On the duality principle by Casazza, Kutyniok, and Lammers

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Ole Christensen, Hong Oh Kim, Rae Young Kim

BK21 Research Report

10 - 03

February 11, 2010

DEPARTMENT OF MATHEMATICAL SCIENCES

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February 2, 2010

Abstract

The R-dual sequences of a frame $\{f_i\}_{i\in I}$, introduced by Casazza, Kutyniok and Lammers in [1], provide a powerful tool in the analysis of duality relations in general frame theory. In this paper we derive conditions for a sequence $\{\omega_j\}_{j\in I}$ to be an R-dual of a given frame $\{f_i\}_{i\in I}$. In particular we show that the R-duals $\{\omega_j\}_{j\in I}$ can be characterized in terms of frame properties of an associated sequence $\{n_i\}_{i\in I}$. We also derive the duality results obtained for tight Gabor frames in [1] as a special case of a general statement for R-duals of frames in Hilbert spaces. Finally we consider a relaxation of the R-dual setup of independent interest. Several examples illustrate the results.

Math Subject Classifications: 42C15, 42C40, 42A38.
Keywords: Duality principle, Frame, Riesz basis, Gabor system, Wexler-Raz theorem.

1 Introduction and notation

Let $\{f_i\}_{i\in I}$ denote a frame for a separable Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. In [1], Casazza, Kutyniok, and Lammers introduced the *Riesz-dual sequence* (R-dual sequence) *of* $\{f_i\}_{i\in I}$ *with respect to a choice of orthonormal bases* $\{e_i\}_{i\in I}$ *and* $\{h_i\}_{i\in I}$ as the sequence $\{\omega_j\}_{j\in I}$ given by

$$\omega_j = \sum_{i\in I} \langle f_i, e_j\rangle h_i, \; j \in I. \tag{1}$$

The paper [1] demonstrates that there is a strong relationship between the frame-theoretic properties of $\{\omega_j\}_{j\in I}$ and $\{f_i\}_{i\in I}$, see Theorem 1.3 below for details. The purpose of this paper is to analyze the concept of R-dual sequence from another angle than it was done in [1]. Technically this is done by considering a dual formulation of (1), namely, for a given frame $\{f_i\}_{i\in I}$ and a (Riesz) sequence $\{\omega_j\}_{j\in I}$ to search for orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \ \ i \in I. \tag{2}$$

Using this approach we state a number of equivalent conditions for $\{\omega_j\}_{j\in I}$ to be an R-dual of $\{f_i\}_{i\in I}$. In particular we introduce a sequence $\{n_i\}_{i\in I}$ that can be used to check whether $\{\omega_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ or not; in fact, the answer is yes if and only if $\{n_i\}_{i\in I}$ is a tight frame sequence with frame bound $E = 1$.

One of the key properties of the R-duals is a certain duality relation that resembles the duality principle in Gabor analysis. The driving force in the article [1] was the question whether the duality principle in Gabor analysis actually can be derived from the theory of the R-duals. The question remains unsolved, but in [1] a positive conclusion is derived in the special case of a tight Gabor frame. The results presented here shed new light on this issue: in fact, the partial result in [1] turns out to be a consequence of a general result about R-duals, valid for any tight frame in any Hilbert space.

In the rest of this section we review some of the needed facts about the R-duals, as well as tools from frame theory. We also state a few basic results about Gabor systems and their relationship to the R-dual concept. Our main results for the R-duals associated with general frames are stated in Section 2. Section 3 deals with an relaxation of the above setup: we show that for the relevant sequences $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ and any orthonormal basis $\{e_i\}_{i\in I}$ we can always find an *orthogonal system* $\{h_i\}_{i\in I}$ such that (2) holds. An additional condition on the relationship between $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ implies that $\{h_i\}_{i\in I}$ can even be chosen as an orthonormal system, i.e., compared to the general agenda only the completeness of $\{h_i\}_{i\in I}$ is missing. Finally, Section 4 deals with a special choice of the R-dual, and Appendix A contains a proof of a technical lemma.

Frames and Riesz bases. It will be essential to distinguish carefully between sequences forming a basis/frame for the entire Hilbert space $\mathcal{H}$ or a subspace hereof. For that reason we begin with the following standard definition:

Definition 1.1 *Let I denote a countable index set.*

(i) A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,||f||^2, \ \forall f \in \mathcal{H}.$$

(ii) A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,||f||^2, \ \forall f \in \mathcal{H}.$$

The numbers A, B are called frame bounds. The frame is tight if we can choose $A = B$.

(iii) A sequence $\{\omega_j\}_{j\in I}$ in $\mathcal{H}$ is a Riesz sequence if there exist constants $C, D > 0$ such that

$$C \sum_{j\in I} |c_j|^2 \leq \left|\left|\sum_{j\in I} c_j\omega_i\right|\right|^2 \leq D \sum_{j\in I} |c_j|^2$$

for all finite sequences $\{c_i\}_{i\in I}$. The numbers C, D are called (Riesz) bounds.

(iv) A Riesz sequence $\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$ if $\overline{span}\{\omega_j\}_{j\in I} = \mathcal{H}$.

Given any sequence $\{\omega_j\}_{j\in I}$ in $\mathcal{H}$, let

$$W := \overline{\text{span}}\{\omega_j\}_{j\in I}.$$

In case $\{\omega_j\}_{j\in I}$ is a Riesz sequence, it is well known that $\{\omega_j\}_{j\in I}$ has a unique *dual Riesz sequence* belonging to W: that is, there exists a unique Riesz sequence $\{\widetilde{\omega_k}\}_{k\in I}$ of elements in W such that

$$\langle \omega_j, \widetilde{\omega_k}\rangle = \delta_{j,k}, \ j, k \in I. \tag{3}$$

If $\{\omega_j\}_{j\in I}$ has Riesz bounds C, D, then the dual Riesz sequence has bounds $1/D, 1/C$.

Recall that the sequence $\{\omega_j\}_{j\in I}$ has *infinite deficit* if

$$\dim(\overline{\operatorname{span}}\{\omega_j\}_{j\in I}^{\perp}) = \infty.$$

The R-duals of a sequence $\{f_i\}_{i\in I}$. We now state the definition of the R-dual sequence, repeated from [1]:

Definition 1.2 *Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which*

$$\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty, \ \forall j \in I. \tag{4}$$

The R-dual of $\{f_i\}_{i\in I}$ with respect to the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ is the sequence $\{\omega_j\}_{j\in I}$ given by

$$\omega_j = \sum_{i\in I} \langle f_i, e_j\rangle h_i, \ j \in I. \tag{5}$$

Note that any given sequence $\{f_i\}_{i\in I}$ has *many* associated R-dual sequences, namely, one for each choice of the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$. We collect the main results about the relationship between $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ from [1].

Theorem 1.3 *Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ denote orthonormal bases for $\mathcal{H}$, and let $\{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which $\sum_{i\in I} |\langle f_i, e_j\rangle|^2 < \infty$ for all $j \in I$. Define the R-dual $\{\omega_j\}_{j\in I}$ as in (5). Then the following hold:*

(i) For all $i \in I$,

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \tag{6}$$

i.e., $\{f_i\}_{i\in I}$ is the R-dual sequence of $\{\omega_j\}_{j\in I}$ w.r.t. the orthonormal bases $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$.

(ii) $\{f_i\}_{i\in I}$ is a Bessel sequence if and only $\{\omega_i\}_{i\in I}$ is a Bessel sequence; the Bessel bounds coincide.

(iii) $\{f_i\}_{i\in I}$ *satisfies the lower frame condition with bound* A *if and only if* $\{\omega_j\}_{j\in I}$ *satisfies the lower Riesz sequence condition with bound* A.

(iv) $\{f_i\}_{i\in I}$ *is a frame for* $\mathcal{H}$ *with bounds* A, B *if and only if* $\{\omega_j\}_{j\in I}$ *is a Riesz sequence in* $\mathcal{H}$ *with bounds* A, B.

(v) Two Bessel sequences $\{f_i\}_{i\in I}$ *and* $\{g_i\}_{i\in I}$ *in* $\mathcal{H}$ *are dual frames if and only if the associated R-dual sequences* $\{\omega_j\}_{j\in I}$ *and* $\{\gamma_j\}_{j\in I}$ *w.r.t. the same choices of orthonormal bases* $\{e_i\}_{i\in I}$ *and* $\{h_i\}_{i\in I}$ *satisfy that*

$$\langle \omega_j, \gamma_k\rangle = \delta_{j,k}, \ \ j, k \in I. \tag{7}$$

The property in Theorem 1.3(v) is a key result and the main motivation for the interest in the R-dual. The next paragraph explains this in more detail.

Gabor systems. For a function $g \in L^2(\mathbb{R})$, the *Gabor system* associated with g and two given parameters a, b is the collection of functions given by

$$\{e^{2\pi imbx} g(x - na)\}_{m,n\in\mathbb{Z}}.$$

We will use the short notation $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to denote the Gabor system.

The *duality principle* is one of the most fundamental results in Gabor analysis. It was discovered almost simultaneously by three groups of researchers: Janssen [6], Daubechies, Landau, and Landau [3], and Ron and Shen [7]. The duality principle concerns the relationship between frame properties for a function g with respect to the lattice $\{(na, mb)\}_{m,n\in\mathbb{Z}}$ and with respect to the so-called *dual lattice* $\{(n/b, m/a)\}_{m,n\in\mathbb{Z}}$:

Theorem 1.4 *Let* $g \in L^2(\mathbb{R})$ *and* $a, b > 0$ *be given. Then the Gabor system* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *with bounds* A, B *if and only if* $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *is a Riesz sequence with bounds* A, B.

Comparing Theorem 1.4 with Theorem 1.3(iv) makes it natural to ask whether $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ can be realized as the R-dual of $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with respect to appropriate choices of orthonormal bases $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ and $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$. Combined with Theorem 1.3(v), the well known *Wexler-Raz theorem* provides strong support for this hypothesis:

Theorem 1.5 *If the Gabor systems* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ *are dual frames, then the Gabor systems* $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *and* $\{\frac{1}{\sqrt{ab}} E_{m/a}T_{n/b}h\}_{m,n\in\mathbb{Z}}$ *are biorthogonal.*

In [1], Casazza, Kutyniok and Lammers proved the following partial result:

Theorem 1.6 *Assuming that* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R})$ *the following hold:*

(i) If $ab = 1$, *then* $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *can be realized as the R-dual of* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *w.r.t. certain choices of orthonormal bases* $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ *and* $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

(ii) If $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *is a tight frame, then* $\{\frac{1}{\sqrt{ab}}E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ *can be realized as the R-dual of* $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ *w.r.t. certain choices of orthonormal bases* $\{e_{m,n}\}_{m,n\in\mathbb{Z}}$ *and* $\{h_{m,n}\}_{m,n\in\mathbb{Z}}$ *for* $L^2(\mathbb{R})$.

Among other results, we will show that Theorem 1.6(ii) is a consequence of a general result that is valid for any tight frame in any separable Hilbert space.

2 Duality for general frames

Our first goal is to find conditions on two sequences $\{f_i\}_{i\in I}, \{\omega_j\}_{j\in I}$ such that $\{\omega_j\}_{j\in I}$ is the R-dual of $\{f_i\}_{i\in I}$ with respect to *some* choice of the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$.We will always assume that $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$. By Theorem 1.3 this implies that any R-dual sequence $\{\omega_j\}_{j\in I}$ is a Riesz sequence in $\mathcal{H}$ and that (6) holds. On the other hand, Theorem 1.3 shows that if $\{\omega_j\}_{j\in I}$ is a Riesz sequence and (6) holds, then $\{\omega_j\}_{j\in I}$ is a R-dual of $\{f_i\}_{i\in I}$. Thus we arrive at the following key question:

Question: Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ and $\{\omega_j\}_{j\in I}$ a Riesz sequence in $\mathcal{H}$. Under what conditions can we find orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ for $\mathcal{H}$ such that (6) holds?

We first show that for any Riesz sequence $\{\omega_j\}_{j\in I}$, any sequence $\{f_i\}_{i\in I}$, and any orthonormal basis $\{e_i\}_{i\in I}$, we can actually find and characterize the sequences $\{h_i\}_{i\in I}$ for which (6) holds; thus, the remaining question is whether at least one of these sequences forms an orthonormal basis for $\mathcal{H}$. The key point in the analysis is the definition of a sequence $\{n_i\}_{i\in I}$, given by

$$n_i := \sum_{k\in I} \langle e_k, f_i\rangle \widetilde{\omega_k},\ i \in I, \tag{8}$$

where $\{\widetilde{\omega_k}\}_{k\in I}$ is the dual Riesz sequence of $\{\omega_j\}_{j\in I}$. Note that under the above assumptions the sequences $\{\widetilde{\omega_k}\}_{k\in I}$ and $\{e_i\}_{i\in I}$ are Bessel sequences, implying that the infinite series defining n_i is convergent. We begin with a simple lemma, relating the involved sequences:

Lemma 2.1 *Let $\{\omega_j\}_{j\in I}$ be a Riesz basis for the subspace W of $\mathcal{H}$, with dual Riesz basis $\{\widetilde{\omega_k}\}_{k\in I}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Given any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$, define $\{n_i\}_{i\in I}$ as in (8). Then*

$$\langle \omega_j, n_i\rangle = \langle f_i, e_j\rangle, \ \forall i,j \in I.$$

Lemma 2.1 is a direct consequence of the definition of n_i and (3). Our starting point is now to characterize the sequences $\{h_i\}_{i\in I}$ for which (6) holds:

Proposition 2.2 *Let $\{\omega_j\}_{j\in I}$ be a Riesz basis for the subspace W of $\mathcal{H}$, with dual Riesz basis $\{\widetilde{\omega_k}\}_{k\in I}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Given any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$, the following hold:*

(i) There exists a sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i\rangle e_j, \ \forall i \in I. \tag{9}$$

(ii) The sequences $\{h_i\}_{i\in I}$ satisfying (9) are characterized as

$$h_i = m_i + n_i, \tag{10}$$

where n_i is given by (8) and $m_i \in W^\perp$.

(iii) If $\{\omega_j\}_{j\in I}$ is a Riesz basis for $\mathcal{H}$, then (9) has the unique solution

$$h_i = n_i, \ i \in I.$$

Proof. Expanding f_i in the orthonormal basis $\{e_j\}_{j\in I}$ and using Lemma 2.1,

$$f_i = \sum_{j\in I} \langle f_i, e_j\rangle e_j = \sum_{j\in I} \langle \omega_j, n_i\rangle e_j, \ i \in I,$$

i.e., the choice $h_i = n_i$ satisfies (9). This proves (i). For $m_i \in W^\perp$ it now follows from $\omega_j \in W$ that the choice $h_i = m_i + n_i$ will satisfy (9) as well. In order to complete the proof of (ii) we only need to show that all solutions $\{h_i\}_{i\in I}$ of (9) are of the form in (10). Let $\{h_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ satisfying (9). Fix any $i \in I$. We can write $h_i = m_i + n_i$ with $m_i := h_i - n_i$. The expansion coefficients of f_i in terms of the basis $\{e_i\}_{i\in I}$ are unique, so from

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j = \sum_{j\in I} \langle \omega_j, n_i \rangle e_j$$

it follows that

$$\langle \omega_j, h_i \rangle = \langle \omega_j, n_i \rangle, \ \forall j \in I,$$

i.e.,

$$\langle \omega_j, m_i \rangle = 0, \ \forall j \in I.$$

This implies that $m_i \in W^{\perp}$. This proves (ii). The result in (iii) is a consequence of (ii). $\square$

With Proposition 2.2 at hand our goal is now to find conditions under which an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ of the form (10) exists. We note that Proposition 2.2 did not use any assumption on $\{f_i\}_{i\in I}$ or any relationship between $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$. The uniqueness statement in Proposition 2.2(iii) makes it easy to find a case where no orthonormal basis of the form (10) exists, even if we assume that $\{f_i\}_{i\in I}$ is a frame:

Example 2.3 Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Let $\{\omega_i\}_{i\in I} := \{e_i\}_{i\in I}$, and

$$\{f_i\}_{i\in I} := \{2e_1, e_2, e_3, \cdots\}.$$

Then

$$\{\widetilde{\omega_k}\}_{k\in I} = \{e_1, e_2, e_3, \cdots\},$$

and n_i in (8) is given by

$$n_i = \begin{cases} 2e_1, & \text{if } i = 1; \\ e_i, & \text{if } i \geq 2. \end{cases}$$

The sequence $\{n_i\}_{i\in I}$ is clearly not an orthonormal basis. The uniqueness statement in Proposition 2.2(iii) now implies that no orthonormal basis $\{h_i\}_{i\in I}$ can satisfy (9). Thus $\{\omega_j\}_{j\in I}$ is not the R-dual of $\{f_i\}_{i\in I}$ w.r.t. $\{e_i\}_{i\in I}$ and any choice of orthonormal basis $\{h_i\}_{i\in I}$; this conclusion could of course also have been derived from Theorem 1.3. $\square$

We will now have a closer look at the properties of the sequence $\{n_i\}_{i\in I}$ in (8).

Lemma 2.4 *Let $\{\omega_j\}_{j\in I}$ be a Riesz sequence in $\mathcal{H}$ with bounds C, D, and let $\{e_i\}_{i\in I}$ an orthonormal basis for $\mathcal{H}$. Given a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ with frame bounds A, B, the sequence $\{n_i\}_{i\in I}$ in* (8) *is a frame for W with frame bounds $A/D, B/C$.*

Proof. It is clear that $n_i \in W,\ \forall i \in I$. Now, for any $f \in W$,

$$\begin{aligned}
\sum_{i\in I} |\langle f, n_i\rangle|^2 &= \sum_{i\in I} \left| \langle f, \sum_{k\in I} \langle e_k, f_i\rangle \widetilde{\omega_k}\rangle \right|^2 \\
&= \sum_{i\in I} \left| \sum_{k\in I} \langle f, \widetilde{\omega_k}\rangle \langle f_i, e_k\rangle \right|^2 \\
&= \sum_{i\in I} \left| \langle f_i, \sum_{k\in I} \langle \widetilde{\omega_k}, f\rangle e_k\rangle \right|^2 .
\end{aligned}$$

Note that $\{\widetilde{\omega_k}\}_{k\in I}$ is a Riesz basis for W with bounds $1/D, 1/C$. Thus the above calculation yields that

$$\begin{aligned}
\sum_{i\in I} |\langle f, n_i\rangle|^2 \geq A \left|\left| \sum_{k\in I} \langle \widetilde{\omega_k}, f\rangle e_k \right|\right|^2 &= A \sum_{k\in I} |\langle \widetilde{\omega_k}, f\rangle|^2 \\
&\geq \frac{A}{D} ||f||^2.
\end{aligned}$$

The proof for the upper bound is similar. $\square$

We will now present a solution to our key question, i.e., characterize the existence of an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ such that (9) holds. We note that the case where the Riesz sequence $\{\omega_j\}_{j\in I}$ spans the entire space $\mathcal{H}$ is solved in Proposition 2.2(iii). Thus, we concentrate on the case where the Riesz sequence $\{\omega_j\}_{j\in I}$ spans a proper subspace of $\mathcal{H}$.

Theorem 2.5 *Let $\{\omega_j\}_{j\in I}$ be a Riesz sequence spanning a proper subspace W of $\mathcal{H}$ and $\{e_i\}_{i\in I}$ an orthonormal basis for $\mathcal{H}$. Given any frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$, the following are equivalent:*

(i) $\{\omega_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ w.r.t. $\{e_i\}_{i\in I}$ and some orthonormal basis $\{h_i\}_{i\in I}$.

(ii) There exists an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ satisfying (9).

(iii) The sequence $\{n_i\}_{i\in I}$ in (8) *is a tight frame for W with frame bound $E = 1$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 2.2.
(ii) $\Rightarrow$(iii). Let P denote the orthogonal projection of $\mathcal{H}$ onto W. The expression in (10) for all solutions to (9) shows that a sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ is a solution if and only if $Ph_i = n_i, \ \forall i \in I$. Now, it is well known that the projection of an orthonormal basis onto a subspace forms a tight frame for that subspace with frame bound equal to one. Thus, if $\{h_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$, then necessarily $\{n_i\}_{i\in I}$ is a tight frame for W with frame bound $E = 1$.
(iii) $\Rightarrow$(ii). If $\{n_i\}_{i\in I}$ is a tight frame for W with frame bound $E = 1$, then Naimark's theorem (see, e.g., [5]) says that there exists an orthonormal basis for a larger Hilbert space such that $Ph_i = n_i$. Since W is assumed to be a proper subspace of $\mathcal{H}$ we can identify the larger Hilbert space with $\mathcal{H}$, which leads to the desired conclusion. □

Using Theorem 2.5 we can now give an example of a frame $\{f_i\}_{i\in I}$ and a Riesz sequence $\{\omega_j\}_{j\in I}$ that can not be an R-dual of $\{f_i\}_{i\in I}$ w.r.t. a given orthonormal basis $\{e_i\}_{i\in I}$ and any choice of $\{h_i\}_{i\in I}$, despite the fact that the bounds for $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ coincide:

Example 2.6 Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ and

$$\{f_i\}_{i\in I} := \{2e_1, e_1, e_2, e_3, \dots\},$$

$$\{\omega_j\}_{j\in I} = \{5e_1, e_3, e_5, \dots\}.$$

Then $\{f_i\}_{i\in I}$ is a frame with bounds $A=1, B=5$, and $\{\omega_j\}_{j\in I}$ is a Riesz sequence with the same bounds. The dual Riesz sequence is

$$\{\widetilde{\omega_k}\}_{k\in I} = \{\frac{1}{5}e_1, e_3, e_5, \dots\}.$$

Direct calculation shows that

$$\{n_i\}_{i\in I} = \{\frac{2}{5}e_1, \frac{1}{5}e_1, e_3, e_5, \dots\}.$$

The frame is clearly not tight, so $\{\omega_j\}_{j\in I}$ is not an R-dual of $\{f_i\}_{i\in I}$ with respect to $\{e_i\}_{i\in I}$ and any choice of an orthonormal basis $\{h_i\}_{i\in I}$. □

Combining Lemma 2.4 and Theorem 2.5, we obtain a partial answer to our key question:

Corollary 2.7 *Assume that $\{\omega_j\}_{j\in I}$ is a Riesz sequence with upper and lower bound A, spanning a proper subspace of $\mathcal{H}$, and that $\{f_i\}_{i\in I}$ is a tight frame for $\mathcal{H}$ with frame bound A. Then $\{\omega_i\}_{i\in I}$ is an R-dual of $\{f_i\}_{i\in I}$.*

Proof. The assumptions imply by Lemma 2.4 that $\{n_i\}_{i\in I}$ is a tight frame for W with frame bound $E=1$, for any choice of the orthonormal basis $\{e_i\}_{i\in I}$. Now the result follows from Theorem 2.5. □

The assumptions in Corollary 2.7 correspond exactly to the known relationship between a tight Gabor frame and the corresponding Gabor system on the dual lattice. Thus Corollary 2.7 is a generalization of the result from [1] that we stated in Theorem 1.6(ii).

The assumption that $\{\omega_j\}_{j\in I}$ spans a proper subspace of $\mathcal{H}$ is essential in Corollary 2.7:

Example 2.8 Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and

$$\{\omega_j\}_{j\in\mathbb{N}} := \{e_1, e_2, \cdots\}.$$

We now construct a tight frame $\{f_i\}_{i\in\mathbb{N}}$ for $\mathcal{H}$ for which $\{\omega_j\}_{j\in\mathbb{N}} := \{e_1, e_2, \cdots\}$ is not an R-dual w.r.t. $\{e_i\}_{i\in\mathbb{N}}$ and any choice of $\{h_i\}_{i\in\mathbb{N}}$. Split $\{e_i\}_{i\in\mathbb{N}}$ into a union of sequences with two elements, e.g.,

$$\{e_i\}_{i\in\mathbb{N}} = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots.$$

Associated with each pair $\{e_{2k-1}, e_{2k}\}$, $k \in \mathbb{N}$, we construct a tight frame for the space span$\{e_{2k-1}, e_{2k}\}$ with frame bound 1 and consisting of 3 vectors, to be denoted by $\{f_{3k-2}, f_{3k-1}, f_{3k}\}$ (this can be done in many ways, e.g., by Daubechies' Mercedes-Benz star). The union of the sequences $\{f_{3k-2}, f_{3k-1}, f_{3k}\}$, $k \in \mathbb{N}$, yields a tight frame $\{f_i\}_{i\in\mathbb{N}}$ for $\mathcal{H}$ with frame bound 1. Note that $\{f_{3k-2}, f_{3k-1}, f_{3k}\}$ does not form an orthonormal system, so $\{f_i\}_{i\in\mathbb{N}}$ is not an orthonormal system either. By Proposition 2.2(iii) the only sequence $\{h_i\}_{i\in\mathbb{N}}$ satisfying that

$$f_i = \sum_{j=1}^{\infty} \langle \omega_j, h_i \rangle e_j, \ \forall i \in \mathbb{N}$$

is $h_i = n_i$, with n_i defined as in (8). Now,

$$\langle n_i, n_j \rangle = \langle \sum_{k=1}^{\infty} \langle e_k, f_i \rangle \widetilde{\omega_k}, \sum_{k=1}^{\infty} \langle e_k, f_j \rangle \widetilde{\omega_k} \rangle = \sum_{k=1}^{\infty} \langle e_k, f_i \rangle \langle f_j, e_k \rangle = \langle f_j, f_i \rangle.$$

We have already argued that $\{f_i\}_{i\in\mathbb{N}}$ can not be an orthonormal system, so $\{n_j\}_{j\in\mathbb{N}}$ can not be an orthonormal system either. By Proposition 2.2(iii) we conclude that $\{\omega_j\}_{j\in\mathbb{N}}$ can not be an R-dual of $\{f_i\}_{i\in\mathbb{N}}$ w.r.t. $\{e_i\}_{i\in\mathbb{N}}$ and any choice of the orthonormal basis $\{h_i\}_{i\in\mathbb{N}}$. □

With Theorem 2.5 and Corollary 2.7 in mind it is natural to ask whether an orthonormal basis $\{h_i\}_{i\in I}$ for $\mathcal{H}$ satisfying (9) can be found if the frame $\{f_i\}_{i\in I}$ is non-tight. Intuitively this sounds unlikely - but there are cases where the answer is yes:

Example 2.9 Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$, and define the sequences $\{f_i\}_{i\in I}$ and $\{\omega_j\}_{j\in I}$ by

$$\{f_i\}_{i\in I} = \{\frac{1}{2}e_1, e_2, e_3, \cdots\},$$

respectively,

$$\{\omega_j\}_{j\in I} = \{\frac{1}{2}e_1, e_2, e_3, \cdots\}.$$

Then

$$\widetilde{\omega_k} = \{2e_1, e_2, e_3, \cdots\},$$

and thus

$$n_i = \sum_{k\in I} \langle e_k, f_i\rangle \widetilde{\omega_k} = e_i, \ \forall i \in I.$$

Thus $\{n_i\}_{i\in I}$ is an orthonormal basis and therefore tight, despite the fact that $\{f_i\}_{i\in I}$ is non-tight. □

Theorem 2.5 leads to a simple criterion for $\{\omega_j\}_{j\in I}$ to be an R-dual of $\{f_i\}_{i\in I}$. The result can be considered as an if and only if version of Proposition 5 in [1]:

Corollary 2.10 *Let $\{\omega_j\}_{j\in I}$ be a Riesz basis for the subspace W of $\mathcal{H}$, with dual Riesz basis $\{\widetilde{\omega_k}\}_{k\in I}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Given any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$, define $\{n_i\}_{i\in I}$ as in (8). For any $\{c_i\}_{i\in I} \in \ell^2(I)$, let the vectors e and ω be related by*

$$e = \sum_{j\in I} \overline{c_j} e_j, \quad \omega = \sum_{j\in I} c_j \omega_j. \tag{11}$$

Then $\{\omega_j\}_{j\in I}$ is an R-dual of $\{f_i\}_{i\in I}$ w.r.t. $\{e_i\}_{i\in I}$ and some orthonormal basis $\{h_i\}_{i\in I}$ if and only if

$$\sum_{i\in I} |\langle f_i, e\rangle|^2 = ||\omega||^2$$

for all choices of the sequence $\{c_i\}_{i\in I} \in \ell^2(I)$.

Proof. By the result in Lemma 2.1 and the relation between e and ω,

$$\langle n_i, \omega\rangle = \sum_{j\in I} \overline{c_j} \langle n_i, \omega_j\rangle = \sum_{j\in I} \overline{c_j} \langle e_j, f_i\rangle = \langle e, f_i\rangle.$$

Thus

$$\sum_{i\in I} |\langle n_i, \omega\rangle|^2 = \sum_{i\in I} |\langle e, f_i\rangle|^2.$$

The result now follows from Theorem 2.5. □

3 Orthonormal sequences $\{h_i\}_{i\in I}$

In Proposition 2.2 we have shown that a Riesz sequence $\{\omega_j\}_{j\in I}$ is an R-dual of a frame $\{f_i\}_{i\in I}$ if there exists orthonormal bases $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \ \forall i \in I. \tag{12}$$

In order to gain further insight in the problem we will now consider a weaker version of this condition: we will assume that $\{e_i\}_{i\in I}$ is a given orthonormal basis, and ask for the existence of an *orthogonal,* resp. *orthonormal* sequence $\{h_i\}_{i\in I}$ such that (12) holds. We will show that these questions have very general answers.

We begin with a lemma, stating an observation of independent interest. For the proof, see Appendix A.

Lemma 3.1 *Assume that $\{f_i\}_{i\in I}$ is a Bessel sequence with bound B. Then for any f_i, f_j,*

$$|\langle f_i, f_j\rangle|^2 \le B\left(B - ||f_i||^2 - ||f_j||^2\right) + ||f_i||^2 ||f_j||^2. \tag{13}$$

Note that the result in Lemma 3.1 is trivial if $B - ||f_i||^2 - ||f_j||^2 \ge 0$. However, under the assumptions given here it can very well happen that $B - ||f_i||^2 - ||f_j||^2 < 0$, and for such elements f_i, f_j the result is an improvement of Cauchy–Schwarz' inequality.

Theorem 3.2 *Let $\{\omega_j\}_{j\in I}$ be a Riesz sequence in $\mathcal{H}$ having infinite deficit, and let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Then the following hold:*

(i) For any sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ there exists an orthogonal sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ such that

$$f_i = \sum_{j\in I} \langle \omega_j, h_i \rangle e_j, \ \forall i \in I. \tag{14}$$

(ii) Assume that $\{f_i\}_{i\in I}$ is a Bessel sequence with bound B and that $\{\omega_j\}_{j\in I}$ has a lower Riesz basis bound $C \ge B$. Then there exists an orthonormal sequence $\{h_i\}_{i\in I}$ such that (14) holds.

(iii) For any Bessel sequence $\{f_i\}_{i\in I}$ and regardless of the lower Riesz bound for $\{\omega_j\}_{j\in I}$, there exist an orthonormal sequence $\{h_i\}_{i\in I}$ in $\mathcal{H}$ and a constant $\alpha > 0$ such that

$$f_i = \sum_{j\in I} \langle \alpha\omega_j, h_i\rangle e_j, \ \forall i \in I. \tag{15}$$

Proof. The proof of (i) is based on Proposition 2.2. We consider again the vectors n_i in (8) and want to find $m_i \in W^\perp, \ i \in I$, such that $h_i := m_i + n_i$ is an orthogonal sequence. For notational convenience, assume that $I = \mathbb{N}$. Note that with such a choice of h_i, we know that (14) is satisfied. Note also that

$$\langle h_i, h_j\rangle = \langle n_i, n_j\rangle + \langle m_i, m_j\rangle, \ \forall i, j \in I. \tag{16}$$

We will use the following inductive procedure. Choose $m_1 \in W^\perp$ arbitrarily. Now, take $m_2 \in W^\perp$ such that

$$\langle h_1, h_2\rangle = 0,$$

i.e., such that

$$\langle m_1, m_2\rangle = -\langle n_1, n_2\rangle.$$

In general, assuming that we have constructed $m_1, \ldots, m_N \in W^\perp$ such that $\{h_i\}_{i=1}^N$ is an orthogonal system, take $m_{N+1} \in W^\perp$ such that

$$\langle h_k, h_{N+1}\rangle = 0, \ k = 1, \ldots, N,$$

i.e., such that

$$\langle m_k, m_{N+1}\rangle = -\langle n_k, n_{N+1}\rangle, \ k = 1, \ldots, N.$$

This can always be done because $\{\omega_j\}_{j\in I}$ is assumed to have infinite deficit. We conclude that $\{h_i\}_{i\in I}$ forms an orthogonal system, as desired.

For the proof of (ii), let B denote an upper frame bound for $\{f_i\}_{i\in I}$ and C a lower bound for the Riesz sequence $\{\omega_j\}_{j\in I}$. By an argument like in the proof of Lemma 2.4, the sequence $\{n_i\}_{i\in I}$ is a Bessel sequence with bound $\frac{B}{C} \leq 1$; in particular, the norms of the vectors n_i are uniformly bounded by $||n_i|| \leq 1$. We now aim at a construction of a sequence $\{h_i\}_{i\in I}$ satisfying

(14) and $||h_i|| = 1, \ \forall i \in I$. We use the inductive procedure outlined in (i), but now paying attention to the norm of the vectors h_i. First we choose $m_1 \in W^\perp$ such that $||h_1|| = 1$, i.e., such that

$$||m_1|| = \sqrt{1 - ||n_1||^2}.$$

We now want to choose $m_2 \in W^\perp$ such that $||h_2|| = 1$ and $\langle h_1, h_2 \rangle = 0$; this means that we want that

$$||m_2|| = \sqrt{1 - ||n_2||^2} \ \text{ and } \ \langle m_1, m_2 \rangle = -\langle n_1, n_2 \rangle. \tag{17}$$

The first condition in (17) can always be satisfied; and the second can be satisfied for a sequence m_2 with $||m_2|| = \sqrt{1 - ||n_2||^2}$ if and only if

$$\sqrt{1 - ||n_1||^2}\sqrt{1 - ||n_2||^2} \geq |\langle n_1, n_2 \rangle|. \tag{18}$$

The condition in (18) is satisfied by Lemma 3.1.

Following the inductive procedure outlined in (i), we see that it is possible to construct an orthonormal sequence $\{h_i\}_{i\in I}$ satisfying (14) if

$$\sqrt{1 - ||n_i||^2}\sqrt{1 - ||n_j||^2} \geq |\langle n_i, n_j \rangle|, \ \forall i, j \in I,$$

which is satisfied by Lemma 3.1.

Finally, the result in (iii) is obtained by scaling of the Riesz sequence $\{\omega_j\}_{j\in I}$ in such a way that we obtain a sequence $\{\alpha\, \omega_j\}_{j\in I}$ to which we can apply (ii). □

4 A special choice of the R-dual of a given frame $\{f_i\}_{i\in I}$

We will now consider an operator theoretic way of constructing an R-dual of a given frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis. Then there exists a bounded, surjective, and linear operator $T : \mathcal{H} \to \mathcal{H}$ such that

$$Te_i = f_i. \tag{19}$$

We need a related operator. Define the mapping $\widetilde{T} : \mathcal{H} \to \mathcal{H}$ as the unique anti-linear bounded operator for which

$$\widetilde{T}e_i = f_i.$$

That is,

$$\widetilde{T}\left(\sum_{i\in I} c_i e_i\right) := \sum_{i\in I} \overline{c_i} f_i,\ \{c_i\}_{i\in I} \in \ell^2(I). \tag{20}$$

The operator $\widetilde{T}$ is clearly bounded and surjective. Also, for any $g \in \mathcal{H}$ the mapping

$$f \mapsto \langle g, \widetilde{T} f\rangle$$

is bounded and linear. The adjoint operator $\widetilde{T}^*$ is introduced as the unique mapping $g \mapsto \widetilde{T}^* g$ for which

$$\langle g, \widetilde{T} f\rangle = \langle f, \widetilde{T}^* g\rangle,\ \forall f, g \in \mathcal{H}.$$

It is easy to check that $\widetilde{T}^*$ is bounded and anti-linear.

Proposition 4.1 *Let $\{f_i\}_{i\in I}$ be a frame and $\{e_i\}_{i\in I}$ an orthonormal basis. Define the linear operator T by* (19) *the anti-linear operator $\widetilde{T}$ by* (20), *and let*

$$\omega_j := \widetilde{T}^* e_j,\ j \in I. \tag{21}$$

Then $\{\omega_j\}_{j\in I}$ is the R-dual of $\{f_i\}_{i\in I}$ with respect to the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I} := \{e_i\}_{i\in I}$.

Proof. Expanding $\widetilde{T}^* e_j$ in the orthonormal basis $\{e_i\}_{i\in I}$ and using the definition of $\widetilde{T}^*$ leads to

$$\begin{aligned}
\widetilde{T}^* e_j &= \sum_{i\in I} \langle \widetilde{T}^* e_j, e_i\rangle e_i \\
&= \sum_{i\in I} \langle \widetilde{T} e_i, e_j\rangle e_i \\
&= \sum_{i\in I} \langle f_i, e_j\rangle e_i.
\end{aligned}$$

By definition of the R-dual this shows that $\{\omega_j\}_{j\in I}$ is the R-dual of $\{f_i\}_{i\in I}$ with respect to the orthonormal bases $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I} := \{e_i\}_{i\in I}$. □

The example below is a concrete construction of an R-dual. It does not play any role in the present paper, but it is included here because it is useful as a nontrivial toy example.

Example 4.2 For any orthonormal basis $\{e_i\}_{i=1}^\infty$, the sequence $\{e_i+e_{i+1}\}_{i=1}^\infty$ is a Bessel sequence with bound 4 (see [2]). Thus, the union of the sequences $\{e_i\}_{i=1}^\infty$ and $\{\frac{e_i+e_{i+1}}{\sqrt{2}}\}_{i=1}^\infty$, i.e., the sequence

$$\{f_i\}_{i=1}^\infty := \{e_1, \frac{e_1+e_2}{\sqrt{2}}, e_2, \frac{e_2+e_3}{\sqrt{2}}, \dots\}$$

is a frame of unit norm vectors, with frame bounds $1, 3$. Note that

$$f_{2i-1} = e_i, \quad f_{2i} = \frac{e_i+e_{i+1}}{\sqrt{2}}, \ i \in \mathbb{N}.$$

The R-dual $\{\omega_j\}_{j\in\mathbb{N}}$ w.r.t. the orthonormal basis $\{e_i\}_{i=1}^\infty$ is given by

$$\omega_j = \sum_{i=1}^\infty \langle f_i, e_j\rangle e_i;$$

direct calculation shows that

$$\omega_1 = e_1 + \frac{1}{\sqrt{2}}e_2, \quad \omega_j = e_{2j-1} + \frac{1}{\sqrt{2}}e_{2j} + \frac{1}{\sqrt{2}}e_{2j-2}, \ j \geq 2. \qquad \square$$

5 Appendix A - proof of Lemma 3.1

Proof of Lemma 3.1: We give the proof for the case $B = 1$; the general case follows from here by replacing $\{f_i\}_{i\in I}$ by $\{f_i/\sqrt{B}\}_{i\in I}$. For notational convenience we take $i = 1, j = 2$.

First, we assume $\langle f_1, f_2\rangle$ is real. Let $f := xf_1 + f_2$ for some $x \in \mathbb{R}$. Then

$$||f||^2 = x^2||f_1||^2 + 2x\langle f_1, f_2\rangle + ||f_2||^2 \tag{22}$$

and

$$\begin{aligned} |\langle f, f_1\rangle|^2 + |\langle f, f_2\rangle|^2 &= ||f_1||^4x^2 + 2\langle f_1, f_2\rangle||f_1||^2x + |\langle f_1, f_2\rangle|^2 \\ &+ |\langle f_1, f_2\rangle|^2x^2 + 2\langle f_1, f_2\rangle||f_2||^2x + ||f_2||^4 \\ &= (||f_1||^4 + |\langle f_1, f_2\rangle|^2)x^2 + 2\langle f_1, f_2\rangle(||f_1||^2 + ||f_2||^2)x \\ &+ ||f_2||^4 + |\langle f_1, f_2\rangle|^2 \end{aligned} \tag{23}$$

Using the upper frame condition on f,

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq ||f||^2;$$

keeping only the terms corresponding to $i = 1, 2$ shows that

$$|\langle f, f_1\rangle|^2 + |\langle f, f_2\rangle|^2 \leq ||f||^2. \tag{24}$$

Putting (22) and (23) into this yields

$$\begin{aligned}&(||f_1||^4 + |\langle f_1, f_2\rangle|^2)x^2 + 2\langle f_1, f_2\rangle(||f_1||^2 + ||f_2||^2)x + ||f_2||^4 + |\langle f_1, f_2\rangle|^2\\ &\leq x^2||f_1||^2 + 2x\langle f_1, f_2\rangle + ||f_2||^2,\end{aligned}$$

or,

$$\begin{aligned}&(||f_1||^2 - ||f_1||^4 - |\langle f_1, f_2\rangle|^2)x^2 + 2\langle f_1, f_2\rangle(1 - ||f_1||^2 - ||f_2||^2)x\\ &+||f_2||^2 - ||f_2||^4 - |\langle f_1, f_2\rangle|^2 \geq 0.\end{aligned} \tag{25}$$

We split into two cases:
(1): Assume $||f_1||^2 - ||f_1||^4 - |\langle f_1, f_2\rangle|^2 = 0$, or,

$$|\langle f_1, f_2\rangle|^2 = ||f_1||^2 - ||f_1||^4. \tag{26}$$

Note that (25) is satisfied for all real values of x. Thus,

$$\langle f_1, f_2\rangle(1 - ||f_1||^2 - ||f_2||^2) = 0.$$

If $\langle f_1, f_2\rangle = 0$, then (13) trivially holds; if $1 - ||f_1||^2 - ||f_2||^2 = 0$, then (26) implies that

$$\begin{aligned}|\langle f_1, f_2\rangle|^2 &= ||f_1||^2 - ||f_1||^4\\ &= (1 - ||f_1||^2)||f_1||^2\\ &= (1 - ||f_1||^2)(1 - ||f_2||^2),\end{aligned}$$

so (13) holds.

(2): Assume that $||f_1||^2 - ||f_1||^4 - |\langle f_1, f_2\rangle|^2 \neq 0$. Let

$$\begin{aligned}&a := ||f_1||^2 - ||f_1||^4 - |\langle f_1, f_2\rangle|^2 \ (\neq 0)\\ &b := \langle f_1, f_2\rangle(1 - ||f_1||^2 - ||f_2||^2)\\ &c := ||f_2||^2 - ||f_2||^4 - |\langle f_1, f_2\rangle|^2.\end{aligned} \tag{27}$$

Then (25) implies that

$$ax^2 + 2bx + c \geq 0.$$

Substitute $x := -b/a$ into this, to obtain

$$-(b^2 - ac)/a \geq 0. \tag{28}$$

The frame condition (24) applied to $f := f_1$ yields that

$$|\langle f_1, f_2\rangle|^2 \leq ||f_1||^2 - ||f_1||^4,$$

so $a > 0$. It follows that

$$b^2 - ac \leq 0 \tag{29}$$

Using (27), a direct calculation shows that

$$\begin{aligned} b^2 - ac &= \left(|\langle f_1, f_2\rangle|^2 - ||f_1||^2||f_2||^2\right) \times \\ &\left(|\langle f_1, f_2\rangle|^2 - (1 - ||f_1||^2 - ||f_2||^2 + ||f_1||^2||f_2||^2)\right). \end{aligned}$$

By Cauchy-Schwarz inequality,

$$|\langle f_1, f_2\rangle|^2 \leq ||f_1||^2||f_2||^2.$$

This and (29) imply

$$|\langle f_1, f_2\rangle|^2 \leq 1 - ||f_1||^2 - ||f_2||^2 + ||f_1||^2||f_2||^2.$$

Thus (13) holds.

Now, we assume $\langle f_1, f_2\rangle$ is complex. Choose $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda\langle f_1, f_2\rangle = |\langle f_1, f_2\rangle|$. Let $\tilde{f} := x\lambda f_1 + f_2$ for $x \in \mathbb{R}$. Then

$$||\tilde{f}||^2 = x^2||f_1||^2 + 2x|\langle f_1, f_2\rangle| + ||f_2||^2$$

and

$$\begin{aligned} |\langle \tilde{f}, f_1\rangle|^2 + |\langle \tilde{f}, f_2\rangle|^2 &= (||f_1||^4 + |\langle f_1, f_2\rangle|^2)x^2 + 2|\langle f_1, f_2\rangle|(||f_1||^2 + ||f_2||^2)x \\ &+ ||f_2||^4 + |\langle f_1, f_2\rangle|^2. \end{aligned}$$

Hence we can apply the partial result just proved to $\tilde{f}$.

□

Note that the correct value of the Bessel bound is essential in (13) :

Example 5.1 Let $\{e_1, e_2\}$ be an orthonormal basis for a 2-dimensional Hilbert space and put $f_1 = \sqrt{1+\epsilon}\, e_1, f_2 = \sqrt{1-\epsilon}\, e_2$ for some $\epsilon \in]0,1[$. Then $\{f_1, f_2\}$ is a Bessel sequence with bound $1+\epsilon$, and

$$\begin{aligned} 1 - ||f_1||^2 - ||f_1||^2 + ||f_1||^2||f_2||^2 &= 1-(1+\epsilon)-(1-\epsilon)+(1+\epsilon)(1-\epsilon) \\ &= -\epsilon^2 < 0. \end{aligned}$$

By Lemma 3.1 the inequality (13) holds with $B = 1+\epsilon$. The above calculation shows that the inequality is false if B is replaced by 1. □

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Ole Christensen
Department of Mathematics
Technical University of Denmark
Building 303

2800 Lyngby
Denmark
Email: Ole.Christensen@mat.dtu.dk

Hong Oh Kim
Department of Mathematical Sciences, KAIST
373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701
Republic of Korea
Email: kimhong@kaist.edu

Rae Young Kim
Department of Mathematics
Yeungnam University
214-1, Dae-dong, Gyeongsan-si, Gyeongsangbuk-do, 712-749
Republic of Korea
Email: rykim@ynu.ac.kr