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# Recovery of Fine Shape Details Using the Generalized Polarization Tensors

Habib Ammari* Hyeonbae Kang† Mikyoung Lim‡ Habib Zribi‡

**Abstract**

With each Lipschitz domain and material parameter, an infinite number of tensors, called the Generalized Polarization Tensors (GPTs), is associated. The GPTs contain significant information on the shape of the domain and its material parameter. They generalize the concept of Polarization Tensor (PT), which can be seen as the first-order GPT. It is known that given an arbitrary shape, one can find an equivalent ellipse or ellipsoid with the same PT. In this paper we consider the problem of recovering finer details of the shape of a given domain using higher-order polarization tensors. We design an optimization approach which solves the problem by simply minimizing a weighted discrepancy functional. In order to compute the shape derivative of this functional, we rigorously derive an asymptotic expansion of the perturbations of the GPTs that are due to a small deformation of the boundary of the domain. Our derivations are based on the theory of layer potentials. We perform some numerical experiments to demonstrate the validity and the limitations of the proposed method. The results clearly show that our approach is very promising in recovering fine shape details.

Mathematics Subject Classification (MSC2000): 35R30, 35B30

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# 1 Introduction

With each shape of a domain, physical and geometric quantities, such as eigenvalues and capacity, are intrinsically associated. The notion of (generalized) polarization tensors (GPTs) is one of them [6]. The GPTs generalize the concepts of classic polarization tensors [23]. The GPTs associated with a domain and a material parameter can be used to describe the perturbations of electric fields due to the presence of a conductivity inclusion. An electrical field present in the free space (homogeneous plane without an inclusion) is perturbed by the presence of inclusions. Then the far-field perturbations can be represented by multipolar expansions which are expressed in terms of the GPTs.

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In order to be more precise, let us consider the following conductivity transmission problem:

$$\begin{cases} \nabla \cdot \big(\chi(\mathbb{R}^2 \setminus \overline{D}) + k\chi(D)\big)\nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\chi(D)$ denotes the characteristic function of the domain $D$ and $H$ is a given harmonic function in $\mathbb{R}^2$.

The coefficient $\chi(\mathbb{R}^2 \setminus \overline{D}) + k\chi(D)$ represents the conductivity distribution. The inclusion $D$ has conductivity $k \neq 1$ while the background $\mathbb{R}^2 \setminus \overline{D}$ has conductivity 1. The function $\nabla H$ is the background electric field and $\nabla u$ is the electric field in the presence of the inclusion $D$. Then the perturbation, $u - H$, is given by the multipolar expansion:

$$(u - H)(x) = \sum_{|\alpha|,|\beta|=1}^{+\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial^\alpha \Gamma(x) M_{\alpha\beta} \partial^\beta H(x_0) \quad \text{as } |x| \to +\infty, \tag{1.2}$$

where $x_0$ is the center of mass of $D$ and $\Gamma$ is the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \frac{1}{2\pi} \ln |x|. \tag{1.3}$$

Here $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are multi-indices and $|\alpha| = \alpha_1 + \alpha_2$.

The quantity $M_{\alpha\beta}$ is called the generalized polarization tensor (GPT). Formula (1.2) shows that through the GPTs we have complete information about the far-field expansion of the perturbation $u - H$.

When $|\alpha| = |\beta| = 1$, we denote $M_{\alpha\beta} = m_{ij}$ and call the matrix $M = (m_{ij})_{i,j=1}^2$ the polarization tensor (PT).

The concepts of PT and GPTs occur in several interesting contexts, in particular in asymptotic models of dilute composites (see [20] and [9]) and in potential theory related to certain questions arising in hydrodynamics [23].

Another important use of these concepts is for imaging diametrically small inclusions from boundary measurements. In fact, the GPTs are the basic building blocks for the asymptotic expansions of the boundary voltage perturbations due to the presence of small conductivity inclusions inside a conductor [18, 3]. Based on this expansion, efficient algorithms to determine the location and some geometric features of the inclusions were proposed. We refer to [5, 6] and the references therein for recent developments of this theory.

According to [12] and [7], the PT associated with an unknown inclusion can be detected from boundary measurements. The detected PT in turn yields the "equivalent ellipse" of a single inclusion. In other words, in terms of the PT associated with an inclusion and a conductivity parameter (or a cluster of inclusions and a set of conductivity parameters) we are able to recover an equivalent ellipse with the same PT. On the other hand, it is proved in [4] that the full set of GPTs uniquely determines the inclusion (and its conductivity). Therefore it is natural to ask the question whether we can recover more shape details than the equivalent ellipse using a finite number of GPTs. The aim of this paper is to investigate this challenging question.

Recall that there is a canonical one-to-one correspondence between the class of PTs and the class of ellipses [12]. That is why one can find easily the equivalent ellipse if one knows the PT. However, there is no (and it is unlikely to have one) known class of geometric shapes which has such a property for higher-order polarization tensors. In this paper, we propose an optimization approach to recover finer shape details using GPTs.

Let $B$ be an unknown domain. Let $M_{\alpha\beta}(k,B)$ denote the GPT associated with $B$ and the conductivity $k$. It is worth emphasizing that the GPT also depends on the conductivity contrast $k$. Suppose that $M_{\alpha\beta}(k,B)$ are known for all $|\alpha|+|\beta| \le K$ for some number $K$. Suppose also that the conductivity is known. Our recursive optimization procedure would be to minimize over $D$

$$J^{(n)}[D] := \frac{1}{2} \sum_{|\alpha|+|\beta|\le K} w^{(n)}_{|\alpha|+|\beta|} \left| \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k,D) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k,B) \right|^2 . \tag{1.4}$$

Here the coefficients $a_\alpha$ and $b_\beta$ are such that $H = \sum a_\alpha x^\alpha$ and $F = \sum b_\beta x^\beta$ are homogeneous harmonic polynomials and $w^{(n)}_{|\alpha|+|\beta|}$ are binary weights, that is, they take two values: 0 and 1 and they determine which GPTs we keep at step $n$.

In step $n$ we use as an initial guess the result of step $n-1$. In the first step we get an equivalent ellipse with the same PT as well as the location of the inclusion. If there are multiple inclusions, we choose in the second step

$$w^{(2)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 3 \le |\alpha|+|\beta| \le K$$

in order to have a better initial guess than an ellipse.

Our method is in the same spirit as the continuation method in frequency [13, 14, 10] which was designed to solve inverse scattering problems for the Helmholtz equation.

In order to minimize the weighted discrepancy functional given in (1.4), we need a shape derivative for the GPTs. It turns out the shape derivative of $\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k,D)$ has a simple form. This is the main reason of choosing as a discrepancy functional the difference between calculated and given harmonic sums of GPTs rather than the difference between individual GPTs. In order to calculate the shape derivative of our discrepancy functional, we derive an asymptotic expansion of the GPTs under small perturbations of the boundary of the inclusion $D$.

The derivation is rigorous and based on layer potential techniques in the same spirit as in [5, 6]. We mention that related asymptotic formulas for boundary measurements, far-field data, and modal measurements have been obtained in a series of recent papers [8, 1, 2, 22]

We implement the proposed optimization procedure to recover both convex or non-convex shapes. The method of this paper is quite promising in the sense that the numerical results clearly exhibit that the shape moves toward the actual shape. They show not only the validity of the method but also that the equivalent ellipse is a good initial guess.

This paper is organized as follows. In Section 2, we review some basic facts on layer potentials which will be used to define the GPTs and to derive their shape derivatives. In Section 3 we review asymptotic formulas for perturbations in boundary integral operators due to small changes of the boundary. Section 4 is to derive a new asymptotic formula for the perturbations of the GPTs. In Section 5 we set up the optimization problem to recover shape details using a set of GPTs. In Section 6 we present results of numerical experiments and discuss the validity and the limitations of our method.

We emphasize that even though we only investigate the problem in two dimensions, the method of this paper works equally well in three dimensions.

## 2 Layer potentials and GPTs

Throughout this paper we assume that the domains under consideration have $\mathcal{C}^2$-smooth boundaries. For a given bounded domain $D$ in $\mathbb{R}^2$, the single and double layer potentials of the density function $\phi \in L^2(\partial D)$ are defined by

$$\mathcal{S}_D[\phi](x) := \int_{\partial D} \Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2,$$

$$\mathcal{D}_D[\phi](x) := \int_{\partial D} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D,$$

where $\nu_y$ is the outward unit normal to $\partial D$ at $y \in \partial D$ and $\Gamma$ is given by (1.3).

For a function $u$ defined on $\mathbb{R}^2 \setminus \partial D$, we denote

$$\frac{\partial u}{\partial \nu}\Big|_{\pm}(x) := \lim_{t \to 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle, \quad x \in \partial D,$$

if the limits exist. The notation $u|_{\pm}$ is understood likewise. The following are the well-known properties of the single and double layer potentials:

- Trace formula [19]:

$$\frac{\partial \mathcal{S}_D[\phi]}{\partial \nu}\Big|_{\pm}(x) = \left(\pm\frac{1}{2}I + \mathcal{K}_D^*\right)[\phi](x), \quad x \in \partial D, \tag{2.1}$$

$$\mathcal{D}_D[\phi]|_{\pm} = \left(\mp\frac{1}{2}I + \mathcal{K}_D\right)[\phi](x), \quad x \in \partial D, \tag{2.2}$$

  where

$$\mathcal{K}_D[\phi](x) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle y-x, \nu(y)\rangle}{|x-y|^2}\phi(y)\, d\sigma(y),$$

  and $\mathcal{K}_D^*$ is the $L^2$-adjoint of $\mathcal{K}_D$, *i.e.*,

$$\mathcal{K}_D^*[\phi](x) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^2}\phi(y)\, d\sigma(y).$$

- For any real number $\lambda$ with $|\lambda| > 1/2$ or $\lambda = -1/2$, $(\lambda I - \mathcal{K}_D^*)$ is invertible on $L^2(\partial D)$. If $|\lambda| \geq 1/2$, then $(\lambda I - \mathcal{K}_D^*)$ is invertible on $L_0^2(\partial D) := \{f \in L^2(\partial D) : \int_{\partial D} f d\sigma = 0\}$. See [19] and [24] (when $D$ has a Lipschitz boundary).

- If $\phi \in \mathcal{C}^{1,\alpha}(\partial D)$ for some $\alpha > 0$, then $\mathcal{D}_D\phi$ is $\mathcal{C}^{1,\alpha}$ on $\overline{D}$ and $\mathbb{R}^2 \setminus D$, and we have, see [15] for example,

$$\frac{\partial(\mathcal{D}_D\phi)}{\partial \nu}\Big|_{-} = \frac{\partial(\mathcal{D}_D\phi)}{\partial \nu}\Big|_{+} \quad \text{on } \partial D.$$

Let $D$ be a bounded domain in $\mathbb{R}^2$ and suppose that the conductivity of $D$ is $k$, $0 < k \neq 1 < +\infty$. Let $\lambda = (k+1)/(2(k-1))$. For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, define $\phi_\alpha$ by

$$\phi_\alpha(y) := (\lambda I - \mathcal{K}_D^*)^{-1}\left[\nu_x \cdot \nabla x^\alpha\right](y), \quad y \in \partial D. \tag{2.3}$$

Here and throughout this paper, we use the conventional notation: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$. Then, the generalized polarization tensors $M_{\alpha\beta}$ for $\alpha$, $\beta \in \mathbb{N}^2$ are defined, equivalently to (1.2), by

$$M_{\alpha\beta}(k, D) := \int_{\partial D} y^\beta \phi_\alpha(y) d\sigma(y). \tag{2.4}$$

Key properties of positivity and symmetry of the GPTs are studied in [6, Chapter 4]. We shall emphasize that what is important is not the individual terms $M_{\alpha\beta}$ but their harmonic combinations. A harmonic combination of GPTs is $\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}$ where $\sum_\alpha a_\alpha x^\alpha$ and $\sum_\beta b_\beta x^\alpha$ are harmonic polynomials. We will call such $(a_\alpha)$ and $(b_\beta)$ harmonic coefficients. For example, the following symmetry property holds:

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D) = \sum_{\alpha,\beta} a_\alpha b_\beta M_{\beta\alpha}(k, D) \tag{2.5}$$

for any pair $(a_\alpha)$, $(b_\beta)$ of harmonic coefficients.

Let us record the following uniqueness theorem.

**Theorem 2.1** *If the GPTs of two domains are the same, i.e.,*

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k_1, D_1) = \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k_2, D_2)$$

*for all pairs $(a_\alpha)$, $(b_\beta)$ of harmonic coefficients, then $D_1 = D_2$ and $k_1 = k_2$.*

In [4], the uniqueness theorem was stated under the assumption that $M_{\alpha\beta}(k_1, D_1) = M_{\alpha\beta}(k_2, D_2)$ for all $\alpha$ and $\beta$. But a quick glance of the proof there reveals that Theorem 2.1 is what was actually proved.

# 3 Asymptotic expansions of boundary integral operators

Let $D$ be a bounded domain with $\mathcal{C}^2$-boundary and let, for $\epsilon$ small, $D_\epsilon$ be an $\epsilon$-perturbation of $D$, *i.e.*, there is a function $h \in \mathcal{C}^1(\partial D)$ such that

$$\partial D_\epsilon := \{\tilde{x} = x + \epsilon h(x)\nu(x) \mid x \in \partial D\}, \tag{3.1}$$

where $\nu$ is the outward unit normal vector field on $\partial D$. Let $\Psi_\epsilon$ be the diffeomorphism from $\partial D$ to $\partial D_\epsilon$ given by

$$\Psi_\epsilon(x) = x + \epsilon h(x)\nu(x). \tag{3.2}$$

In view of (2.3) and (2.4), we need to get an asymptotic expansion of the operator $\mathcal{K}^*_{D_\epsilon}$ in order to get that of $M_{\alpha\beta}(k, D_\epsilon)$. A complete asymptotic expansion of the boundary integral operator $\mathcal{K}^*_{D_\epsilon}$ on $L^2(\partial D_\epsilon)$ is derived in terms of $\epsilon$ in [8, Theorem 2.1]. Especially, the first order approximation is as follows.

**Lemma 3.1** *For $\tilde{\phi} \in L^2(\partial D_\epsilon)$ let $\phi := \tilde{\phi} \circ \Psi_\epsilon$. There exists a constant $C$ depending only on the $\mathcal{C}^2$-norm of $\partial D$ and $\|h\|_{\mathcal{C}^1}$ such that*

$$\left\| \left( \mathcal{K}^*_{D_\epsilon}[\tilde{\phi}] \right) \circ \Psi_\epsilon - \mathcal{K}^*_D[\phi] - \epsilon \mathcal{K}^{(1)}_D[\phi] \right\|_{L^2(\partial D)} \le C\epsilon^2 ||\phi||_{L^2(\partial D)}, \tag{3.3}$$

*with the operator* $\mathcal{K}_D^{(1)}$ *defined for any* $\phi \in L^2(\partial D)$ *by*

$$\mathcal{K}_D^{(1)}[\phi](x) = \text{p.v.} \int_{\partial D} \mathbb{k}_1(x,y)\phi(y)d\sigma(y) \quad x \in \partial D,$$

*where*

$$\begin{aligned}
\mathbb{k}_1(x,y) \quad &= -2\frac{\langle x-y, \nu(x)\rangle\langle x-y,\ h(x)\nu(x) - h(y)\nu(y)\rangle}{|x-y|^4} \\
&+\frac{\langle h(x)\nu(x) - h(y)\nu(y),\ \nu(x)\rangle}{|x-y|^2} \\
&-\frac{\langle x-y,\ \tau(x)h(x)\nu(x) + h'(x)T(x)\rangle}{|x-y|^2} \\
&+\frac{\langle x-y, \nu(x)\rangle}{|x-y|^2}\big(h(x)\tau(x) - h(y)\tau(y)\big).
\end{aligned} \tag{3.4}$$

*Here,* $\tau(x)$ *denotes the curvature of* $\partial D$ *at* $x$*,* $T$ *the unit tangential vector field on* $\partial D$*,* p.v. *the Cauchy principal value, and* $h'$ *the derivative of* $h$ *on* $\partial D$*, i.e.,* $h' = \frac{\partial h}{\partial T}$*.*

We shall emphasize that $\mathcal{K}_D^{(1)}$ is bounded on $L^2(\partial D)$. In fact, the 1st, 2nd, and 4th kernels on the right-hand side of (3.4) are bounded since $\partial D$ is of class $\mathcal{C}^2$, while the 3rd kernel defines a singular integral operator which is bounded on $L^2(\partial D)$ by the theorem of Coifman-McIntosh-Meyer [16].

Moreover, the following expansions of $\tilde{\nu}$ and $\tilde{\sigma}$ hold:

$$\tilde{\nu}(\tilde{x}) = \nu(x) - \epsilon h'(x)T(x) + O(\epsilon^2), \tag{3.5}$$

and

$$d\tilde{\sigma}(\tilde{x}) = d\sigma(x) - \epsilon\tau(x)h(x)d\sigma(x) + O(\epsilon^2). \tag{3.6}$$

Here, the remainder $O(\epsilon^2)$ is bounded by $C\epsilon^2$ for some $C$ which depends only on the $\mathcal{C}^2$-norm of $\partial D$ and $\|h\|_{\mathcal{C}^1(\partial D)}$.

The following lemma was also obtained in [8, Lemma 3.1].

**Lemma 3.2** *Let* $\tilde{\phi}_\epsilon = \big(\lambda I - \mathcal{K}_{D_\epsilon}^*\big)^{-1}(\tilde{\nu}\cdot\nabla H)$*,* $\phi_\epsilon = \tilde{\phi}_\epsilon \circ \Psi_\epsilon$*, and* $\phi = \big(\lambda I - \mathcal{K}_D^*\big)^{-1}(\nu\cdot\nabla H)$*. Then we have*

$$\|\phi_\epsilon - \phi - \epsilon\phi_1\|_{L^2(\partial D)} \le C\epsilon^2\|\phi\|_{L^2(\partial D)},$$

*where* $C$ *is a constant depending only on the* $\mathcal{C}^2$*-norm of* $\partial D$ *and* $\|h\|_{\mathcal{C}^1}$ *and*

$$\phi_1 = (\lambda I - \mathcal{K}_D^*)^{-1}\left[\mathcal{K}_D^{(1)}[\phi] + h\langle(\nabla^2 H)\nu, \nu\rangle - h'\frac{\partial H}{\partial T}\right]. \tag{3.7}$$

We now rewrite the operator $\mathcal{K}_D^{(1)}$ in terms of more familiar operators. For $x, y \in \partial D$ ($x \neq y$), we have

$$\frac{\partial}{\partial T(x)}\Gamma(x-y) = \frac{1}{2\pi}\frac{\langle x-y, T(x)\rangle}{|x-y|^2},$$

$$\begin{aligned}
\frac{\partial^2}{\partial T(x)^2}\Gamma(x-y) &= \frac{1}{2\pi}\left[\frac{1}{|x-y|^2} + \frac{\langle x-y,\nu(x)\rangle\tau(x)}{|x-y|^2} - \frac{2\left|\langle x-y, T(x)\rangle\right|^2}{|x-y|^4}\right] \\
&= \frac{1}{2\pi}\left[-\frac{1}{|x-y|^2} + \frac{\langle x-y,\nu(x)\rangle\tau(x)}{|x-y|^2} + \frac{2\left|\langle x-y, \nu(x)\rangle\right|^2}{|x-y|^4}\right],
\end{aligned}$$

and

$$\frac{\partial^2}{\partial\nu(x)\nu(y)}\Gamma(x-y) = \frac{1}{2\pi}\left[-\frac{\langle\nu(x),\nu(y)\rangle}{|x-y|^2} + \frac{2\langle x-y,\nu(x)\rangle\langle x-y,\nu(y)\rangle}{|x-y|^4}\right].$$

It then follows that

$$\begin{aligned}
&-\frac{\partial}{\partial T(x)}\left(h(x)\frac{\partial}{\partial T(x)}\right)\Gamma(x-y) + h(y)\frac{\partial^2}{\partial\nu(x)\nu(y)}\Gamma(x-y)\\
&= -2\frac{\langle x-y,\nu(x)\rangle\langle x-y,\ h(x)\nu(x)-h(y)\nu(y)\rangle}{|x-y|^4} + \frac{\langle h(x)\nu(x)-h(y)\nu(y),\ \nu(x)\rangle}{|x-y|^2}\\
&\quad - \frac{\langle x-y,\ \tau(x)h(x)\nu(x)+h'(t)T(x)\rangle}{|x-y|^2}\\
&= \Bbbk_1(x,y) - \frac{\langle x-y,\nu(x)\rangle}{|x-y|^2}\big(h(x)\tau(x)-h(y)\tau(y)\big).
\end{aligned}$$

Define $H^s(\partial D), s = 1, 2$, to be the usual Sobolev spaces on $\partial D$. If $\phi \in H^1(\partial D)$, then $\mathcal{S}_D[\phi] \in H^2(\partial D)$ and $\frac{\partial}{\partial\nu}\mathcal{D}_D[h\phi] \in L^2(\partial D)$. Note that the left-hand side of the first identity is the integral kernel of the operator

$$\phi \mapsto -\frac{\partial}{\partial T}(h\frac{\partial}{\partial T})\mathcal{S}_D[\phi] + \frac{\partial}{\partial\nu}\mathcal{D}_D[h\phi].$$

Thus the second identity shows that

$$\mathcal{K}_D^{(1)}[\phi] = -\frac{\partial}{\partial T}\Big(h\frac{\partial\mathcal{S}_D[\phi]}{\partial T}\Big) + \frac{\partial\mathcal{D}_D[h\phi]}{\partial\nu} + h\tau\mathcal{K}_D^*[\phi] - \mathcal{K}_D^*[h\tau\phi] \tag{3.8}$$

for all $\phi \in H^1(\partial D)$. It is interesting to observe that the above identity tells us that the operator

$$\phi \mapsto -\frac{\partial}{\partial T}(h\frac{\partial}{\partial T})\mathcal{S}_D[\phi] + \frac{\partial}{\partial\nu}\mathcal{D}_D[h\phi]$$

may be extended as a bounded operator on $L^2(\partial D)$.

# 4 Asymptotic expansions of the GPTs

We now derive asymptotic expansions of the GPTs.

**Proposition 4.1** *For multi-indices $\alpha$ and $\beta$, let $F(x) = x^\beta$ and $H(x) = x^\alpha$. Let*

$$\phi = (\lambda I - \mathcal{K}_D^*)^{-1}\left[\frac{\partial H}{\partial\nu}\Big|_{\partial D}\right], \tag{4.1}$$

$$\psi = (\lambda I - \mathcal{K}_D)^{-1}[F|_{\partial D}]. \tag{4.2}$$

*The following asymptotic expansion holds:*

$$M_{\alpha\beta}(k, D_\epsilon) - M_{\alpha\beta}(k, D) = \epsilon\Big\langle h, p_{\alpha\beta}(k, D)\Big\rangle_{L^2(\partial D)} + O(\epsilon^2), \tag{4.3}$$

*where*

$$p_{\alpha\beta}(k, D) = \frac{\partial\psi}{\partial T}\frac{\partial(H+\mathcal{S}_D[\phi])}{\partial T} + \phi\frac{\partial(F+\mathcal{D}_D[\psi])}{\partial\nu} + \psi\left(\langle(\nabla^2 H)\nu,\nu\rangle + \langle(\nabla^2 H)T,T\rangle\right). \tag{4.4}$$

*Proof.* Since

$$F(x+\epsilon h(x)\nu(x)) = F(x) + \epsilon h(x)\frac{\partial F}{\partial \nu}(x) + O(\epsilon^2), \quad x \in \partial D,$$

it follows from (3.6) and Lemma 3.2 that

$$\begin{aligned}
M_{\alpha\beta}(k, D_\epsilon) &= \int_{\partial D_\epsilon} F(\tilde{x})\tilde{\phi}_\epsilon(\tilde{x})\, d\sigma_\epsilon(\tilde{x}) \\
&= \int_{\partial D} \Big(F(x) + \epsilon h(x)\frac{\partial F}{\partial \nu}(x)\Big)\big(\phi(x) + \epsilon\phi_1(x)\big)\big(1 - \epsilon\tau(x)h(x)\big)\, d\sigma(x) + O(\epsilon^2) \\
&= M_{\alpha\beta}(k, D) + \epsilon\int_{\partial D} F\phi_1\, d\sigma + \epsilon\int_{\partial D}\big(\frac{\partial F}{\partial \nu} - \tau F\big)\phi h\, d\sigma + O(\epsilon^2).
\end{aligned}$$

Hence the definition (4.2) yields

$$M_{\alpha\beta}(k, D_\epsilon) = M_{\alpha\beta}(k, D) + \epsilon\int_{\partial D}(\lambda I - \mathcal{K}_D)[\psi]\phi_1\, d\sigma + \epsilon\int_{\partial D}\big(\frac{\partial F}{\partial \nu} - \tau F\big)\phi h\, d\sigma + O(\epsilon^2). \quad (4.5)$$

Let us now calculate the term $\int_{\partial D}(\lambda I - \mathcal{K}_D)[\psi]\phi_1\, d\sigma$. From (3.7) we get

$$\begin{aligned}
\int_{\partial D}(\lambda I - \mathcal{K}_D)[\psi]\phi_1\, d\sigma &= \int_{\partial D}\psi(\lambda I - \mathcal{K}_D^*)[\phi_1]\, d\sigma \\
&= \int_{\partial D}\psi\left[\mathcal{K}_D^{(1)}[\phi] + h\langle(\nabla^2 H)\nu, \nu\rangle - h'\frac{\partial H}{\partial T}\right] d\sigma.
\end{aligned}$$

Next, because of (3.8), we have

$$\int_{\partial D}\psi\mathcal{K}_D^{(1)}[\phi]\, d\sigma = \int_{\partial D}\psi\left[-\frac{\partial}{\partial T}\Big(h\frac{\partial \mathcal{S}_D[\phi]}{\partial T}\Big) + \frac{\partial \mathcal{D}_D[h\phi]}{\partial \nu} + h\tau\mathcal{K}_D^*[\phi] - \mathcal{K}_D^*[h\tau\phi]\right] d\sigma.$$

We claim that

$$\int_{\partial D}\psi\frac{\partial \mathcal{D}_D[h\phi]}{\partial \nu}\, d\sigma = \int_{\partial D}\frac{\partial \mathcal{D}_D[\psi]}{\partial \nu}h\phi\, d\sigma. \quad (4.6)$$

In fact, let $\Lambda_D$ denote the Dirichlet-to-Neuman map on $D$, that is, $\Lambda_D[\psi] = \partial u/\partial\nu$, where $\Delta u = 0$ in $D$ and $u = \psi$ on $\partial D$. Then Green's theorem yields

$$\begin{aligned}
\int_{\partial D}\psi\frac{\partial \mathcal{D}_D[h\phi]}{\partial \nu}\, d\sigma &= \int_{\partial D}\Lambda_D[\psi]\mathcal{D}_D[h\phi]\big|_-\, d\sigma \\
&= \int_{\partial D}\Lambda_D[\psi](\frac{1}{2}I + \mathcal{K}_D)[h\phi]\, d\sigma \\
&= \int_{\partial D}(\frac{1}{2}I + \mathcal{K}_D^*)\Lambda_D[\psi]h\phi\, d\sigma.
\end{aligned}$$

In view of (2.2), the solution to the Dirichlet problem $\Delta u = 0$ in $D$ and $u = \psi$ on $\partial D$ is given by

$$u(x) = \mathcal{D}_D\big(\frac{1}{2}I + \mathcal{K}_D\big)^{-1}[\psi](x), \quad x \in D.$$

Therefore, we have

$$\Lambda_D[\psi] = \frac{\partial}{\partial \nu}\mathcal{D}_D\big(\frac{1}{2}I + \mathcal{K}_D\big)^{-1}[\psi] \quad \text{on } \partial D.$$

It then follows from (2.1) that

$$(\frac{1}{2}I + \mathcal{K}_D^*)\Lambda_D[\psi] = \frac{\partial}{\partial \nu}\mathcal{S}_D\left[\frac{\partial}{\partial \nu}\mathcal{D}_D\big(\frac{1}{2}I + \mathcal{K}_D\big)^{-1}[\psi]\right]\Big|_+.$$

One can easily see, using again Green's theorem and (2.2), that for $x \in \mathbb{R}^2 \setminus \overline{D}$

$$\mathcal{S}_D\left[\frac{\partial}{\partial \nu}\mathcal{D}_D\big(\frac{1}{2}I + \mathcal{K}_D\big)^{-1}[\psi]\right](x) = \mathcal{D}_D\left[\mathcal{D}_D\big(\frac{1}{2}I + \mathcal{K}_D\big)^{-1}[\psi]\Big|_-\right](x) = \mathcal{D}[\psi](x).$$

Thus we get

$$(\frac{1}{2}I + \mathcal{K}_D^*)\Lambda_D[\psi] = \frac{\partial \mathcal{D}_D[\psi]}{\partial \nu},$$

and hence (4.6) holds.

With this result in hand, we now obtain

$$\int_{\partial D}\psi \mathcal{K}_D^{(1)}[\phi]\, d\sigma = \int_{\partial D} h\left[\frac{\partial \psi}{\partial T}\frac{\partial \mathcal{S}_D[\phi]}{\partial T} + \frac{\partial \mathcal{D}[\psi]}{\partial \nu}\phi + \tau\psi\mathcal{K}_D^*[\phi] - \tau\mathcal{K}_D[\psi]\phi\right] d\sigma,$$

and hence

$$\begin{aligned}\int_{\partial D}(\lambda I - \mathcal{K}_D)[\psi]\phi_1\, d\sigma = \int_{\partial D} h\Bigg[&\frac{\partial \psi}{\partial T}\frac{\partial \mathcal{S}_D[\phi]}{\partial T} + \frac{\partial \mathcal{D}[\psi]}{\partial \nu}\phi \\ &+ \tau\psi\mathcal{K}_D^*[\phi] - \tau\mathcal{K}_D[\psi]\phi + \psi\langle(\nabla^2 H)\nu, \nu\rangle + \frac{\partial}{\partial T}\big(\psi\frac{\partial H}{\partial T}\big)\Bigg] d\sigma.\end{aligned}$$

It then follows from (4.5) that

$$M_{\alpha\beta}(k, D_\epsilon) - M_{\alpha\beta}(k, D) = \epsilon\int_{\partial D} h(x)p_{\alpha\beta}(k, D)(x)\, d\sigma + O(\epsilon^2), \tag{4.7}$$

where

$$\begin{aligned}p_{\alpha\beta}(k, D) = &\frac{\partial \psi}{\partial T}\frac{\partial \mathcal{S}_D[\phi]}{\partial T} + \frac{\partial \mathcal{D}[\psi]}{\partial \nu}\phi + \tau\psi\mathcal{K}_D^*[\phi] - \tau\mathcal{K}_D[\psi]\phi \\ &+ \psi\langle(\nabla^2 H)\nu, \nu\rangle + \frac{\partial}{\partial T}\big(\psi\frac{\partial H}{\partial T}\big) + \big(\frac{\partial F}{\partial \nu} - \tau F\big)\phi.\end{aligned}$$

But,

$$\frac{\partial}{\partial T}\big(\psi\frac{\partial H}{\partial T}\big) = \frac{\partial \psi}{\partial T}\frac{\partial H}{\partial T} + \psi\langle(\nabla^2 H)T, T\rangle + \psi\tau\frac{\partial H}{\partial \nu}.$$

Note also that because of (4.1) and (4.2),

$$\mathcal{K}_D^*[\phi] + \frac{\partial H}{\partial \nu} = \lambda\phi \quad \text{and} \quad \mathcal{K}_D[\psi] + F = \lambda\psi.$$

Thus we arrive at

$$p_{\alpha\beta}(k, D) = \frac{\partial \psi}{\partial T}\frac{\partial (H + \mathcal{S}_D[\phi])}{\partial T} + \phi\frac{\partial (F + \mathcal{D}_D[\psi])}{\partial \nu} + \psi\left(\langle(\nabla^2 H)\nu, \nu\rangle + \langle(\nabla^2 H)T, T\rangle\right),$$

as desired. This completes the proof. ■

Let us now suppose that $a_\alpha$ and $b_\beta$ are constants such that $H = \sum_\alpha a_\alpha x^\alpha$ and $F = \sum_\beta b_\beta x^\beta$ are harmonic polynomials. Then it can be easily seen that

$$\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D_\epsilon) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D) = \epsilon \Big\langle h, \sum_{\alpha,\beta} a_\alpha b_\beta p_{\alpha\beta}(k, D) \Big\rangle_{L^2(\partial D)} + O(\epsilon^2), \quad (4.8)$$

and

$$\begin{aligned} &\sum_{\alpha,\beta} a_\alpha b_\beta p_{\alpha\beta}(k, D) \\ &= \frac{\partial \psi}{\partial T} \frac{\partial (H + \mathcal{S}_D[\phi])}{\partial T} + \phi \frac{\partial (F + \mathcal{D}_D[\psi])}{\partial \nu} + \psi \left( \langle (\nabla^2 H)\nu, \nu \rangle + \langle (\nabla^2 H) T, T \rangle \right), \end{aligned}$$

where $\phi$ and $\psi$ satisfy (4.1) and (4.2) with new (harmonic functions) $H$ and $F$. Since $H$ is harmonic,

$$\langle (\nabla^2 H)\nu, \nu \rangle + \langle (\nabla^2 H) T, T \rangle = \Delta H = 0,$$

and hence

$$\sum a_\alpha b_\beta p_{\alpha\beta}(k, D) = \frac{\partial \psi}{\partial T} \frac{\partial (H + \mathcal{S}_D[\phi])}{\partial T} + \phi \frac{\partial (F + \mathcal{D}_D[\psi])}{\partial \nu}. \quad (4.9)$$

Let

$$u(x) := H(x) + \mathcal{S}_D[\phi](x) \quad \text{and} \quad v(x) := F(x) + \mathcal{D}_D[\psi](x), \quad x \in \mathbb{R}^2. \quad (4.10)$$

Then one can see using the jump relations (2.1) and (2.2) that $u$ and $v$ are respectively solutions to the following transmission problems:

$$\begin{cases} \Delta u = 0, & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\ u|_+ - u|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ - k \dfrac{\partial u}{\partial \nu}\Big|_- = 0, & \text{on } \partial D, \\ (u - H)(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \quad (4.11)$$

and

$$\begin{cases} \Delta v = 0, & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\ k v|_+ - v|_- = 0, & \text{on } \partial D, \\ \dfrac{\partial v}{\partial \nu}\Big|_+ - \dfrac{\partial v}{\partial \nu}\Big|_- = 0, & \text{on } \partial D, \\ (v - F)(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases} \quad (4.12)$$

Moreover, according to [21], we have

$$\phi = (k-1) \frac{\partial u}{\partial \nu}\Big|_- \quad \text{on } \partial D.$$

Similarly,

$$\psi = \frac{k-1}{k} v|_- \quad \text{on } \partial D,$$

and hence

$$\frac{\partial \psi}{\partial T} = \frac{k-1}{k} \frac{\partial v}{\partial T}\Big|_- .$$

In fact, from (4.2) we obtain that

$$v|_- = F + (\frac{1}{2}I + \mathcal{K}_D)[\psi] = (\lambda + \frac{1}{2})\psi = \frac{k}{k-1}\psi.$$

So far we proved the following theorem which is the main theoretical result of this paper.

**Theorem 4.2** *Suppose that $a_\alpha$ and $b_\beta$ are constants such that $H = \sum_\alpha a_\alpha x^\alpha$ and $F = \sum_\beta b_\beta x^\beta$ are harmonic polynomials. Then*

$$\begin{aligned} &\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D_\epsilon) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D) \\ &= \epsilon(k-1) \int_{\partial D} h(x) \left[ \frac{\partial v}{\partial \nu}\Big|_- \frac{\partial u}{\partial \nu}\Big|_- + \frac{1}{k} \frac{\partial u}{\partial T}\Big|_- \frac{\partial v}{\partial T}\Big|_- \right] (x)\, d\sigma(x) + O(\epsilon^2), \end{aligned} \tag{4.13}$$

*where $u$ and $v$ satisfy (4.11) and (4.12), respectively.*

A few remarks are in order regarding the dependency of the remainder $O(\epsilon^2)$ term. It is bounded by $C\epsilon^2$ for some $C$ depending only on the $\mathcal{C}^2$-norm of $\partial D$ and $\|h\|_{\mathcal{C}^1(\partial D)}$. It also depends on the degrees of the harmonic polynomials $H$ and $F$. As the degree gets larger, the remainder gets larger. However, the remainder does not depend on the conductivity contrast $k$. Then formula (4.13) holds for also the extreme cases $k = 0$ and $k = +\infty$. This important fact is because of the estimate

$$\left\| \Big( \frac{k+1}{2(k-1)} I - \mathcal{K}_D^* \Big)^{-1} [f] \right\|_{L^2(\partial D)} \le C \|f\|_{L^2(\partial D)}$$

with a constant $C$ independent of $k$, which was first proved in [6].

Note also that formula (4.13) gives the shape derivative of $\sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D)$.

Finally, it is quite interesting to observe the similarity between the asymptotic formula (4.13) and the one for eigenvalue perturbations obtained in [1] (see [2] for the elasticity case).

# 5 Reconstruction of shape details using GPTs

## 5.1 Recursive scheme

According to Theorem 2.1, one can (approximately) reconstruct the shape of $B$ by recursively minimizing at each step $n$ the functional $J^{(n)}[D]$ given in (1.4) over $D$. For doing so, we need to compute the shape derivative of $J^{(n)}[D]$.

Let $H = \sum a_\alpha x^\alpha$ and $F = \sum b_\beta x^\beta$ be homogeneous harmonic polynomials and let

$$\phi_{HF}(x) = (k-1) \left[ \frac{\partial v}{\partial \nu}\Big|_- \frac{\partial u}{\partial \nu}\Big|_- + \frac{1}{k} \frac{\partial u}{\partial T}\Big|_- \frac{\partial v}{\partial T}\Big|_- \right],$$

where $u$ and $v$ satisfy (4.11) and (4.12), respectively. Theorem 4.2 shows that the shape derivative of $J^{(n)}[D]$ is given by

$$\langle d_S J^{(n)}[D], h \rangle_{L^2(\partial D)} = \sum_{|\alpha|+|\beta| \le K} w^{(n)}_{|\alpha|+|\beta|} \delta_{HF} \langle \phi_{HF}, h \rangle_{L^2(\partial D)}, \tag{5.1}$$

where

$$\delta_{HF} = \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, D) - \sum_{\alpha,\beta} a_\alpha b_\beta M_{\alpha\beta}(k, B).$$

Given the PT of the inclusion $B$ we can find an ellipse with the same PT but not its location since the PT is invariant under translation. We can locate the inclusion provided that its GPTs with $|\alpha| + |\beta| = 3$ are known. This can be used as an (initial) guess, and more shape details for $B$ can be reconstructed by minimizing (1.4) for $n = 2$. The result of step $n-1$ is used as an initial guess for step $n$.

The weights $w_{|\alpha|+|\beta|}$ determine the GPTs we keep at each step. We choose

$$w^{(1)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 2 \leq |\alpha| + |\beta| \leq 3 \quad \text{and 0 elsewhere,}$$

$$w^{(2)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 2 \leq |\alpha| + |\beta| \leq 4 \text{ and 0 elsewhere,}$$

and, more generally, in step $K - 2 \geq n \geq 3$,

$$w^{(n)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 2 \leq |\alpha| + |\beta| \leq n + 2 \text{ and 0 elsewhere.}$$

If there are multiple inclusions, we choose in the second step

$$w^{(2)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 3 \leq |\alpha| + |\beta| \leq K$$

in order to have a better initial guess than an ellipse and, in step $K \geq n \geq 3$,

$$w^{(n)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 2 \leq |\alpha| + |\beta| \leq n.$$

Our algorithm is in the same spirit as the continuation method in frequency for solving inverse scattering problems [13, 14, 10]. Since the high-frequency oscillations of the boundary of an inclusion are only contained in its high-order GPTs, our recursive optimization scheme yields a stable way to reconstruct such information.

### 5.2 Deformations undetectable from the GPTs

It follows from the expression of the shape derivative of $J^{(n)}$ that if a shape deformation is orthogonal to the functions $\phi_{HF}$, then it is undetectable.

As we can see from [8], if $D$ is a disk, then using $M_{\alpha\beta}$, $|\alpha| + |\beta| \leq K$, one can only detect the Fourier coefficients of the deformation up to $K$.

Figure 1 and Figure 2 are the (orthogonalized) $\phi_{HF}$ for $K = 3$.

## 6 Numerical Results

In the following examples, we use the GPTs up to $|\alpha|+|\beta| \leq 6$, i.e., $K = 6$. The conductivity inside the inclusion is given to be 3. To reconstruct multiple inclusions as well as a single inclusion, we use in the second step $w^{(2)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 3 \leq |\alpha|+|\beta| \leq 6$ and, in step $3 \leq n \leq 6$, $w^{(n)}_{|\alpha|+|\beta|} = 1 \quad \text{for } 2 \leq |\alpha| + |\beta| \leq n$. The gray curve is the target domain ($B$) and the black curve ($D$) is the reconstructed one.

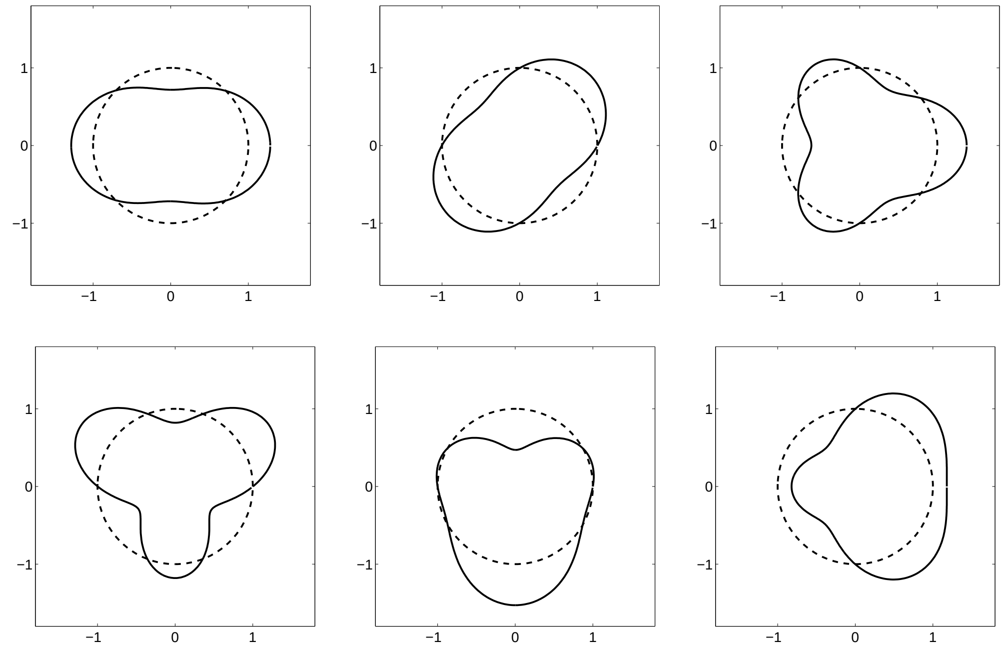


Figure 1: $\phi_{HF}$ corresponding to $\sum a_\alpha b_\beta M_{\alpha\beta}$, $|\alpha| + |\beta| \leq 3$ for the disk $D$, which is the dotted curve. The solid curves are $\partial D +$ a linear combination of $\phi_{HF}\nu$.

**Example 1**. The example in Figure 3 shows that the equivalent ellipse is gradually modified toward the target domain. The first image is the equivalent ellipse and the others are the reconstructed images for $n = 2, \ldots, 6$.
**Example 2**. This example in Figure 4 reveals the limit of the shape we can reconstruct when we use the GPTs up to $|\alpha| + |\beta| = K$. When the target function is a sinusoidal perturbation of a disk, we can reconstruct the shape perturbation when the angular frequency is smaller than or equal to $K$. High-frequency information is undetectable.
**Example 3**. Using higher-order GPTs we can better detect multiple inclusions; see Figure 5.

# 7 Conclusion

In this paper we have proposed a new recursive optimization scheme to recover fine shape details from the GPTs. We have presented some numerical experiments to demonstrate the validity and the limitations of the proposed approach which is in the same spirit as the continuation method in frequency. Since the high-frequency oscillations of the boundary of an inclusion are only contained in its high-order GPTs, the recursive method yields a stable way to reconstruct such information.

Other schemes can be designed by choosing different weights in the discrepancy functional

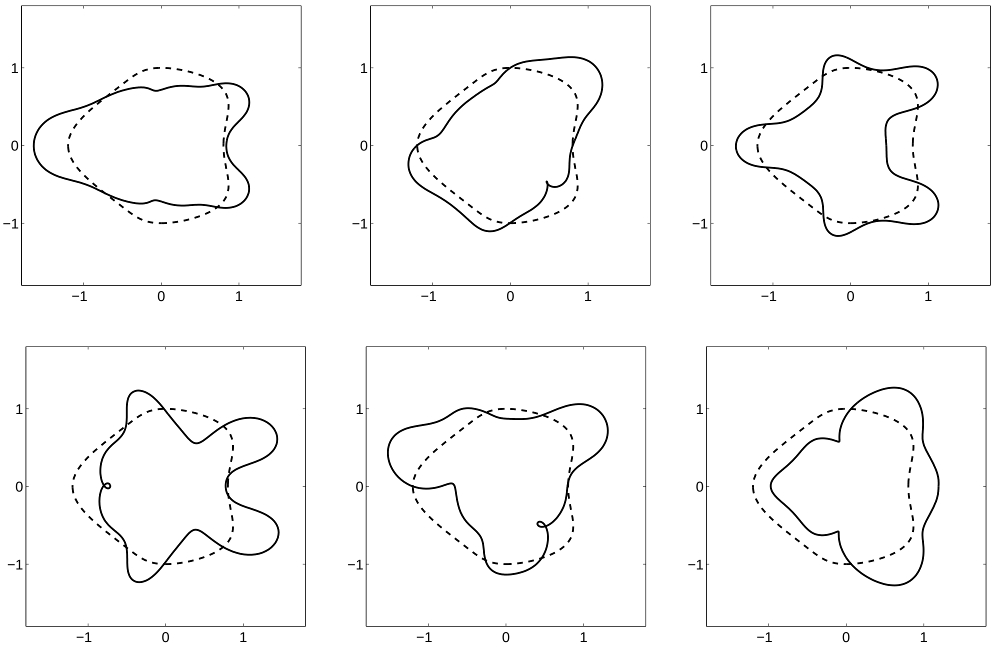

Figure 2: $\phi_{HF}$ corresponding to $\sum a_\alpha b_\beta M_{\alpha\beta}$, $|\alpha|+|\beta|\leq 3$ for a domain $D$, which is the dotted curve. The solid curves are $\partial D+$ a linear combination of $\phi_{HF}\nu$.

(1.4). For example, choosing

$$
\begin{aligned}
w^{(1)}_{|\alpha|+|\beta|} &= 1 \quad \text{for } 2\leq|\alpha|+|\beta|\leq l_1 \quad \text{and 0 elsewhere,}\\
w^{(2)}_{|\alpha|+|\beta|} &= 1 \quad \text{for } l_1+1\leq|\alpha|+|\beta|\leq l_2 \text{ and 0 elsewhere,}\\
w^{(3)}_{|\alpha|+|\beta|} &= 1 \quad \text{for } l_2+1\leq|\alpha|+|\beta|\leq l_3 \text{ and 0 elsewhere,}
\end{aligned}
$$

and so on, where $2<l_1<l_2<l_3<\ldots$, yields a scheme that is closely related to the one developed in [11]. It could have better resolution than the one implemented in this paper but clearly is less stable. It requires a very good initial guess. A detailed resolution and stability analysis for both schemes will be reported elsewhere.


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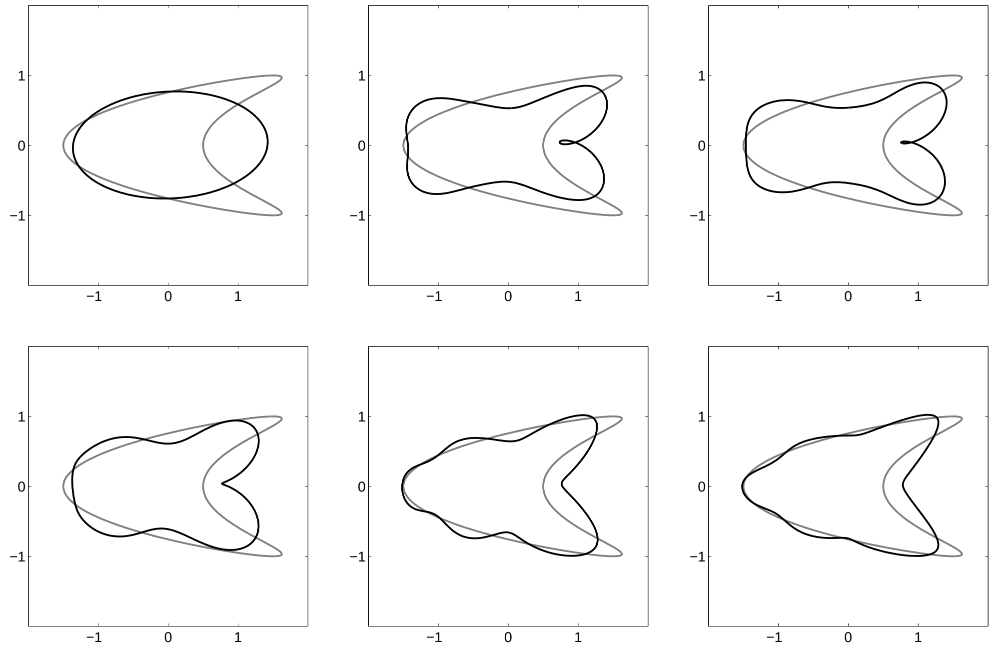

Figure 3: Gray curve is the target domain ($B$) and the black curve ($D$) is the reconstructed shape. The first image is the equivalent ellipse and the others are the reconstructed images for $n = 2, \ldots, 6$.

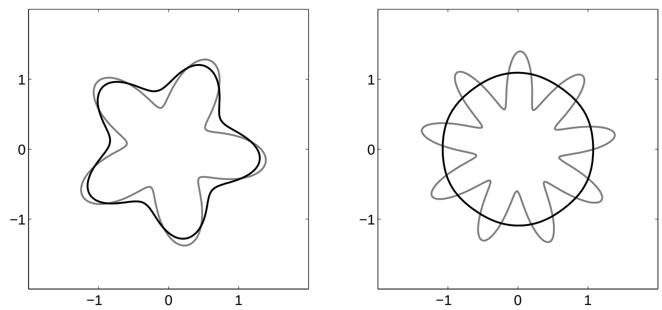

Figure 4: Gray curve is the target domain ($B$) and the black curve ($D$) is the reconstructed shape for $n = 6$ starting from the equivalent ellipse.

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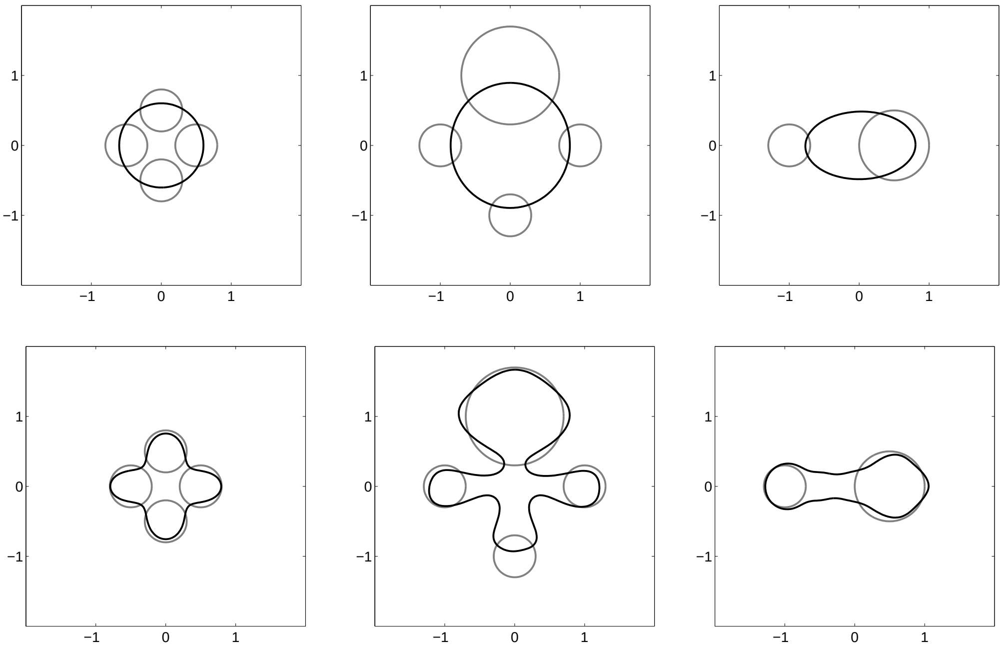

Figure 5: Reconstruction results for multiple inclusions. The upper images show the equivalent ellipse, and the lower ones show the reconstructed image for $n = 6$.

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