M-IDEAL PROPERTIES IN MARCINKIEWICZ SPACES

ANNA KAMIŃSKA AND HAN JU LEE

Dedicated to Professor Julian Musielak on his 75th birthday

ABSTRACT. We study M-ideal properties of function and sequence Marcinkiewicz spaces. In particular we calculate the duals of the space $\Sigma = L^1 + L^\infty$ equipped with two standard norms and investigate when its subspace of order continuous elements is an M-ideal in Σ.

A closed subspace Y of a Banach space X is called an *M-ideal* of X if there is a bounded projection $\mathcal{P} : X^* \to X^*$ with range $Y^\perp$ such that for each $x^* \in X^*$,

$$\|x^*\| = \|\mathcal{P}x^*\| + \|(I - \mathcal{P})x^*\| .$$

A Banach space X is said to be *M-embedded* if X is an M-ideal of its bidual X^{**}. It is well known and easy to verify that c_0 is an M-ideal in its bidual ℓ^∞. In this paper we investigate the Marcinkiewicz function and sequence spaces (M_Ψ and m_Ψ respectively), called also weak Lorentz spaces. M-ideal properties of these spaces have been studied for particular functions Ψ for instance in [2].

Here we provide several results for spaces M_Ψ or m_Ψ generated by an arbitrary quasi-concave funcion Ψ. In sections one and three we characterize among others when the subspace M_Ψ^0 (resp. m_Ψ^0) of order continuous elements of M_Ψ (resp. m_Ψ) is an M-ideal in M_Ψ (resp. m_Ψ^0), and when it is M-embedded. In section two we investigate the space $\Sigma = L^1 + L^\infty$, which coincides to space M_Ψ where $\Psi(t) = \max(1, t)$, $t > 0$. We calculate the dual norms to $(\Sigma, \|\cdot\|)$ and $(\Sigma, |||\cdot|||)$, where $\|\cdot\|$ and $|||\cdot|||$ are equivalent usual norms employed in Σ. Consequently we obtain that $(\Sigma_0, \|\cdot\|)$ is not an M-ideal in $(\Sigma, \|\cdot\|)$, while $(\Sigma_0, |||\cdot|||)$ is an M-ideal in $(\Sigma, |||\cdot|||)$.

Let $(\Omega, \mu) = (\Omega, \mathcal{B}, \mu)$ be a measure space with a complete σ-finite measure μ on a σ-algebra $\mathcal{B}$ of subsets of Ω. Let $L^0(\mu)$ denote the space of all μ-equivalence classes of $\mathcal{B}$-measurable $\mathbb{F}$-valued functions on Ω with the topology of convergence in measure on μ-finite sets.

A Banach space $(X, \|\cdot\|)$ is said to be a *Banach function space* on (Ω, μ) if it is a subspace of $L^0(\mu)$ such that there is $h \in L^0(\mu)$ with $h > 0$ a.e. in Ω and it has the *ideal property* that is if $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ a.e. then $f \in X$ and $\|f\| \le \|g\|$. If in addition the unit ball B_X is closed in $L^0(\mu)$, then we say that X has the *Fatou property.* A Banach function space defined on $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with the counting measure μ is called a *Banach sequence space*. In this case $e_i \in X$ for

The second named author acknowledges support from the Project Brain Korea 21.

all $i \in \mathbb{N}$, where e_i denotes a standard unit vector, that is $e_i = (0, \dots, 0, 1, 0, \dots)$ with 1 as the ith component.

A Banach function space X on (Ω, μ) is said to be *rearrangement invariant* (*r.i.,or symmetric*) if for every $f \in L^0(\mu)$ and $g \in X$ with $\mu_f = \mu_g$, we have $f \in X$ and $\|f\| = \|g\|$, where for any $h \in L^0(\mu)$, μ_h is a *distribution function* of h defined by

$$\mu_h(t) = \mu\{\omega \in \Omega : |h(\omega)| > t\}, \quad t \geq 0.$$

If X is a Banach function space on (Ω, μ), then the *associate space* X' of X is a Banach function space, which can be identified with the space of all functionals possessing an integral representation, that is,

$$X' = \{g \in L^0(\mu) : \|g\|_{X'} = \sup_{\|f\| \leq 1} \int_\Omega |fg| d\mu < \infty\}.$$

It is well known that if X has the Fatou property, then $(X'', \|\cdot\|_{X''})$ coincides with $(X, \|\cdot\|)$ [1, 5, 6].

An element $f \in X$ is said to be *order continuous* if $\|f_n\| \downarrow 0$ for every sequence $\{f_n\}$ with $|f_n| \leq |f|$ a.e. and $|f_n| \downarrow 0$ a.e. on Ω. A Banach function space X is said to be *order continuous* if every element of X is order continuous. It is well known that if X is an order continuous Banach function space, then X^* is order isometric to X', and this identification will be denoted by $X^* \simeq X'$.

Suppose for the moment that X is a Banach function space consisting of real valued functions. An element $\phi \in X^*$ is called an *integral functional* if for any $\{f_n\} \subset X$ with $0 \leq f_n \downarrow 0$ a.e., $\phi(f_n) \to 0$. A linear functional $\phi_s \in X^*$ is called a *positive singular linear functional* whenever $\phi_s(f) \geq 0$ holds for all non-negative f in X and for every integral linear functional ϕ, $0 \leq \phi(f) \leq \phi_s(f)$ for all non-negative f in X implies $\phi = 0$. A *singular linear functional* in X^* means the difference of two positive singular linear functionals in X^*. It is known that the space of integral linear functionals in X^* is order isometric to X' and a dual space X^* is order isometric to $X' \oplus X_s^*$, where X_s^* is the space of singular functionals on X [5, 6, 7].

Whenever X is a Banach function space, X_0 (or X^0) will denote the set of all order continuous elements of X. It is easy to show that X_0 is an *order ideal*, which means that it is a closed subspace with the ideal property. Note that X_0 is contained in the closure of the family of all simple functions in X with support of finite measure [1]. It is well known that if X is a Banach function space with the Fatou property and X_0 contains all simple functions with support of finite measure, then $(X_0)^* \simeq X'$. In this case $X^* \simeq (X_0)^* \oplus X_0^\perp$, where $X_0^\perp$ coincides with X_s^* when X is a Banach function space consisting of real valued functions (cf. Theorem 102.6, Theorem 102.7 in [6]).

We will use the following facts about M-ideals [2].

Theorem 0.1. *Suppose Y is a closed subspace of a Banach space X.*

(*i*) *(The 3-ball property)* Y *is an* M*-ideal of* X *if and only if for all* $y_1, y_2, y_3 \in B_Y$*, all* $x \in B_X$ *and* $\epsilon > 0$ *there is* $y \in Y$ *satisfying*

$$\|x + y_i - y\| \leq 1 + \epsilon \quad \textit{for all } i = 1, 2, 3.$$

(*ii*) *A Banach space* X *is* M*-embedded if and only if every separable subspace of* X *is also* M*-embedded.*

(*iii*) *If* X *is an* M*-embedded space, then every separable subspace of* X *has a separable dual.*

For any real functions F and G, we say that F is equivalent to G and we write it as $F \approx G$ whenever there are constants $C_1, C_2 > 0$ such that $C_1|F(u)| \leq |G(u)| \leq C_2|F(u)|$ for all u in the domain of the functions. Recall also that for $z \in \mathbb{C}$, $\operatorname{sign} z = \overline{z}/|z|$ if $z \neq 0$ and $\operatorname{sign} z = 1$ if $z = 0$.

In this paper, We examine M-ideal properties of Marcinkiewicz spaces, including the space $L^1 + L^\infty$.

1. Marcinkiewicz function spaces M_Ψ

Let $L^0 = L^0(I, \mathcal{B}, \mu)$ be the space of all Lebesgue measurable functions on I, where $I = (0, 1)$ or $I = (0, \infty)$, μ is the Lebesgue measure on σ-algebra $\mathcal{B}$ of the Lebesgue measurable subsets of I. For any $f \in L^0$ the *decreasing rearrangement* of f is the function f^* defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},$$

where μ_f is the distribution function of f.

Definition 1.1. Let $\Psi : [0, \infty) \to [0, \infty)$, $\Psi(0) = 0$, Ψ be increasing, and $\Psi(u) > 0$ for $u > 0$. Then the *Marcinkiewicz space* M_Ψ (called also *weak Lorentz space*) is the collection of all functions $f \in L^0$ such that

$$\|f\| = \|f\|_{M_\Psi} = \sup_{t>0} \frac{\int_0^t f^*}{\Psi(t)} < \infty.$$

We will assume further without loss of generality that the function $\Psi(t)/t$ is decreasing on $(0, \infty)$. In fact for any function Ψ from Definition 1.1 that defines non-trivial space M_Ψ, there exists a function $\widehat{\Psi}$ such that $\widehat{\Psi}(t)/t$ is decresing, $\widehat{\Psi}$ has the same properties as Ψ, and the identity operator between M_Ψ and $M_{\widehat{\Psi}}$ is an isometry. Indeed, let

$$\widehat{\Psi}(t) = t \inf\{\Psi(s)/s : 0 < s \leq t\}, \quad t > 0.$$

It is clear that $\widehat{\Psi}(t)/t$ is decreasing for $0 < t_1 < t_2$ we have

$$\begin{aligned}
\widehat{\Psi}(t_2) &= t_2 \min\{\inf\{\Psi(s)/s : 0 < s \leq t_1\}, \inf\{\Psi(s)/s : t_1 \leq s \leq t_2\}\} \\
&= \min\{t_2 \inf\{\Psi(s)/s : 0 < s \leq t_1\}, \Psi(t_1)\} \\
&\geq t_1 \min\{\inf\{\Psi(s)/s : 0 < s \leq t_1\}, \Psi(t_1)/t_1\} = \widehat{\Psi}(t_1),
\end{aligned}$$

which shows that $\widehat{\Psi}$ is increasing. It is also easy to see that M_Ψ is non-trivial if and only if $\widehat{\Psi}(t) > 0$ for $t > 0$. Finally, since $\widehat{\Psi}(t) \le \Psi(t)$, $\|f\|_{M_\Psi} \le \|f\|_{M_{\widehat{\Psi}}}$. On the other hand for any $0 < s \le t$,

$$t\,\frac{1}{s}\int_0^s f^* = t\,\frac{\Psi(s)}{s}\frac{\int_0^s f^*}{\Psi(s)} \le t\,\frac{\Psi(s)}{s}\|f\|_{M_\Psi},$$

and so

$$\int_0^t f^* = t \inf\{\frac{1}{s}\int_0^s f^* : 0 < s \le t\} \le t \inf_{0<s\le t} \frac{\Psi(s)}{s}\,\|f\|_{M_\Psi} = \widehat{\Psi}(t)\|f\|_{M_\Psi},$$

which yields $\|f\|_{M_{\widehat{\Psi}}} \le \|f\|_{M_\Psi}$. Thus M_Ψ and $M_{\widehat{\Psi}}$ coincide and have equivalent norms.

In view of the above remarks we assume further in this section that $\Psi : [0,\infty) \to [0,\infty)$, $\Psi(0) = 0$, $\Psi(t) > 0$ for $t > 0$, Ψ is increasing and $\Psi(t)/t$ is decreasing on $(0,\infty)$ i.e., Ψ is *quasi-concave*. It is well known and easy to show that M_Ψ is a r.i. space with the Fatou property (cf. [1, 5]).

Definition 1.2. M_Ψ^0 is a subspace of M_Ψ consisting of all $f \in M_\Psi$ satisfying

$$\lim_{t\to 0^+} \frac{\int_0^t f^*}{\Psi(t)} = 0 \quad \text{in case when} \quad I = (0,1),$$

and

$$\lim_{t\to 0^+,\infty} \frac{\int_0^t f^*}{\Psi(t)} = 0 \quad \text{in case when} \quad I = (0,\infty).$$

We have the following basic results on M_Ψ and M_Ψ^0 (cf. [5]).

Theorem 1.3. *(i) $M_\Psi^0 \ne \{0\}$ if and only if*

$$\inf_{t>0} \frac{t}{\Psi(t)} = 0 \quad \textit{for} \quad I = (0,1), \tag{1.1}$$

and

$$\inf_{t>0} \frac{t}{\Psi(t)} = 0 \quad \textit{and} \quad \sup_{t>0} \Psi(t) = \infty \quad \textit{for} \quad I = (0,\infty). \tag{1.2}$$

(ii) Let $M_\Psi^0 \ne \{0\}$. Then the three sets: M_Ψ^0, the subspace of all order continuous elements of M_Ψ, and the closure of all simple (or bounded) functions with support of finite measure, coincide.

Proof. Condition (i) is clear since for $0 < t < a$

$$\frac{\int_0^t \chi_{(0,a)}}{\Psi(t)} = \frac{t}{\Psi(t)},$$

and for $t > a$

$$\frac{\int_0^t \chi_{(0,a)}}{\Psi(t)} = \frac{a}{\Psi(t)}.$$

We shall show (ii) only in the case when $I = (0, \infty)$. Since $M_\Psi^0 \neq \{0\}$, the conditions in 1.2 are satisfied. Let $0 < f_n \leq f \in M_\Psi^0$ and $f_n \downarrow 0$. Given $\epsilon > 0$, there exist $0 < t_0 < t_1 < \infty$ such that

$$\sup_{0<t<t_0} \frac{\int_0^t f^*}{\Psi(t)} < \epsilon \quad \text{and} \quad \sup_{t_1<t<\infty} \frac{\int_0^t f^*}{\Psi(t)} < \epsilon.$$

By the Dominated Lebesgue Theorem, there exists N such that for all $n > N$

$$\int_0^{t_1} f_n^* < \epsilon \Psi(t_0).$$

Hence for $n > N$,

$$\|f_n\| \leq \sup_{0<t<t_0} \frac{\int_0^t f^*}{\Psi(t)} + \sup_{t_1<t<\infty} \frac{\int_0^t f^*}{\Psi(t)} + \frac{\int_0^{t_1} f_n^*}{\Psi(t_0)} < 3\epsilon.$$

So every element in M_Ψ^0 is order continuous. Then M_Ψ^0 is contained in the closure of all simple (or bounded) functions with support of finite measure ([1], Theorem 3.11). Finally by conditions 1.2 and by Lemma 5.4 in [5], which is also valid under our assumptions, the closure of the set of all simple functions with support of finite measure coincides to M_Ψ^0.

□

Now, we investigate when M_Ψ^0 is an M-ideal in M_Ψ. The next theorem extends the already known result for some functions Ψ (cf. [2]).

Theorem 1.4. *If $I = (0,1)$, then M_Ψ^0 is an M-ideal in M_Ψ. If $I = (0, \infty)$ and*

$$\inf_{t>0} \Psi(t)/t = 0,$$

then M_Ψ^0 is an M-ideal in M_Ψ.

Proof. In the proof we shall use the 3-ball property (see Theorem 0.1), that is we show that for every $f \in B_{M_\Psi}$, every $f_i \in B_{M_\Psi^0}$, $i = 1, 2, 3$, and $\epsilon > 0$ there exists $g \in B_{M_\Psi^0}$ such that $\|f + f_i - g\| \leq 1 + \epsilon$, $i = 1, 2, 3$.

We asuume that $M_\Psi^0 \neq \{0\}$, otherewise there is nothing to prove. Let first $I = (0, 1)$. Then by Theorem 1.3, $\inf_{t>0} t/\Psi(t) = 0$. By density of bounded functions in M_Ψ^0 we can take f_i bounded. Thus there exists $b > 0$ such that for all $0 < t \leq b$

$$\frac{\int_0^t f_i^*}{\Psi(t)} \leq \frac{Mt}{\Psi(t)} \leq \frac{Mb}{\Psi(b)} < \epsilon,$$

where $|f_i(x)| \leq M$, $x \in (0, 1)$, $i = 1, 2, 3$. We then choose $0 < c \leq b$ such that

$$\frac{\int_0^c f^*}{\Psi(b)} \leq \epsilon.$$

Setting

$$g = f\chi_{\{s:|f(s)|\leq f^*(c)\}},$$

it is clear that $g \in B_{M_\Psi^0}$. Moreover, for $0 < t \leq b$, $i = 1, 2, 3$,

$$\frac{\int_0^t (f_i + f - g)^*}{\Psi(t)} \leq \frac{\int_0^t f_i^*}{\Psi(t)} + \frac{\int_0^t (f-g)^*}{\Psi(t)} \leq \epsilon + \frac{\int_0^t f^*}{\Psi(t)} \leq 1 + \epsilon.$$

We also have for $s \in (0, 1)$,

$$(f - g)^*(s) \leq f^* \chi_{(0,c)}(s).$$

Hence for $t \geq b$, $i = 1, 2, 3$,

$$\frac{\int_0^t (f_i + f - g)^*}{\Psi(t)} \leq \|f_i\| + \frac{\int_0^c f^*}{\Psi(b)} \leq 1 + \epsilon.$$

Combining the above inequalities we get $\|f_i + f - g\| \leq 1 + \epsilon$.

Now let $I = (0, \infty)$. Then in view of $M_\Psi^0 \neq \{0\}$ and Theorem 1.3, conditions 1.2 have to be satisfied. For every $f \in M_\Psi$

$$\limsup_{t\to\infty} \frac{\int_0^t f^*}{\Psi(t)} = \limsup_{t\to\infty} \frac{\frac{1}{t}\int_0^t f^*}{\frac{\Psi(t)}{t}} \leq \sup_{t>0} \frac{\int_0^t f^*}{\Psi(t)} < \infty,$$

and thus in view of the assumption $\inf_{t>0} \Psi(t)/t = 0$ we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f^* = \lim_{t\to\infty} f^*(t) = 0.$$

Since $f_i \in M_\Psi^0$, there are $0 < b_1 < b_2$ such that for all $t < b_1$ or all $t > b_2$,

$$\frac{\int_0^t f_i^*}{\Psi(t)} < \epsilon,$$

for $i = 1, 2, 3$. Choose then $\eta > 0$ so small that $\eta \frac{b_2}{\Psi(b_1)} < \epsilon$ and take $0 < c \leq b_1$ such that

$$\frac{\int_0^c f^*}{\Psi(b_1)} \leq \epsilon.$$

Setting

$$g = f\chi_{\{s:\eta<|f(s)|\leq f^*(c)\}},$$

we have $g \in M_\Psi^0$. Indeed, there is $T > 0$ such that

$$f^*(T) = \inf\{s > 0 : \mu_f(s) \leq T\} < \eta,$$

and so there exists $0 < s < \eta$ such that $\mu_f(s) \leq T$. Hence $\mu_f(\eta) = \mu\{|f| > \eta\} \leq T$ and

$$\lim_{t\to\infty} \frac{\int_0^t g^*}{\Psi(t)} \leq \lim_{t\to\infty} \frac{\int_0^T f^*}{\Psi(t)} = 0.$$

Moreover,

$$\lim_{t\to0^+} \frac{\int_0^t g^*}{\Psi(t)} \leq \lim_{t\to0^+} \frac{tf^*(c)}{\Psi(t)} = 0.$$

For $i = 1, 2, 3$ and $0 < t \le b_1$ or $t \ge b_2$,

$$\frac{\int_0^t (f_i + f - g)^*}{\Psi(t)} \le \frac{\int_0^t f_i^*}{\Psi(t)} + \frac{\int_0^t f^*}{\Psi(t)} \le 1 + \epsilon.$$

Finally for $i = 1, 2, 3$ and $b_1 \le t \le b_2$,

$$\begin{aligned}
\frac{\int_0^t (f_i + f - g)^*}{\Psi(t)} &\le \frac{\int_0^t (f_i + f\chi_{\{|f|\le\eta\}\cup\{|f|>f^*(c)\}})^*}{\Psi(t)} \\
&\le \frac{\int_0^t f_i^* + \int_0^t (f\chi_{\{|f|\le\eta\}})^* + \int_0^t (f\chi_{\{|f|>f^*(c)\}})^*}{\Psi(t)} \\
&\le \frac{\int_0^{b_1} f_i^* + \int_{b_1}^t f_i^* + \int_0^c f^* + t\eta}{\Psi(t)} \\
&\le \frac{\int_0^{b_1} f_i^* + b_1\eta}{\Psi(t)} + \frac{\int_{b_1}^t f_i^* + (t - b_1)\eta}{\Psi(t)} + \frac{\int_0^c f^*}{\Psi(b_1)} \\
&\le \frac{\int_0^{b_1} f_i^* + b_1\eta}{\Psi(b_1)} + \frac{\int_{b_1}^t f_i^* + \eta(b_2 - b_1)}{\Psi(t)} + \epsilon \\
&\le \epsilon + \eta\frac{b_1}{\Psi(b_1)} + 1 + \eta\frac{(b_2 - b_1)}{\Psi(b_1)} + \epsilon \\
&< 1 + 4\epsilon.
\end{aligned}$$

Combining the above inequalities we complete the proof. $\square$

We will see in the next section (Remark 2.3) that the assumption $\inf_{t>0} \Psi(t)/t = 0$ for $I = (0, \infty)$ in the above theorem cannot be removed.

It is well known that if Ψ is quasi-concave, then there exists an increasing concave function $\widetilde{\Psi}$ on I such that $\Psi(t) \le \widetilde{\Psi}(t) \le 2\Psi(t)$ on I (cf. Proposition 5.10 in [1]). It is easy to show that $\| \ \|_{M_{\widetilde{\Psi}}} \approx \| \cdot \|_{M_\Psi}$. So we can obtain an equivalent norm on M_Ψ, which is induced by an increasing concave function on I.

Theorem 1.5. *Let $M_\Psi^0 \neq \{0\}$ that is the conditions 1.1 or 1.2 are satisfied.*

(i) If $I = (0, 1)$ then M_Ψ is the bidual of M_Ψ^0.

(ii) If $I = (0, \infty)$ and

$$\inf_{t>0} \Psi(t) = 0$$

then M_Ψ is the bidual of M_Ψ^0.

Proof. Assume first that Ψ is concave. By Theorem 1.3 (ii), $(M_\Psi^0)^* = (M_\Psi)'$. Let $\|f\|_{M_\Psi} \le 1$ and g be a simple function such that $g^* = \sum_{i=1}^n a_i \chi_{(0,t_i]}$, where $0 < t_1 < \cdots < t_n$, and $a_i \ge 0$. Then for all $t > 0$, $\int_0^t f^* \le \Psi(t)$ and so

$$\int_I g^* f^* \le \sum_{i=1}^n a_i \Psi(t_i) = \int_I g^* d\Psi,$$

where the Lebesgue-Stieltjes integral is well-defined since Ψ is continuous on $[0,\infty)$. By the Fatou property of M_Ψ we have (cf. Proposition 4.2 in [1])

$$\|g\|_{(M_\Psi)'} = \sup\Big\{\int_I f^* g^* : \|f\|_{M_\psi} \le 1\Big\}.$$

Thus for all g in L^0,

$$\|g\|_{(M_\Psi)'} \le \int_I g^* d\Psi.$$

Since Ψ is continuous and concave, there exists $h \in L^0$ such that for $t \in I$,

$$\Psi(t) = \int_0^t h^*(s)ds.$$

Then $\|h\|_{M_\Psi} \le 1$, and for any $g \in L^0$, $\int_I h^* g^* = \int_I g^* d\Psi$. So we get the reverse inequality

$$\|g\|_{(M_\Psi)'} \ge \int_I g^* d\Psi,$$

which yields that $\|g\|_{(M_\Psi)'} = \int_I g^* d\Psi$. Therefore the associate space

$$(M_\Psi)' = \Big\{g \in L^0 : \int_I g^* d\Psi < \infty\Big\}$$

is a Lorentz space, and thus is order continuous [5]. In general, if Ψ is not concave then $\|\cdot\|_{M_\Psi} \approx \|\ \|_{M_{\tilde\Psi}}$, and hence $\|\cdot\|_{(M_\Psi)'} \approx \|\cdot\|_{(M_{\tilde\Psi})'}$. Since $(M_{\tilde\Psi})'$ is order continuous, $(M_\Psi)'$ is order continuous too. Then order continuity of $(M_\Psi)'$ implies $(M_\Psi^0)^{**} \simeq (M_\Psi)'^* \simeq (M_\Psi)'' = M_\Psi$ by the Fatou property of $\|\cdot\|_{M_\Psi}$. This completes the proof.

□

Notice that the assumption $\inf_{t>0}\Psi(t) = 0$ cannot be skipped in the above theorem (cf. Remark 2.5).

2. The spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$

In this section we will investigate M-ideal properties of $\Sigma = L^1 + L^\infty$ and $\Delta = L^1 \cap L^\infty$ on $I = (0,\infty)$ equipped with the following norms.

$$\|f\|_\Sigma = \inf\{\|g\|_1 + \|h\|_\infty : f = g+h, g \in L^1, h \in L^\infty\} = \int_0^1 f^*, \tag{2.1}$$

$$\|\!|f|\!\|_\Sigma = \inf\{\max\{\|g\|_1, \|h\|_\infty\} : f = g+h, g \in L^1, h \in L^\infty\},$$

$$\|f\|_\Delta = \max\{\|f\|_1, \|f\|_\infty\},$$

$$\|\!|f|\!\|_\Delta = \|f\|_1 + \|f\|_\infty .$$

It is clear that $\|\cdot\|$ and $|||\cdot|||$ are equivalent. The equality in (2.1) is well known and can be found e.g. in [1]. It is also well known [3] that $(\Sigma, \|\cdot\|_\Sigma)' = (\Delta, \|\cdot\|_\Delta)$ and $(\Sigma, |||\cdot|||_\Sigma)' = (\Delta, |||\cdot|||_\Delta)$. Moreover,

$$\Sigma_0 = \{f \in \Sigma : \lim_{t\to\infty} f^*(t) = 0\},$$

where Σ_0 is a subspace of all order continuous elements of Σ (cf. [1, 5]).

It appears that for certain choice of Ψ, the Marcinkiewicz space M_Ψ coincides with Σ, and M_Ψ^0 with Σ_0. In fact we have the following result.

Proposition 2.1. *The norms $\|\cdot\|_{M_\Psi}$ and $\|\cdot\|_\Sigma$ are equal if and only if for all $t > 0$*

$$\Psi(0) = 0 \quad \textit{and} \quad \Psi(t) = \max\{t, 1\},$$

and they are equivalent if and only if for all $t > 0$

$$\Psi(0) = 0 \qquad \textit{and} \qquad \Psi(t) \approx \max\{t, 1\}.$$

Consequently if $I = (0, \infty)$ and $\lim_{t\to 0+} \Psi(t) > 0$ and $\lim_{t\to\infty} \Psi(t)/t > 0$ then the spaces M_Ψ^0 and Σ_0 coincide as sets with equivalent norms.

Proof. If $\|\cdot\|_{M_\Psi}$ and $\|\cdot\|_\Sigma$ are equal, then for $t > 0$,

$$\|\chi_{(0,t)}\|_{M_\Psi} = \frac{t}{\Psi(t)} = \|\chi_{(0,t)}\|_\Sigma = \min\{t, 1\}.$$

Hence $\Psi(t) = \max\{t, 1\}$, for $t > 0$. Conversely suppose that $\Psi(t) = \max\{t, 1\}$ for $t > 0$. Then

$$\|f\|_{M_\Psi} = \sup_{t>0} \frac{\int_0^t f^*}{\max\{t, 1\}} = \max\left\{\sup_{0<t\le 1} \int_0^t f^*, \sup_{t>1} \frac{1}{t}\int_0^t f^*\right\} = \int_0^1 f^* = \|f\|_\Sigma.$$

Analogously we show the conditions for the equivalence of the norms. □

Let $\|\cdot\|$ be an equivalent norm to $\|\cdot\|_\Sigma$ or to $|||\cdot|||_\Sigma$. Then it is not difficult to see that ℓ^1 is isomorphically embedded in $(\Sigma_0, \|\cdot\|)$. Therefore (see Theorem 0.1) $(\Sigma_0, \|\cdot\|)$ is not an M-embedded space.

In the next two theorems we calculate the exact norms of the duals $(\Sigma, \|\cdot\|_\Sigma)^*$ and $(\Sigma, |||\ |||_\Sigma)^*$. In consequence we answer the question when Σ_0 is an M-ideal in Σ. In the sequel $\|\cdot\|_1$ and $\|\cdot\|_\infty$ will denote as usual the norms in L^1 and L^∞, respectively.

Theorem 2.2. *The following equalities hold true.*

$$(\Sigma, \|\cdot\|_\Sigma)^* = \Sigma_0^* \oplus \Sigma_0^\perp \simeq (\Delta, \|\cdot\|_\Delta) \oplus \Sigma_0^\perp.$$

Moreover for any $F \in \Sigma^*$,

$$F = F_1 + F_2$$

with $F_2 \in \Sigma_0^\perp$ and

$$F_1(g) = \int g f_1$$

for some $f_1 \in (\Delta, \|\cdot\|_\Delta)$, *and*

$$\|F\| = \max\{\|f_1\|_\infty, \|f_1\|_1 + \|F_2\|\}.$$

Consequently, Σ_0 *is not an* M*-ideal of* $(\Sigma, \|\ \|_\Sigma)$.

Proof. The equalities $(\Sigma, \|\cdot\|_\Sigma)^* = \Sigma_0^* \oplus \Sigma_0^\perp \simeq (\Delta, \|\cdot\|_\Delta) \oplus \Sigma_0^\perp$ up to equivalence in norms is a consequence of the well known results on duals in Banach function spaces (cf. Theorem 102.6, Theorem 102.7 in [6]).

Now let $F \in \Sigma^*$ and let $\widetilde{F}_1 = F|_{\Sigma_0}$. There exists $f_1 \in \Sigma'$ such that $\widetilde{F}_1(g) = \int f_1 g$ for all $g \in \Sigma_0$ and $\|\widetilde{F}_1\| = \|f_1\|_{\Sigma'} = \|f_1\|_\Delta$. Define $F_1(g) = \int f_1 g$ for all $g \in \Sigma$, and let $F_2 = F - F_1$. Then $F_2|_{\Sigma_0} = 0$ and $\|\widetilde{F}_1\| = \|F_1\|$.

For each $f = g + h$ with $g \in L^1$ and $h \in L^\infty$, we have $F_2(g) = 0$, and so

$$\begin{aligned}
|F(g+h)| &\le \Big|\int f_1 g\Big| + \Big|\int f_1 h\Big| + |F_2(h)| \\
&\le \|f_1\|_\infty \|g\|_1 + \|f_1\|_1 \|h\|_\infty + \|F_2\| \|h\|_\Sigma \\
&\le \|f_1\|_\infty \|g\|_1 + (\|f_1\|_1 + \|F_2\|)\|h\|_\infty \\
&\le (\|g\|_1 + \|h\|_\infty) \max\{\|f_1\|_\infty, \|f_1\|_1 + \|F_2\|\}
\end{aligned}$$

Therefore, $\|F\| \le \max\{\|f_1\|_\infty, \|f_1\|_1 + \|F_2\|\}$.

Conversely, given $\epsilon > 0$ there exist $g \in L^1$, $h \in L^\infty$ such that $\|g\|_1 + \|h\|_\infty \le 1+\epsilon$ and $\|F_2\| \le \operatorname{Re} F_2(h) + \epsilon$. For each $N \ge 1$, Let $f = \operatorname{sign}(f_1)\chi_{[0,N)} + h\chi_{[N,\infty)}$. Then $|f| = \chi_{[0,N)} + |h|\chi_{[N,\infty)}$, and so $\|f\|_\Sigma = \int_0^1 f^* \le 1 + \epsilon$. Thus

$$\begin{aligned}
\operatorname{Re} F(f) &= \int_0^N |f_1| + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \operatorname{Re} F_2(\operatorname{sign}(f_1)\chi_{[0,N)} + h\chi_{[N,\infty)}) \\
&= \int_0^N |f_1| + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \operatorname{Re} F_2(h) \\
&\ge \int_0^N |f_1| + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \|F_2\| - \epsilon.
\end{aligned}$$

Therefore

$$\|F\| \ge \frac{1}{1+\epsilon}\Big(\|F_2\| - \epsilon + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \int_0^N |f_1|\Big)$$

for all $\epsilon > 0$ and all $N \ge 1$. Since $\int_N^\infty f_1 h \to 0$ as $N \to \infty$, so $\|F\| \ge \|F_2\| + \|f_1\|_1$. Clearly, $\|F\| \ge \|\tilde{F}_1\| = \|f_1\|_\Delta \ge \|f_1\|_\infty$. Hence $\|F\| = \max\{\|f\|_\infty, \|f_1\|_1 + \|F_2\|\}$.

Now suppose that Σ_0 is an M-ideal of Σ. Then there is a projection $P : \Sigma^* \to \Sigma^*$ such that the range of P is $\Sigma_0^\perp$ and for each $F \in \Sigma^*$, $\|F\| = \|PF\| + \|(I-P)F\|$. Note that $PF = F_2$ and $(I-P)F = F_1$ so that we can choose $f_1 = \chi_{[0,1/2)}$ and F_2 with $\|F_2\| = 1$. Then by the above calculations $\|F\| = 3/2$. But on the other hand we must have $\|F\| = \|PF\| + \|(I-P)F\| = \|F_2\| + \|f_1\|_\Delta = 2$, which is a contradiction. □

Remark 2.3. By Proposition 2.1, $(\Sigma, \|\cdot\|_\Sigma) = M_\Psi$, where $\Psi(t) = \max\{t, 1\}$, $t > 0$. Thus $\inf_{t>0} \Psi(t)/t = 1$, and so the assumption in Theorem 1.4 is not satisfied. Since Σ_0 is not an M-ideal in $(\Sigma, \| \|_\Sigma)$, we see that the assumption $\inf_{t>0} \Psi(t)/t = 0$ cannot be omitted in Theorem 1.4.

The next theorem shows that if we use another equivalent norm $|\!|\!|\cdot|\!|\!|_\Sigma$ in Σ, the M-ideal properties are remarkably changed.

Theorem 2.4. *The following equalities are satisfied*

$$(\Sigma, |\!|\!|\cdot|\!|\!|_\Sigma)^* = \Sigma_0^* \oplus \Sigma_0^\perp = (\Delta, |\!|\!|\cdot|\!|\!|_\Delta) \oplus_1 \Sigma_0^\perp.$$

Moreover for $F \in \Sigma^*$,

$$F = F_1 + F_2$$

where $F_2 \in \Sigma_0^\perp$ *and*

$$F_1(g) = \int g f_1$$

for some $f_1 \in (\Delta, |\!|\!|\cdot|\!|\!|_\Delta)$, *and*

$$\|F\| = \|F_1\| + \|F_2\| = \|f_1\|_\infty + \|f_1\|_1 + \|F_2\|.$$

Therefore Σ_0 *is an* M*-ideal of* $(\Sigma, |\!|\!| \; |\!|\!|_\Sigma)$.

Proof. By the same method as in the proof of the previous theorem, we get a decomposition $F = F_1 + F_2$ with $F_2|_{\Sigma_0} = 0$, $F_1(g) = \int f_1 g$ for all $g \in \Sigma$, and $\|F_1\| = |\!|\!|f_1|\!|\!|_\Delta$.

Now for each $f = g + h \in \Sigma$ with $g \in L^1$ and $h \in L^\infty$ we have

$$\begin{aligned} |F(g+h)| &\le \left| \int f_1(g+h) \right| + |F_2(h)| \\ &\le (\|f\|_1 + \|f_1\|_\infty) \max\{\|g\|_1, \|h\|_\infty\} + \|F_2\| \, |\!|\!|h|\!|\!|_\Sigma \\ &\le (\|f\|_1 + \|f_1\|_\infty) \max\{\|g\|_1, \|h\|_\infty\} + \|F_2\| \|h\|_\infty \\ &\le \max\{\|g\|_1, \|h\|_\infty\}(\|f_1\|_\infty + \|f_1\|_1 + \|F_2\|). \end{aligned}$$

Hence $\|F\| \le \|f_1\|_\infty + \|f_1\|_1 + \|F_2\|$.

Conversely, suppose that $\|f_1\|_\infty \ne 0$. For large enough $n \in \mathbb{N}$, choose $E_n \subset \{|f_1| > \|f_1\|_\infty - 1/n\}$ with $0 < \mu E_n < \infty$. Let

$$g_n = \operatorname{sign}(f_1) \frac{\chi_{E_n}}{\mu E_n}.$$

Given $\epsilon > 0$, choose $g \in L^1$ and $h \in L^\infty$ so that $\max\{\|g\|_1, \|h\|_\infty\} \le 1 + \epsilon$ and $\|F_2\| \le \operatorname{Re} F_2(h) + \epsilon$. Let

$$h_n = h\chi_{[n,\infty)} + \operatorname{sign}(f_1)\chi_{[0,n)}.$$

Then $\|h_n\|_\infty \le 1+\epsilon$ and $\|g_n\|_1 \le 1$. Hence for $f_n = g_n + h_n$, we have $\|\!|f_n|\!\|_\Sigma \le 1+\epsilon$. Consequently

$$
\begin{aligned}
\operatorname{Re} F(f_n) &= \operatorname{Re}\int f_1 g_n + \operatorname{Re}\int f_1 h_n + \operatorname{Re} F_2(h_n) \\
&= \int_{E_n} \frac{|f_1|}{\mu E_n} + \int_0^n |f_1| + \operatorname{Re}\int_n^\infty f_1 h + \operatorname{Re} F_2(h_n - \operatorname{sign}(f_1)\chi_{[0,n)} + h\chi_{[0,n)}) \\
&\ge \|f\|_\infty - \frac{1}{n} + \int_0^n |f_1| + \operatorname{Re}\int_n^\infty f_1 h + \operatorname{Re} F_2(h) \\
&\ge \|f\|_\infty - \frac{1}{n} + \int_0^n |f_1| + \operatorname{Re}\int_n^\infty f_1 h + \|F_2\| - \epsilon.
\end{aligned}
$$

Therefore $\|F\| \ge \frac{1}{1+\epsilon}(\|f\|_\infty - \frac{1}{n} + \int_0^n |f_1| + \operatorname{Re}\int_n^\infty f_1 h + \|F_2\| - \epsilon)$. Note that h is independent of n. Since $\lim_{n\to\infty}\int_n^\infty f_1 h = 0$ and ϵ is arbitrary we obtain $\|F\| \ge \|f_1\|_\infty + \|f_1\|_1 + \|F_2\|$, and this completes the proof. □

Remark 2.5. (1) Note that we have the following equalities (with equivalence of norms)

$$\Sigma_0^{**} \simeq (\Sigma')^* = \Delta^* \simeq \Delta' \oplus \Delta_s^* = \Sigma \oplus \Delta_s^*,$$

where $\Delta_s^* \ne \{0\}$ since Δ is not order continuous. Thus the bidual of $\Sigma_0 = M_\Psi^0$ with $\Psi(t) = \max\{t,1\}$, $t > 0$, is not equal to $\Sigma = M_\Psi$. It shows that the assumption $\inf_{t>0}\Psi(t) = 0$ in Theorem 1.5 cannot be omitted.

(2) We observe also that since $(\Delta, \|\cdot\|_\Delta)$ is not order continuous, it contains an isomorphic copy of ℓ^∞ (cf. [6]), and so it contains an isomorphic copy of ℓ^1 which has a non-separable dual. Therefore Δ with any equivalent norm to $\|\cdot\|_\Delta$ is not M-embedded.

3. Marcinkiewicz sequence spaces

In this section we will consider Marcinkiewicz sequence spaces. Assume further that $\Psi = \{\Psi(n)\} = \{\Psi(n)\}_{n=0}^\infty$ is a sequence such that $\Psi(0) = 0$, $\{\Psi(n)\}$ is increasing, $\Psi(n) > 0$ for $n > 0$ and $\{\Psi(n)/n\}$ is decreasing. Given a sequence $x = \{x(n)\} = \{x(n)\}_{n=1}^\infty$ define its decreasing rearrangement $x^* = \{x^*(n)\}$ as

$$x^*(n) = f^*(n-1), \quad n \in \mathbb{N},$$

where $f(t) = \sum_{k=1}^\infty x(k)\chi_{[k-1,k)}(t)$, $t \ge 0$.

Definition 3.1. The *Marcinkiewicz sequence space* m_Ψ consists of all sequences $x = \{x(n)\} = \{x(n)\}_{n=1}^\infty$ such that

$$\|x\| = \|x\|_{m_\Psi} = \sup_{n\ge 1} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} < \infty.$$

Let m_Ψ^0 be a subspace of m_Ψ consisting of all $x \in m_\Psi$ satisfying

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} = 0.$$

We have the following basic facts about m_Ψ and m^0_Ψ.

Theorem 3.2. (1) *m_Ψ is a r.i. Banach sequence space with the Fatou property.*

(2) *$m^0_\Psi \neq \{0\}$ if and only if $\lim_{n\to\infty} \Psi(n) = \infty$.*

(3) *If $\lim_{n\to\infty} \Psi(n) = \infty$, then m^0_Ψ is a non-trivial subspace of all order continuous elements of m_Ψ.*

(4) *The following conditions are equivalent.*

  (*a*) *$\|x\|_{m_\Psi} = \|x\|_\infty$ for all $x \in \ell^\infty$ (resp. $\|x\|_{m_\Psi} \approx \|x\|_\infty$ for all $x \in \ell^\infty$).*

  (*b*) *$\|x\|_{m_\Psi} = \|x\|_\infty$ for all $x \in c_0$ (resp. $\|x\|_{m_\Psi} \approx \|x\|_\infty$ for all $x \in c_0$).*

  (*c*) *$\Psi(n) = n$ for all $n \in \mathbb{N}$ (resp. $\Psi(n) \approx n$ for all $n \in \mathbb{N}$).*

Proof. Condition (1) is immediate and (2) is clear if we note that $e_1 \in m^0_\Psi$ is equivalent to $\lim_{n\to\infty} 1/\Psi(n) = 0$. For (3), note that m^0_Ψ contains all characteristic functions with support of finite measure by (2), so it contains all order continuous elements [1]. The proof that any $x \in m^0_\Psi$ is order continuous is very similar to the function case, so we omit it. Finally we shall prove that $4(a)$ is equivalent to $4(c)$. Let's assume first that two norms are equal. Then for $n \in \mathbb{N}$,

$$\|e_1 + \cdots + e_n\|_{m_\Psi} = \frac{n}{\Psi(n)} = 1.$$

For the converse, if we assume $\Psi(n) = n$ for $n \in \mathbb{N}$, then for any $x \in \ell^\infty$,

$$\|x\|_\infty = x^*(1) = \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^{n} x^*(k) = \|x\|_{m_\Psi}.$$

The remaining equivalences can be proved in a similar way. □

Before we state the main results of this section we need to prove the following simple lemma. Given the sequence $\{\Psi(n)\}$ define the function $\Psi(t) = \sum_{i=0}^\infty \Psi(i)\chi_{[i,i+1)}(t)$ on $[0,\infty)$. Obviously $\Psi|_{\mathbb{N}\cup\{0\}}$ coincides with $\{\Psi(n)\}$.

Lemma 3.3. *There is a concave continuous function $\widetilde{\Psi}$ on $[0,\infty)$ such that $\Psi \leq \widetilde{\Psi} \leq 3\Psi$ on $[1,\infty)$ and $\widetilde{\Psi}(0) = 0$.*

Proof. Fix $s \geq 1$. For $0 < t \leq s$,

$$\frac{\Psi(t)}{t} \leq \frac{\Psi(s)}{t},$$

and for $[s] \leq [t]$,

$$\frac{\Psi(t)}{t} \leq \frac{\Psi([t])}{[t]} \leq \frac{\Psi([s])}{[s]} = \frac{s}{[s]}\frac{\Psi(s)}{s} \leq 2\frac{\Psi(s)}{s},$$

where for real $r \in \mathbb{R}$, $[r]$ is the greatest integer less than or equal to r. Hence for every $t \geq 0$ and $s \geq 1$,

$$\Psi(t) \leq (1 + \frac{2t}{s})\Psi(s) \quad \text{and} \quad \Psi(t) \leq t\Psi(1).$$

Therefore there is a minimal concave function $\widetilde{\Psi}$ such that for each $t \geq 0$, $s \geq 1$,

$$\Psi(t) \leq \widetilde{\Psi}(t) \leq \min\{(1+\frac{2t}{s})\Psi(s), t\Psi(1)\}.$$

Then for every $s \geq 1$ and $t > 0$,

$$\widetilde{\Psi}(s) \leq (1+\frac{2s}{s})\Psi(s) = 3\Psi(s) \quad \text{and} \quad \widetilde{\Psi}(t) \leq t\Psi(1).$$

So $\lim_{t\to 0^+} \widetilde{\Psi}(t) = 0$. Therefore $\widetilde{\Psi}$ is a continuous concave function on $[0,\infty)$. $\square$

Now, we are ready to investigate when m_Ψ is the bidual of m_Ψ^0 and when m_Ψ^0 is an M-ideal of m_Ψ. The following theorems show that the situation in sequence case is simpler than in the non-atomic case.

Theorem 3.4. *The space m_Ψ is a bidual of m_Ψ^0 if and only if* $\lim_{n\to\infty} \Psi(n) = \infty$.

Proof. If $\lim_{n\to\infty} \Psi(n) < \infty$, then by Theorem 3.2 (2), $m_\Psi^0 = \{0\}$. So m_Ψ cannot be bidual of m_Ψ^0 since $m_\Psi \neq \{0\}$.

For the converse, suppose that $\lim_{n\to\infty} \Psi(n) = \infty$. Then by Theorem 3.2 (2) and (3), m_Ψ^0 is the order continuous subspace of m_Ψ and it contains all simple functions with support of finite measure. Hence $(m_\Psi^0)^* \simeq (m_\Psi)'$. So if we show that $(m_\Psi)'$ is order continuous, then $(m_\Psi^0)^{**} \simeq ((m_\Psi)')^* \simeq (m_\Psi)'' = m_\Psi$, and the proof is done.

Note that by Lemma 3.3, there is an equivalent norm in m_Ψ induced by the concave function $\widetilde{\Psi}$, that is

$$\|x\|_{m_{\widetilde{\Psi}}} = \sup_{n\geq 1} \frac{\sum_{k=1}^n x^*(k)}{\widetilde{\Psi}(n)}.$$

If $\|x\|_{m_{\widetilde{\Psi}}} \leq 1$, then

$$\sum_{k=1}^n x^*(k) \leq \widetilde{\Psi}(n),$$

for all $n \geq 1$. For any decreasing sequence $y^* = (y^*(1), \cdots, y^*(n), 0, \cdots)$, the summation by parts shows that

$$\sum_{k=1}^n x^*(k)y^*(k) \leq \sum_{k=1}^n y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)).$$

Then by the Fatou property, for any $y = \{y(k)\}$,

$$\|y\|_{(m_{\widetilde{\Psi}})'} \leq \sum_{k=1}^\infty y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)).$$

Note that there is an integral representation $\widetilde{\Psi}(t) = \int_0^t h^*(s)ds$ for some $h \in L^0$. This shows that, if we take $x(k) = \widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)$ for all $k \in \mathbb{N}$, then the sequence $\{x(k)\}$ is decreasing and for each $n \in \mathbb{N}$,

$$\frac{\sum_{k=1}^n x^*(k)}{\widetilde{\Psi}(n)} = \frac{\widetilde{\Psi}(n)}{\widetilde{\Psi}(n)} = 1.$$

This means that $\|x\| = 1$ and for all y,

$$\sum_{k=1}^{\infty} x^*(k)y^*(k) = \sum_{k=1}^{\infty} y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)).$$

Hence

$$\|y\|_{(m_{\widetilde{\Psi}})'} \geq \sum_{k=1}^{\infty} y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)),$$

for all y. Therefore we obtain the following formula

$$\|y\|_{(m_{\widetilde{\Psi}})'} = \sum_{k=1}^{\infty} y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1))$$

and this implies that $(m_{\widetilde{\Psi}})'$ and hence $(m_\Psi)'$ is order continuous [5]. $\square$

In view of Theorem 3.2 (4), if $\Psi(n) = n$, then $m_\Psi^0 = c_0$ and $m_\psi = \ell^\infty$ with equality of norms, and thus m_Ψ^0 is an M-ideal of m_Ψ. The next theorem extends this result to a broader class of functions Ψ and improves already existing results in certain class of m_Ψ (cf. [2]).

Theorem 3.5. *Assume that* $\lim_{n\to\infty} \frac{\Psi(n)}{n} = 0$ *and* $\lim_{n\to\infty} \Psi(n) = \infty$. *Then* m_Ψ^0 *is an* M*-ideal in its bidual* m_Ψ.

Proof. First observe that if $x \in m_\Psi$, then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} = \limsup_{n\to\infty} \frac{\frac{1}{n}\sum_{k=1}^n x^*(k)}{\frac{1}{n}\Psi(n)} \leq \sup_n \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} < \infty,$$

and in view of the assumption $\lim_{n\to\infty} \frac{\Psi(n)}{n} = 0$,

$$\lim_{n\to\infty} x^*(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n x^*(k) = 0.$$

In the proof we shall use the 3-ball property (cf. Theorem 0.1) and the same technique as in [2], that is we show that for every $x = \{a(n)\} \in B_{m_\Psi}$, every $x_i = \{x_i(n)\} \in B_{m_\Psi^0}$ with finite support, $i = 1, 2, 3$, and $\epsilon > 0$ there is $y \in m_\Psi^0$ such that $\|x + x_i - y\| \leq 1 + \epsilon$, $i = 1, 2, 3$. First assume that for all $i = 1, 2, 3$,

$$\max\{j : x_i^*(j) \neq 0\} =: k_i = k,$$

and

$$\sum_{j=1}^k x_i^*(j) \leq \sum_{j=1}^k a^*(j).$$

Next pick up N such that for all $n \geq N$, $x_i(n) = 0$ and

$$|a(n)| \leq \min\{\delta, a^*(k)\},$$

where $\delta = \min_i x_i^*(k)$. Then define the sequence $y = \{y(n)\}$ by $y(n) = a(n)$ if $n \leq N$ and $y(n) = 0$ otherwise. If $z_i(n) = a(n) + x_i(n) - y(n)$, then $z_i^*(j) = x_i^*(j)$ for $j \leq k$ and $z_i^*(j) \leq a^*(j)$ for $j > k$. Hence for $n \leq k$,

$$\frac{\sum_{j=1}^n z_i^*(j)}{\Psi(n)} \leq 1,$$

and for $n > k$,

$$\frac{\sum_{j=1}^n z_i^*(j)}{\Psi(n)} \leq \frac{\sum_{j=1}^n a^*(j)}{\Psi(n)} \leq 1.$$

Therefore $\|x + x_i - y\| \leq 1$.

In general case, we may assume that x is not an element of m_Ψ^0. In this case, we cannot have $x \in \ell^1$. Hence we can find $l \geq k_i$ for all $i = 1, 2, 3$, such that

$$\sum_{j=1}^{k_i} x_i^*(j) < \sum_{j=1}^{l} a^*(j).$$

Define ξ as follows: If $x_i(n) \neq 0$ then let $\xi_i(n) = x_i(n)$. At $l - k_i$ indices where $x_i(n) = 0$, let $\xi_i(n) = \alpha$ ($\alpha > 0$ is chosen later), otherwise let $\xi_i(n) = 0$. The number α should be chosen so small that for all $i = 1, 2, 3$, $\|x_i - \xi_i\| \leq \epsilon$ and

$$\sum_{j=1}^{n} \xi_i^*(j) \leq \sum_{j=1}^{l} a^*(j).$$

By the first part of the proof, there exists $y \in m_\Psi^0$ such that

$$\|x + \frac{\xi_i}{1+\epsilon} - y\| \leq 1.$$

Hence $\|x + x_i - y\| \leq 1 + 2\epsilon$, which completes the proof. □

Remark 3.6. Theorem 3.2 (4) shows that $\lim_{n\to\infty} \frac{\Psi(n)}{n} > 0$ if and only if $m_\Psi^0 = c_0$ up to equivalent norms. Therefore if $\lim_{n\to\infty} \frac{\Psi(n)}{n} > 0$, then m_Ψ can be renormed so that m_Ψ^0 is an M-ideal of its bidual m_Ψ, since c_0 is an M-ideal of ℓ^∞. But m_Ψ^0 with its original norm does not need to be an M-ideal of m_Ψ if we drop the assumption $\lim_{n\to\infty} \Psi(n)/n = 0$, as we can see in the following example.

Let $\Psi(0) = 0$, $\Psi(n) = \max\{\frac{2n}{3}, 1\}$ for $n \in \mathbb{N}$. Then $m_\Psi = \ell^\infty$ with norm

$$\|x\|_\Psi = \sup\left\{x^*(1), \frac{3(x^*(1) + x^*(2))}{4}, \dots, \frac{3\sum_{k=1}^n x^*(k)}{2n}, \dots\right\}$$

that is equivalent to $\|\cdot\|_\infty$-norm. Then $(c_0, \|\cdot\|_\Psi)$ is not an M-ideal of $(\ell^\infty, \|\cdot\|_\Psi)$.

Indeed, let $x_1 = e_1 + \frac{1}{3}e_2$, $x_2 = e_1 - \frac{1}{3}e_2$, $x_3 = -e_1 + \frac{1}{3}e_2$, and let $x \equiv 2/3$. Note that $\|x_i\| = \|x\| = 1$. Then there is no $y \in c_0$ such that $\|x_i + x - y\|_\Psi < \frac{5}{4}$. Observe the following formulas for any $y \in c_0$,

$$|x_1 + x - y| = (|5/3 - y(1)|, |1 - y(2)|, |2/3 - y(3)|, \ldots),$$
$$|x_2 + x - y| = (|5/3 - y(1)|, |0 - y(2)|, |2/3 - y(3)|, \ldots),$$
$$|x_3 + x - y| = (|1/3 + y(1)|, |1 - y(2)|, |2/3 - y(3)|, \ldots).$$

Then $\max\{|5/3 - y(1)|, |1/3 + y(1)|\} \geq 1$ for all scalars $y(1)$. Therefore for each $y \in c_0$ there is i such that $(x_i + x - y)^*(1) \geq 1$ and note that $\lim_{n\to\infty} |2/3 - y(n)| = 2/3$, so that $(x_i + x - y)^*(2) \geq 2/3$ for all $i = 1, 2, 3$. This means that for every $y \in c_0$ there is some i such that $\|x_i + x - y\|_\Psi \geq 3/4(1 + 2/3) = 5/4$.

This example shows that we cannot omit the additional conditions in Theorem 3.5.

Acknowledgment. The research on this project started when the first named author visited KAIST. She wishes to thank the Institute for its hospitality and support. The second named author would like to express deep gratitude to his supervisor Chang Sun Choi and his companions in KAIST whose guidance and support were crucial for the successful completion of this project.

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press 1988.
[2] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics **1547**, Springer-Verlag 1993.
[3] H. Hudzik, A. Kamińska and M. Mastyło, *Local geometry of $L^1 \cap L^\infty$ and $L^1 + L^\infty$*, Arch. Math. **68** (1997), 159–168.
[4] A. Kamińska and M. Mastyło, *The Dunford Pettis property for symmetric spaces*, Canad. J. Math **52(4)** (2000), 789–803.
[5] S.G. Krein, Ju.I. Petunin and E.M. Semenov, *Interpolation of Linear Operators*, AMS Translations of Math. Monog. **54**, 1982.
[6] W.A.J. Luxemburg and A.C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.
[7] M. Väth, *Ideal Spaces*, Lecture Notes in Mathematics **1664**, Springer-Verlag 1997.

Anna Kamińska
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152, USA
E-mail: kaminska@memphis.edu

Han Ju Lee
Department of Mathematics
Korea Advanced Institute of Science and Technology
373-1, Kusong-Dong, Yusong-Gu, Taejon, 305-701, Republic of Korea

E-mail: hjlee@amath.kaist.ac.kr