \to y_+} h(y + \frac{n_+}{b} - n_+) = 0 \tag{3.25}$$

and

$$\lim_{L_h \ni y \to y_-} h(y - \frac{n_-}{b} + n_-) = 0. \tag{3.26}$$

Note that $y_+ + \frac{n_+}{b} - n_+,\ y_- - \frac{n_-}{b} + n_- + 1 \in [0,1]$. If

$$y_+ + \frac{n_+}{b} - n_+ = y_- - \frac{n_-}{b} + n_- + 1,$$

then by (3.21),

$$b = \lim_{L_h \ni y \to y_+} \left\{ g(y + \frac{n_+}{b} - n_+ - 1)h(y + \frac{n_+}{b} - n_+ - 1) + g(y + \frac{n_+}{b} - n_+)h(y + \frac{n_+}{b} - n_+) \right\};$$

however, this contradicts to (3.25) and (3.26). Hence

$$y_+ + \frac{n_+}{b} - n_+ \neq y_- - \frac{n_-}{b} + n_- + 1,$$

i.e., (v) holds.

(3) $\Rightarrow$ (2) : Assume that (i)-(v) in Theorem 3.4(3) hold. We construct $h(x)$ on $[-1,1]$ satisfying the hypotheses described in Lemma 3.2. For $m, n = 1, 2, \cdots, N-1$, we define the sets Y_n and W_m by

$$Y_n = \{ y_{n,i} \in]0, (N-n)(\frac{1}{b} - 1)] \ : g(y_{n,i}) = 0 \text{ and } \lim_{y \to y_{n,i}} |R_n(y)| = \infty \}_{i=1,2,\cdots,r_n}$$

and

$$W_m = \{ w_{m,j} \in [-(N-n)(\frac{1}{b} - 1), 0[: g(w_{m,j}) = 0 \text{ and } \lim_{y \to w_{m,j}} |L_m(y)| = \infty \ \}_{j=1,2,\cdots,l_m},$$

where r_n and l_m are the cardinalities of Y_n and W_m, respectively. We denote the open interval of radius $\epsilon > 0$ centered at x by

$$B(x;\epsilon) =]x - \epsilon, x + \epsilon[.$$

Let $y_{n,i} \in Y_n$, $w_{m,j} \in W_m$ for $n, m = 1, 2, \cdots, N-1$ and

$$\tilde{y}_{n,i} := y_{n,i} - n + \frac{n}{b},\ \hat{w}_{m,j} := w_{m,j} - \frac{m}{b} + m.$$

The definitions of Y_n and W_m imply

$$\tilde{y}_{n,i} \in]\frac{n}{b} - n, \frac{N}{b} - N], \ \hat{w}_{m,j} \in [N - \frac{N}{b}, m - \frac{m}{b}[. \tag{3.27}$$

Since $g(y_{n,i}) = g(w_{m,j}) = 0$, the condition (ii) implies

$$\hat{w}_{m,j} < w_{m,j} < \frac{N}{b} - N - 1 \leq 0 \leq -\frac{N}{b} + N + 1 < y_{n,i} < \tilde{y}_{n,i}. \tag{3.28}$$

By the conditions (iii), (iv) and (v),

$$g(\tilde{y}_{n,i} - 1) \neq 0 \neq g(\hat{w}_{m,j} + 1), \tag{3.29}$$

and

$$\tilde{y}_{n,i} \neq \hat{w}_{m,j} + 1. \tag{3.30}$$

Then we can choose $\epsilon_0 > 0$ so that $g(x) \neq 0$ for

$$x \in B(\tilde{y}_{n,i} - 1; \epsilon_0) \bigcup B(\hat{w}_{m,j} + 1; \epsilon_0), \tag{3.31}$$

$$B(\tilde{y}_{n,i}; \epsilon_0) \bigcap B(\hat{w}_{m,j} + 1; \epsilon_0) = \emptyset \tag{3.32}$$

and

$$\Big(B(\tilde{y}_{n,i}; \epsilon_0) \bigcup B(\hat{w}_{m,j}; \epsilon_0)\Big) \subseteq [-1, 0]$$

for $m, n = 1, 2, \cdots, N-1$, and $i = 1, 2, \cdots, r_n$ and $j = 1, 2, \cdots, l_m$.

First, we define $h(x)$ on $[-1, -\frac{N}{b} + N] \bigcup [\frac{N}{b} - N - 1, N - \frac{N}{b} + 1] \bigcup [\frac{N}{b} - N, 1]$ by

$$h(x) := \begin{cases} 0, & x \in [-1, -\frac{N}{b} + N] \bigcup [\frac{N}{b} - N, 1]; \\ \frac{b}{g(x)}, & x \in [\frac{N}{b} - N - 1, -\frac{N}{b} + N + 1], \end{cases}$$

which is well-defined by the condition (ii). Then h satisfies the condition (3.3) and

$$g(x)h(x) + g(x+1)h(x+1) = b, \ x \in [-1, -\frac{N}{b} + N] \bigcup [\frac{N}{b} - N - 1, 0].$$

Secondly, we define $h(x)$ on $B(\tilde{y}_{n,i} - 1; \epsilon_0) \bigcup B(\tilde{y}_{n,i}; \epsilon_0)$. Note that

$$\tilde{y}_{n,i} \notin [0, N - \frac{N}{b} + 1] \bigcup (\frac{N}{b} - N, 1]$$

by (3.27) and (3.28). We can choose $h(x)$ continuously on $B(\tilde{y}_{n,i};\epsilon_0)$ so that

$$h(\tilde{y}_{n,i}) := 0$$

and the limit

$$\lim_{y\to y_{n,i}} \left\{h(y+\frac{n}{b}-n)R_n(y)\right\}$$

do exist; if $\tilde{y}_{n,i} = \frac{N}{b} - N$, then we choose $h(x)$ continuously on $B(\tilde{y}_{n,i};\epsilon_0)$ so that

$$h(\tilde{y}_{n,i}) := 0$$

and

$$\lim_{y\to (y_{n,i})^-} \left\{h(y+\frac{n}{b}-n)R_n(y)\right\} = 0$$

by the above first case. Now, define $h(x)$ on $B(\tilde{y}_{n,i}-1;\epsilon_0)$ by

$$h(x) = \frac{b - g(x+1)h(x+1)}{g(x)},$$

which is well-defined by (3.31). Then

$$g(x)h(x) + g(x+1)h(x+1) = b,\ x \in B(\tilde{y}_{n,i}-1;\epsilon_0).$$

Thirdly, we define $h(x)$ on $B(\hat{w}_{m,j};\epsilon_0)\bigcup B(\hat{w}_{m,j}+1;\epsilon_0)$. Note that

$$\hat{w}_{m,j} \notin [-1, -\frac{N}{b}+N)\bigcup[-N+\frac{N}{b}-1, 0]$$

by (3.27) and (3.28). Choose $h(x)$ continuously on $B(\hat{w}_{m,j};\epsilon_0)$ so that

$$\lim_{y\to w_{m,j}} h(y-\frac{m}{b}+m) = 0 =: h(\hat{w}_{m,j})$$

and the limit

$$\lim_{y\to w_{m,j}} \left\{h(y-\frac{m}{b}+m)L_m(y)\right\}$$

do exist; if $\hat{w}_{m,j} = N - \frac{N}{b}$, then we choose $h(x)$ continuously on $B(\hat{w}_{m,j};\epsilon_0)$ so that

$$h(\hat{w}_{m,j}) := 0$$

and

$$\lim_{y\to (w_{m,j})^+} \left\{h(y-\frac{m}{b}+m)L_m(y)\right\} = 0$$

by the above first case. Now, define $h(x)$ on $B(\hat{w}_{m,j}+1;\epsilon_0)$ by

$$h(x)=\frac{b-g(x-1)h(x-1)}{g(x)},$$

which is well-defined by (3.31). Then

$$g(x)h(x)+g(x+1)h(x+1)=b,\ \ x\in B(\hat{w}_{m,j};\epsilon_0).$$

To summarize all these, let

$$A:=\begin{cases}[-1,N-\frac{N}{b}]\bigcup[\frac{N}{b}-N-1,0]\bigcup\\ \left(\bigcup_{n=1}^{N-1}\bigcup_{i=1}^{r_n}B(\tilde{y}_{n,i}-1;\epsilon_0)\right)\bigcup\\ \left(\bigcup_{m=1}^{N-1}\bigcup_{j=1}^{l_m}B(\hat{w}_{m,j};\epsilon_0)\right),\end{cases}$$

Note that $A\subset[-1,0]$. We have defined $h(x)$ on $A\bigcup(A+1)$ so that

$$g(x)h(x)+g(x+1)h(x+1)=b,\ \ x\in A.$$

Finally, we choose $h(x)$ on $[-1,1]\setminus(A\bigcup(A+1))$ such that $h(x)$ is continuous on $[-1,1]$ and

$$g(x)h(x)+g(x+1)h(x+1)=b,\ \ x\in[-1,0]\setminus A,$$

by the condition (i).

By Proposition 3.3, the function h can be extended to a continuous function supported on [-N,N] that is a dual window.

(2) $\Rightarrow$ (1) : This is trivial.

□

Remark 3.5 In Theorem 3.4, $h(x)$ only have flexibility for $x\in]-\frac{N}{b}+N,\frac{N}{b}-N-1[\bigcup]-\frac{N}{b}+N+1,\frac{N}{b}-1[$ In fact, $h(x)$ is determined on $[-1,-\frac{N}{b}+N]\bigcup[\frac{N}{b}-N-1,-\frac{N}{b}+N+1]\bigcup[\frac{N}{b}-1,1]$ by

$$h(x)=\begin{cases}0, & x\in[-1,-\dfrac{N}{b}+N]\cup[\dfrac{N}{b}-N,1];\\ \dfrac{b}{g(x)}, & x\in[\dfrac{N}{b}-N-1,-\dfrac{N}{b}+N+1],\end{cases}$$

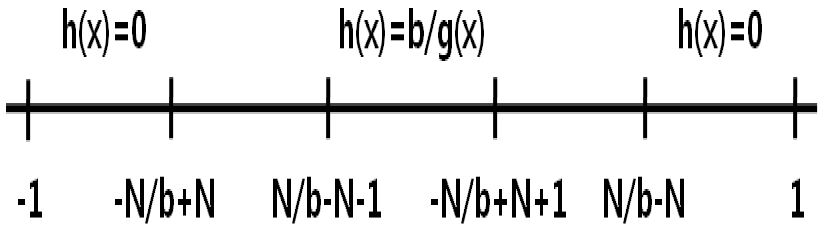

Figure 2:

We now fix $b \in [\frac{M-1}{M}, \frac{M}{M+1}[$ for some $M \in \mathbb{N}$. Let $n_+ \in \{1, 2, \cdots, M-1\}$, and define the function R_{n_+} on (a subset of) $[0, n_+ - \frac{n_+}{b} + 1]$ by

$$R_{n_+}(y) := \begin{cases} \frac{1}{g(y)}, & \text{if } n_+ = 1; \\ \frac{\prod_{n=1}^{n_+-1} g(y+\frac{n}{b}-n-1)}{\prod_{n=0}^{n_+-1} g(y+\frac{n}{b}-n)}, & \text{if } n_+ = 2, \cdots, M-1. \end{cases}$$

Similarly, for $n_- \in \{1, 2, \cdots, M-1\}$, we define the function $L_{n_-}(y)$ on (a subset of) $[-n_- + \frac{n_-}{b} - 1, 0]$ by

$$L_{n_-}(y) := \begin{cases} \frac{1}{g(y)}, & \text{if } n_- = 1; \\ \frac{\prod_{n=1}^{n_--1} g(y-\frac{n}{b}+n+1)}{\prod_{n=0}^{n_--1} g(y-\frac{n}{b}+n)}, & \text{if } n_- = 2, \cdots, M-1. \end{cases}$$

The following result appeared in [4]:

Proposition 3.6 *Let $M \in \mathbb{N} \setminus \{1\}$ and $b \in [\frac{M-1}{M}, \frac{M}{M+1}[$. Assume that $g \in V$. Then the following assertions are equivalent:*

(1) *The function g generates a Gabor frame $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$;*

(2) *There exists a continuous dual window h with* supp $h \subseteq [-M, M]$*;*

(3) *The following four conditions are satisfied:*

(i) $|g(x)| + |g(x+1)| > 0, \ x \in [-1, 0]$;

(ii) *If there exist* $n_+ \in \{1, 2, \cdots, M-1\}$ *and* $y_+ \in [0, n_+ - \frac{n_+}{b} + 1]$ *such that* $g(y_+) = 0$ *and* $\lim_{y \to y_+} |R_{n_+}(y)| = \infty$, *then*

$$g(y_+ + \frac{n_+}{b} - n_+ - 1) \neq 0; \tag{3.33}$$

(iii) *If there exist* $n_- \in \{1, 2, \cdots, M-1\}$ *and* $y_- \in [-n_- + \frac{n_-}{b} - 1, 0]$ *such that* $g(y_-) = 0$ *and* $\lim_{y \to y_-} |L_{n_-}(y)| = \infty$, *then*

$$g(y_- - \frac{n_-}{b} + n_- + 1) \neq 0;$$

(iv) *For* y_+, y_-, n_+, n_- *as in* (ii) *and* (iii),

$$y_+ + \frac{n_+}{b} - n_+ \neq y_- - \frac{n_-}{b} + n_- + 1.$$

Proof of Theorem 2.2: ($\Rightarrow$) This is trivial by the condition (ii) of (3) in Theorem 3.4.
($\Leftarrow$) It suffices to check the conditions (iii)-(v) of (3) in Theorem 3.4.
(iii) : Choose $M \in \{N+1, N+2, \cdots, 2N\}$ such that

$$\frac{M-1}{M} \leq b < \frac{M}{M+1}.$$

Assume that there exist $n_+ \in \{1, 2, \cdots, N-1\}$ and $y_+ \in [0, (N-n_+)(\frac{1}{b} - 1)]$ such that $g(y_+) = 0$ and $\lim_{y \to y_+} |R_{n_+}(y)| = \infty$. Note that

$$b \geq \frac{N}{N+1} \ \Leftrightarrow \ (N - n_+)(\frac{1}{b} - 1) \leq n_+ - \frac{n_+}{b} + 1.$$

Since $N \leq M$, we have

$$g(y_+ + \frac{n_+}{b} - n_+ - 1) \neq 0$$

by Proposition 3.6. This leads to (iii)
(iv) : This is similar to the proof of (iii) by symmetry.
(v) : This is follows from the condition (iv) of (3) in Proposition 3.6.

□

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