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by

B. van Brunt , Hong Oh Kim and Gregory Derfel

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B. van Brunt [*], Hong Oh Kim [†‡] & Gregory Derfel[§]

Running Title:
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Address for Correspondence:
Bruce van-Brunt,
Institute of Fundamental Sciences,
Mathematics
Massey University,
Palmerston North, NEW ZEALAND
+64-6-350 9099 ext.3575
Fax:+64-6-350 5682
e-mail:B.vanBrunt@massey.ac.nz

[*]Institute of Fundamental Sciences, Mathematics, Massey University, New Zealand
[†]Department of Mathematical Sciences, KAIST, Korea, email: kimhong@kaist.edu
[‡]Supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00031)
[§]Department of Mathematics, Ben Gurion University, Israel

Abstract

The pantograph equation is perhaps one of the most heavily studied class of functional differential equations owing to its numerous applications in mathematical physics, biology, and problems arising in industry. This equation is characterized by a linear functional argument. Heard [4] considered a generalization of this equation that included a nonlinear functional argument. His work focussed on the asymptotic behaviour of solutions for a real variable x as $x \to \infty$. In this paper, we revisit Heard's equation, but study it in the complex plane. Using results from complex dynamics we show that any nonconstant solution that is holomorphic at the origin must have the unit circle as a natural boundary. We consider solutions that are holomorphic on the Julia set of the nonlinear argument. We show that the solutions are either constant or have a singularity at the origin. There is a special case of Heard's equation that includes only the derivative and the functional term. For this case we construct solutions to the equation and illustrate the general results using classical complex analysis.

1 Introduction

In this paper we study initial-value problems of the form

$$y'(z) + ay(z) = \lambda y(z^n), \tag{1.1}$$

$$y(\tilde{z}) = y_0, \tag{1.2}$$

where a, $\lambda \neq 0$ and y_0 are constants, $n > 1$ is an integer, and $\tilde{z}$ is a fixed point for $g(z) = z^n$. Specifically, we look at global properties of solutions that are holomorphic at $\tilde{z}$. The function g has one attracting fixed point at $z_0 = 0$, and $n-1$ repelling fixed points on the circle $C(0;1) = \{z : |z| = 1\}$. The global nature of the solutions differs greatly depending on whether or not $\tilde{z}$ is attracting or repelling. We show that solutions holomorphic at the attracting fixed point must have $C(0;1)$ as a natural boundary; whereas, solutions holomorphic at repelling fixed point can be continued throughout the complex plane except at the attracting fixed point, which must be a branch point.

The work here is motivated by that of Heard [4], who studied the asymptotics of real solutions to equation (1.1) as $x \to \infty$. His work was motivated by the close relationship to pantograph type equations

$$y'(z) + ay(z) = \lambda y(\alpha z), \tag{1.3}$$

which have found numerous applications (cf. [5]).

The holomorphic continuation of solutions to initial-value problems such as (1.1), (1.2) has been studied in [7], where the focus is primarily upon solutions that are holomorphic at an attracting fixed point. In that work, results from complex dynamics and complex analysis were exploited to establish the existence of natural boundaries.

In the next section we study solutions that are holomorphic at fixed points (attracting or repelling). We begin with repelling fixed points and show that holomorphy places very strong global conditions on solutions. We use the techniques from [7] to deduce the existence of a natural boundary for solutions holomorphic at the attracting fixed point.

In the third section we look at the special case when $a = 0$. Much of the analysis from the second section can be used in this case, but it has the decisive merit that some of the solutions can be found explicitly and thus used to illustrate the results. In addition, the case $a = 0$ has some special features, and as Heard [4] noted for this case, the growth of solutions is different.

2 The Case $a \neq 0$

Equation (1.1) provides a relationship between the values of a function evaluated at z and at z^n. If y is a holomorphic solution then this equation also provides a direct mechanism for holomorphic continuation. This, in turn, places severe limits on the types of solutions available. Any process of holomorphic continuation involves iterates of $g(z) = z^n$, and this brings to the fore ideas from complex dynamics. The results in [7] apply to this equation; however, the functional argument g is particularly simple and it is possible to extend this work to deduce some properties of solutions holomorphic at repelling fixed points.

The fixed points of g in $\mathbb{C}$ are evidently $z_0 = 0$ and the $n-1$ solutions $z_1, \ldots z_{n-1}$ of the cyclotomic equation $z^{n-1} = 1$. Since $g'(z) = nz^{n-1}$ it is clear that z_0 is an attracting fixed point and the other fixed points are repelling. The basin of attraction for z_0 is the disc $D(0;1) = \{z : |z| < 1)$. The Julia set for g, $J(g)$, is the unit circle $C(0;1)$. The only critical point for g is at z_0.

A crucial property of the Julia set is that iterations of g in some neighbourhood of a point $z^* \in J(g)$ cover the complex plane with at most one exception. Specifically, let

$$\begin{aligned} g_1 &= g(z) \\ g_{k+1} &= g(g_k), \end{aligned}$$

and for any set U let

$$g_k(U) = \{g_k(z) : z \in U\}.$$

If $N(z^*)$ is a neighbourhood of a point z^*, then the set

$$G(z^*) = \cup_{k=1}^{\infty} g_k(N(z^*))$$

omits at most one point in $\mathbb{C}$. If there exists a point $p \in \mathbb{C}$ such that $G(z^*) = \mathbb{C} - \{p\}$, then p is called an exceptional point. For our case, $g(z) = z^n$, and the nature of the exceptional point (if any) is particularly tractable. If P is any polynomial of degree n and p is an exceptional point, then it can be shown g can be written in the form

$$P(z) = p + b(z-p)^n,$$

where $b \neq 0$. More detailed statements and proofs of these results can be found in [2] and [8]. We thus see that if there is an exceptional point for g it must be at the attracting fixed point z_0.

Let $\Omega \subseteq \mathbb{C}$ and denote the set of functions holomorphic in Ω by $\mathcal{H}(\Omega)$. Equation (1.1) provides a simple mechanism for continuation. In particular, we have

$$\frac{dy}{dg} = \frac{ay' + y''}{g'}.$$

Suppose that $y \in \mathcal{H}(N(\hat{z}))$, then the above expression shows that $y \in \mathcal{H}(g(N(\hat{z})))$ provided that $g'(z) \neq 0$ in $N(\hat{z})$. Now, g' vanishes only at z_0, which is also a fixed point. Thus $g(N(\hat{z}))$ contains z_0 only if $N(\hat{z})$ contains z_0. If $z_0 \notin N(\hat{z})$, then above expression shows that y must also be holomorphic in a set that is distinct from $N(\hat{z})$. If $\hat{z} \neq 0$ we can always choose a neighbourhood $N(\hat{z})$ that does not contain z_0. With such a choice of neighbourhood the process can be repeated any number of times to show that $y \in \mathcal{H}(g_k(N(\hat{z})))$ for $k \in \mathbb{N}$. In this manner we see that if $y \in \mathcal{H}(N(\hat{z}))$ is a solution to equation (1.1) then $y \in \mathcal{H}(G(\hat{z}))$, where $G(\hat{z}) = \cup_{k=1}^{\infty} g_k(N(\hat{z}))$.

Suppose now that $y \in \mathcal{H}(N(z^*))$, and $z^* \in J(g)$. The above arguments show that $y \in \mathcal{H}(G(z^*))$, where $G(z^*)$ omits at most one point in $\mathbb{C}$ and that point must be the origin. In summary, we have the following result.

Theorem 2.1 *Suppose that y is a solution to equation (1.1) that is holomorphic at $z^* \in J(g)$. Then y can be holomorphically continued to all points in $\mathbb{C} - \{0\}$.*

We show that the only entire solutions to equation (1.1) must be constant functions, which are available only if $a = \lambda$. We first establish a lemma.

Lemma 2.2 *Let $f \in \mathcal{H}(\mathbb{C})$ and*

$$M_f(R) = \sup_{|z|=R} |f(z)|.$$

Suppose that there exist numbers $\mu > 0$, $\delta > 0$, and $n > 1$ such that

$$M_f(R^n) \leq \mu M_f(R+\delta), \tag{2.1}$$

for all $R > 1$. Then f must be constant.

Proof: The proof is similar to that given in [7]. Suppose that f is not a constant and let

$$h(z) = \frac{f(z) - f(0)}{z}.$$

Since $f \in \mathcal{H}(\mathbb{C})$ and f is not constant, we have $h \in \mathcal{H}(\mathbb{C})$. The triangle inequality gives

$$\begin{aligned} M_h(R) &\leq \frac{M_f(R) + |f(0)|}{R} && (2.2) \\ M_f(R) &\leq R M_h(R) + |f(0)|. && (2.3) \end{aligned}$$

Inequalities (2.1) and (2.2) yield

$$\begin{aligned} M_h(R^n) &\leq \frac{M_f(R^n) + |f(0)|}{R^n} \\ &\leq \frac{\mu M_f(R+\delta) + |f(0)|}{R^n}, \end{aligned}$$

and inequality (2.3) shows that

$$\begin{aligned} M_h(R^n) &\leq \frac{\mu\left((R+\delta) M_h(R+\delta) + |f(0)|\right) + |f(0)|}{R^n} \\ &= \frac{1}{R^n}\left(R + \delta + \frac{(\mu+1)|f(0)|}{M_h(R+\delta)}\right) M_h(R+\delta), \end{aligned}$$

for all $R > 1$. Since $n > 1$ and $M_h(R+\delta)$ is increasing with R by the Maximum Modulus Theorem, we have

$$\lim_{R\to\infty} \frac{1}{R^n}\left(R + \delta + \frac{(\mu+1)|f(0)|}{M_h(R+\delta)}\right) = 0;$$

consequently, there exists a $\hat{R}$ such that for all $R > \hat{R}$

$$\frac{1}{R^n}\left(R+\delta+\frac{(\mu+1)|f(0)|}{M_h(R+\delta)}\right) < 1.$$

For such choices of R

$$R^n > R+\delta,$$

and

$$M_h(R^n) < M_h(R+\delta).$$

The last two inequalities contradict the Maximum Modulus Theorem. Therefore f must be a constant.

□

Theorem 2.3 *The only entire solutions to equation (1.1) are constant functions.*

Proof: Suppose $y \in \mathcal{H}(\mathbb{C})$ is a solution to equation (1.1). Then

$$|\lambda| M_y(R^n) \leq |a| M_y(R) + M_{y'}(R).$$

For any $\delta > 0$ the Cauchy integral formula can be used to show that

$$M_{y'}(R) \leq \frac{M_y(R+\delta)}{\delta};$$

hence,

$$\begin{aligned} M_y(R^n) &\leq \frac{1}{|\lambda|}\left(|a| M_y(R) + \frac{M_y(R+\delta)}{\delta}\right) \\ &\leq \frac{1}{|\lambda|}\left(|a| + \frac{1}{\delta}\right) M_y(R+\delta). \end{aligned}$$

Choose $\delta > 1$. Then y satisfies inequality (2.1) with

$$\mu = \frac{|a|+1}{|\lambda|}.$$

Lemma 2.2 implies that y must be constant.

□

Corollary 2.4 *Suppose that y is a nonconstant solution to equation (1.1) that is holomorphic at $z^* \in J(g)$. Then z_0 must be a (non removable) singularity for y. Moreover, if $a \neq 0$ then $z_0 = 0$ must be a branch point.*

Proof: That z_0 must be a singularity follows immediately from Theorems 2.1 and 2.3, since y is not a constant. Suppose that $a \neq 0$ and that there exists an annulus $A(0; r, R) = \{z : r < |z| < R\}$ such that $y \in \mathcal{H}(A(0; r, R))$. Laurent's theorem implies that y can be represented in the form

$$y(z) = \sum_{k=1}^{\infty} \frac{c_k}{z^k} + H(z), \tag{2.4}$$

for $z \in A(0; r, R)$. Here, H denotes a function holomorphic in $D(0; R)$. Substituting expression (2.4) into equation (1.1) and equating principal parts gives

$$\sum_{k=1}^{\infty} \frac{-kc_k}{z^{k+1}} + a \sum_{k=1}^{\infty} \frac{c_k}{z^k} = \lambda \sum_{k=1}^{\infty} \frac{c_k}{z^{nk}}.$$

The above expression implies that

$$ac_1 = 0,$$

and

$$-(k-1)c_{k-1} + ac_k = \begin{cases} \lambda c_m & \text{if } k = nm,\ m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Since $a \neq 0$ we have $c_1 = 0$, and it follows immediately from the recursive relation for the coefficients c_k that the principal part must be zero and therefore $y(z) = H(z)$ for all $z \in A(0; r, R)$. But y cannot be holomorphic at $z_0 = 0$. We therefore conclude that there is no annulus centred at z_0 wherein y is holomorphic. This means that z_0 cannot be an isolated singularity such as a pole or essential isolated singularity. Theorem 2.1 precludes the existence of other isolated singularities and natural boundaries. The singularity at z_0 must consequently be a branch point.

□

The case $a = -2$ and $\lambda = -1$ provides a simple example that illustrates the above result. The initial-value problem

$$\begin{aligned} y'(z) - 2y(z) &= -y(z^2) \\ y(1) &= 1. \end{aligned}$$

has the solution

$$y(z) = 2 \log z + \frac{1}{z},$$

which is holomorphic at $z = 1 \in J(g)$ and has a logarithmic branch point at $z_0 = 0$.

It is an open question whether the initial-value problem (1.1), (1.2) has holomorphic solutions at fixed points in $J(g)$ for general values of a and λ. Certainly, if a solution exists then it must be unique because the differential equation (1.1) determines the derivatives of y at a fixed point uniquely. We show in the next section that for the case $a = 0$, the initial-value problem has solutions holomorphic at a fixed point in $J(g)$ only for special values of λ. This result suggests that the initial-value problem (1.1), (1.2) has holomorphic solutions only for special values of a and λ.

We now look at solutions that are holomorphic at the origin. The analysis for the initial-value problem in a neighbourhood of an attracting fixed point is more complete making this case more tractable.

Theorem 2.5 *For all values of a and λ there exists a unique solution to the initial-value problem (1.1), (1.2) that is holomorphic at $z_0 = 0$. The solution can be continued throughout the disc $D(0;1)$, and the circle $C(0;1) = J(g)$ forms a natural boundary for nonconstant solutions.*

Proof: The existence and uniqueness of a local solution to the initial-value problem follows immediately from Theorem 2-2 in [7]. Let y be a local solution. Then y can be represented by a power series centred at 0 with a radius of convergence $\rho > 0$. Suppose that $\rho < 1$. Then $y \in \mathcal{H}(D(0;\rho))$ and y has a singularity at some point $\hat{z} \in C(0;\rho)$. The function $e^{az}y(z)$ is therefore singular at $\hat{z}$ and equation (1.1) implies that $e^{az}y(z^n)$ must be singular at $\hat{z}$. We thus have that y is singular at $\hat{z}^2$. But $|\hat{z}| = \rho < 1$; therefore, $\hat{z}^2 < \rho$ and consequently we have the contradiction $y \notin \mathcal{H}(D(0;\rho))$. Therefore $\rho \geq 1$.

Suppose that y is a solution that is holomorphic at 0 and at a point $z^* \in J(g)$. Theorem 2.1 implies that $y \in \mathcal{H}(\mathbb{C})$ and Theorem 2.3 implies that y must be a constant function. If y is a nonconstant solution, then y cannot be holomorphic at any point in $J(g)$; hence, the Julia set must form a natural boundary.

$\square$

3 The Case $a = 0$

In this section we focus on the initial-value problem

$$y'(z) = \lambda y(z^n), \tag{3.1}$$

$$y(\tilde{z}) = y_0 \tag{3.2}$$

Most of the analysis of the previous section can be applied to this problem; however, we can establish some of these results directly, and for this problem we can find the power series solution explicitly thereby illustrating Theorem 2.5.

We note that Heard's work (*op. cit.*) on the asymptotic behaviour of real solutions to the initial value problem for the case $a \neq 0$ and the case $a = 0$ indicates that the solution classes are somewhat different. For example, in the former case if say $n = 2$, he showed that the solutions are $O((\log x)^\kappa)$ as $x \to \infty$, where

$$\kappa = \mathrm{Re}\left(\frac{\log(-a) - \log\lambda}{\log 2}\right).$$

For $n = 2$ he showed that the solutions to the latter problem are $O(x^{-1}(\log x)^\kappa f(\log\log x))$ as $x \to \infty$, where

$$\kappa = \mathrm{Re}\left(\frac{-\log\lambda}{\log 2}\right)$$

and f is a function of period $\log 2$. Although his work concerned real solutions and no assumptions regarding holomorphicity were imposed, it does signal that the solutions behave asymptotically different for the two cases.

Following Heard, we note that equation (3.1) can be transformed into the well-known pantograph equation. Specifically, let $\mu = (n-1)^{-1}$, and

$$y(z) = \frac{1}{z^\mu} f(\log z). \tag{3.3}$$

Then, equation (3.1) transforms to the pantograph equation

$$f'(w) - \mu f(w) = \lambda f(nw), \tag{3.4}$$

where

$$w = \log z. \tag{3.5}$$

There is a formidable body of research on pantograph equations (e.g. [1], [5], [6]). One feature of these equations is that they admit solutions holomorphic at $w = 0$ only for special values of λ, and the resulting solutions are polynomials.

Lemma 3.1 *The only solutions to equation (3.4) that are holomorphic at $w = 0$ are polynomial solutions. Such solutions exist only if*

$$\lambda = -\frac{\mu}{n^k}, \tag{3.6}$$

for some $k = 0, 1, 2, \ldots$.

Proof: The proof is given in [6], but it is simple enough to include here for completeness. If f is holomorphic at $w = 0$ then f can be represented by a power series

$$f(w) = \sum_{k=0}^{\infty} c_k w^k.$$

Substituting this power series into equation (3.4) gives

$$c_{k+1} = \frac{\mu + \lambda n^k}{k+1} c_k.$$

Suppose that equation (3.6) is not satisfied for any k and that $c_0 \neq 0$. Then the recursive relationship shows that

$$\frac{c_{k+1}}{c_k} = \frac{\mu + \lambda n^k}{k+1} \to \infty,$$

as $k \to \infty$, since $n > 1$.

□

Lemma 3.1 can be exploited to show that equation (3.1) does not have solutions that are holomorphic at fixed points in $J(g)$ unless λ satisfies equation (3.6).

Theorem 3.2 *Suppose that y is a solution of equation (3.1) that is holomorphic at some point of the Julia set. Then y must be of the form*

$$y(z) = \frac{1}{z^{\mu}} P_{n,k}(\log z), \tag{3.7}$$

where $P_{n,k}$ is a polynomial of degree k in $\log z$ and k, λ and n satisfy equation (3.6).

Proof: If y is holomorphic at a point in $J(g)$ then we know that it must be holomorphic at all points on $J(g)$. Without loss of generality we can assume that it is holomorphic at $z_1 = 1$. Now, y is holomorphic at $z_1 = 1$; hence, $f(w) = e^{\mu w} y(e^w)$ must be holomorphic at $w = 0$. Since y is a solution to (3.1), f must be a solution to (3.4). Lemma 3.1 thus implies that there is a k such that k, λ and n satisfy equation (3.6) and that f must be a polynomial in w. The transformation (3.5) shows that y must be of the form (3.7). In fact, it can be shown that

$$P_{n,k}(\log z) = \sum_{j=0}^{k} \frac{\prod_{m=k-j+1}^{k} \left(1 - \frac{1}{n^m}\right)}{(n-1)^{j-1} j!} (\log z)^j.$$

□

The proof of Corollary (2.4) shows that, generically, the singularity at the origin must be a branch point. The solution form (3.7) shows that this branch point is logarithmic in character. The proof of Corollary (2.4), however, specified that $a \neq 0$. The impact of this assumption was to ensure that $c_1 = 0$ in the Laurent expansion. This in turn forced all the other Laurent coefficients to vanish. For $a = 0$ the recursive relation is

$$-(k-1)c_{k-1} = \begin{cases} \lambda c_m & \text{if } k = nm,\ m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2 shows that the only possible meromorphic solution is $y = 1/z^{\mu}$. If $n > 2$ then we have $c_1 = 0$ and it follows readily from the above relation that all the other coefficients must vanish. We thus see that if $n > 2$ then there are no meromorphic solutions. The special case is $n = 2$. We have the solution $y(z) = y_0/z$, provided $\lambda = -1$.

We now look at solutions that are holomorphic in a neighbourhood of the attracting fixed point $z_0 = 0$. We know from Theorem (2.5) that there is a holomorphic solution and that the unit circle is a natural boundary. For the case $a = 0$, we can find the power series solution explicitly and use the Hadamard Gap Theorem [9] to establish a natural boundary. We focus on the case $n = 2$.

We know that y can be represented by a power series of the form

$$y(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{3.8}$$

Substituting the series (3.8) into equation (3.1)with $n = 2$ and equating coefficients of z^k gives

$$y(z) = \sum_{k=0}^{\infty} c_k z^{2^k - 1}, \tag{3.9}$$

where

$$c_0 = y_0,$$

and, for $n \geq 0$,

$$c_{k+1}(2^k - 1) = \lambda c_k. \tag{3.10}$$

If $y_0 = 0$, then y is the trivial solution. If $y_0 \neq 0$, then $c_k \neq 0$ for all k. The ratio test can thus be used to show that the series converges uniformly in the closed disc $\bar{D}(0; 1) = \{z : |z| \leq 1\}$. The function defined by the series (3.9) is thus holomorphic in $D(0; 1)$ and continuous on $\bar{D}(0; 1)$.

In detail, the solution is given by

$$y(z) = y_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{\lambda^k z^{2^k - 1}}{\prod_{j=1}^{k} (2^j - 1)} \right\}. \tag{3.11}$$

A feature of this series is that there are large gaps between nonzero coefficients. The Hadamard Gap Theorem can be invoked to show that the series has a natural boundary on the circle $C(0; 1)$. For this case, we thus have an alternative proof of the existence of a natural boundary.

Although every point on the unit circle is a singularity for y, the series defines a function continuous on the unit circle and, for any $\phi \in \mathbb{R}$, Abel's theorem shows that

$$\lim_{z \to e^{i\phi}} y(z) = y_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{\lambda^k e^{i(2^k - 1)\phi}}{\prod_{j=1}^{k} (2^j - 1)} \right\},$$

where z approaches $e^{i\phi}$ along any path not tangent to the unit circle. In particular, at the repelling fixed point $z_1 = 1$, we have

$$y(1) = y_0 F_2(\lambda),$$

where

$$F_2(\lambda) = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{\prod_{j=1}^{k} (2^j - 1)}. \tag{3.12}$$

The function F is an entire function in λ and, in fact, it is a partition function that can be recast as an infinite product by use of the Euler identity

$$\prod_{k=0}^{\infty} (1 + \beta x^{2k+1}) = 1 + \sum_{k=1}^{\infty} \frac{\beta^k x^{k^2}}{\prod_{j=1}^{k} (1 - x^{2j})} \tag{3.13}$$

(cf. [3] pg. 278). Using $x = p = 1/\sqrt{2}$ and $\beta = \lambda p$ in expression (3.13) yields

$$F_2(\lambda) = \prod_{k=1}^{\infty} \left(1 + \frac{\lambda}{2^k}\right). \tag{3.14}$$

The above expression gives and eigenvalue character to the problem. For example, the boundary-value problem

$$\begin{aligned} y'(z) &= \lambda y(z^2), \\ y(0) &= 1, \\ y(1) &= 0, \end{aligned}$$

is well posed and leads to eigenvalues $\lambda_m = -2^m$, where m is a nonnegative integer. It is also of interest to note that for such a choice of λ it is possible to construct a real solution that is valid over $[-1, \infty)$ such that it has compact support and is infinitely differentiable. The power series (3.9) can be used to define the solution in $[-1, 1]$ and we can set $y = 0$ for $x > 1$. The differential equation ensures that the function is infinitely differentiable at $x = 1$.

The above results are not specific to the case $n = 2$. For any integer $n \geq 2$, the same analysis gives

$$y(z) = y_0 \left\{ 1 + \sum_{k=1}^{\infty} \frac{\lambda^k z^{Q_k}}{\prod_{j=1}^{k} Q_j} \right\},$$

where

$$Q_k = \frac{n^k - 1}{n - 1}.$$

The Hadamard Gap Theorem shows that $C(0; 1)$ must be a natural boundary for the holomorphic function defined by this series. At the repelling fixed point $z_1 = 1$ we have

$$y(1) = y_0 F_n(\lambda),$$

where,

$$F_n(\lambda) = \prod_{k=1}^{\infty} \left(1 + \frac{\lambda(n-1)}{n^k}\right).$$

4 Conclusions

Solutions to the initial value problem (1.1), (1.2) have some interesting features. In this paper we showed that nonconstant solutions that are holomorphic at the origin, which is an attracting fixed point for z^n, must have the circle $|z| = 1$ as a natural boundary. This work builds on that of Marshall *et al.*[7], but they did not consider the case when the condition of holomorphy at the attracting fixed point is replaced by the condition that solution be holomorphic somewhere on the Julia set. For the initial-value problem (1.1), (1.2) we showed that if y is a nonconstant solution that is holomorphic at some point of the Julia set, then the solution must have a branch point at the origin. It is not clear, however, whether there are such solutions for general values of a and λ.

The special case $a = 0$ provides a detailed example that illustrates the general results. This equation can be transformed into the familiar pantograph equation, and it is possible to construct nonconstant solutions that are holomorphic on the Julia set. A feature here is there are solutions only for certain values of λ. The proof of this relies on finding holomorphic solution to the pantograph equation. Although the example suggests a similar result for more general values of a, it is an open question whether such solutions exist and if so under what restrictions on the parameter λ. Future work includes resolving this question and extending the results to more general polynomial functional arguments.

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