ON UNIQUENESS OF EXTENSION OF HOMOGENEOUS POLYNOMIALS

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Abstract. We study the uniqueness of norm-preserving extension of n-homogeneous polynomials in Banach spaces. We show that norm-preserving extensions of n-homogeneous polynomials do not need to be unique for $n \geq 2$ in real Banach spaces, and for $n \geq 3$ in a large class of complex Banach function spaces. We find further a geometric condition, which in particular yields that a unit ball in X does not possess any complex extreme point, under which for every norm-attaining 2-homogeneous polynomial on a complex symmetric sequence space X there exists a unique norm-preserving extension from X to its bidual X^{**}. In particular, if m_Ψ is a Marcinkiewicz sequence space and m_Ψ^0 is its subspace of order continuous elements, we show that every norm-attaining 2-homogeneous polynomial on m_Ψ^0 has a unique norm-preserving extension to its bidual m_Ψ if and only if no element of a unit ball of m_Ψ is a complex extreme point. We then apply these results to obtain some necessary conditions for the uniqueness of extension of 2-homogeneous polynomials from a complex symmetric space X to its bidual X^{**}.

1. Introduction and Preliminaries

In the late seventies, Aron and Berner [3] showed that a continuous extension of a bounded homogeneous polynomial from a subspace of a Banach space to the entire space may not always exist. However they also showed that such extension always exists from a Banach space X to its bidual X^{**}. More than ten years later, Davie and Gamelin [8] proved that this canonical extension constructed in [3] is norm preserving, i.e. it is a Hahn-Banach extension. Very recently, Aron, Boyd and Choi [4] have studied the question when the extension of n-homogeneous polynomials from c_0 to its bidual ℓ_∞ is unique. In the case of 1-homogeneous polynomials, which in fact are linear functionals, it is clear that a Hahn-Banach extension from c_0 to ℓ_∞ is unique, since c_0 is an M-ideal in ℓ_∞. They showed however that it is no longer true for $n \geq 2$ in real spaces as well as for $n \geq 3$ in complex spaces. They also showed that any norm-attaining 2-homogeneous polynomial on a complex c_0 has a unique Hahn-Banach extension to ℓ_∞. Later on similar results were obtained by Choi, Han and Song in [6] for some Marcinkiewicz spaces.

In this article we study analogous problems in more general spaces. We employ and develop some ideas from papers [4, 6], particularly in sections 2 and 3. We start in section 2, by showing that lack of uniqueness of norm-preserving extensions of n-homogeneous polynomial is a very common feature. In fact the uniqueness does not occur for $n \geq 2$ for n-homogeneous polynomials in real Banach spaces, neither for $n \geq 3$ in a large class of complex Banach spaces, including symmetric spaces. The remaining part of the paper is devoted to investigation of the uniqueness of the extension of 2-homogeneous norm-attaining polynomials from complex symmetric sequence spaces X to their biduals X^{**}. We observe among other things that the lack of complex extreme points in a unit ball of X is a crucial property in order to obtain a unique extension of a 2-homogeneous polynomial to X^{**}. In fact we prove in section 3, that if the unit ball of X satisfies a certain geometric condition ((3.1) in Theorem 3.2), which yields in particular that the ball does not contain

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any complex extreme point, then a 2-homogeneous norm-attaining polynomial depends only on finite coordinates, and thus has a unique Hahn-Banach extension to X^{**}. In the case of Marcinkiewicz spaces m^0_Ψ and its bidual m_Ψ, called also weak Lorentz spaces, we can say more. The main result of section 4, Theorem 4.4, states that a 2-homogeneous norm-attaining polynomial on m^0_Ψ has a unique extension to m_Ψ if and only if the unit ball of m_Ψ (or m^0_Ψ) has no complex extreme points, which in turn is equivalent to a simple condition that the sequence $\{\Psi(n)\}$ is strictly increasing. In the proof we apply a strong version of the Maximum Modulus Theorem [17]. Finally, in section 5, we apply these results to r.i. complex symmetric sequence spaces X and we obtain some necessary conditions for uniqueness of extension of 2-homogeneous polynomials from X to X^{**}. This application is based on the well known fact that a symmetric sequence space X is embedded into a Marcinkiewicz space m_Ψ, where $\Psi(n) = n/\Phi(n)$ and Φ is a fundamental function of X.

Marcinkiewicz sequence spaces have appeared earlier in a similar context. In [10], Gowers showed that the space of all norm-attaining bounded operators $NA(m^0_\Psi, \ell_p)$ from m^0_Ψ to ℓ_p, $1 < p < \infty$, is not dense in the space of all bounded operators $L(m^0_\Psi, \ell_p)$, where $\Psi(n) = \sum_{i=1}^n i^{-1}$. Later on in [1], the same Marcinkiewicz space was used for showing that the Bishop-Phelps theorem does not hold for multilinear mappings. This result was recently improved in [7].

Let further X be a Banach space over a scalar field $\mathbb{F}$, where $\mathbb{F}$ is either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. By B_X and S_X we will denote a unit ball and a unit sphere of X, respectively. A bounded multi-linear form is an n-linear mapping $L : X^n \to \mathbb{F}$ for $n \in \mathbb{N}$, with a finite norm $\|L\|$, which is defined as

$$\|L\| = \sup\{|L(x_1, \cdots, x_n)| : x_i \in B_X, i = 1, \cdots, n\}.$$

Then a map $P(x) = L(x, \cdots, x) : X \to \mathbb{F}$ is called an *n-homogeneous polynomial* [2, 9] on X and its norm is defined by

$$\|P\| = \sup\{|P(x)| : x \in B_X\}.$$

Given a Banach space X, if $x \in X$ and $x^* \in X^*$ then $\langle x^*, x\rangle$ denotes $x^*(x)$. We also denote by $[x_1, \dots, x_n]$ a linear span of vectors $\{x_i\}_{i=1}^n \subset X$. For each subset M of X, let $M^\perp$ be the set of all bounded linear functionals which vanish on M. A point x of B_X in a complex Banach space X is said to be a *complex extreme point* whenever $\{x + \zeta y : |\zeta| \le 1, \zeta \in \mathbb{C}\} \subset B_X$ for y in X yields $y = 0$. It is easy to check that every extreme point of B_X is also its complex extreme point. The converse however is not true, since every point of S_{ℓ_1} is a complex extreme point of B_{ℓ_1} ([17]).

Let $(\Omega, \mu) = (\Omega, \mathcal{B}, \mu)$ be a measure space with a complete σ-finite measure μ on a σ-algebra $\mathcal{B}$ of subsets of Ω. Let $L^0(\mu)$ denote the space of all μ-equivalence classes of $\mathcal{B}$-measurable $\mathbb{F}$-valued functions on Ω with the topology of convergence in measure on μ-finite sets.

A Banach space $(X, \|\cdot\|)$ is said to be a *Banach function space* on (Ω, μ) if it is a subspace of $L^0(\mu)$ such that there is $h \in L^0(\mu)$ with $h > 0$ a.e. in Ω and it has the *ideal property*; that is if $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ a.e. then $f \in X$ and $\|f\| \le \|g\|$. If in addition the unit ball B_X is closed in $L^0(\mu)$, then we say that X has the *Fatou property*. A Banach function space defined on $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with the counting measure μ is called a *Banach sequence space*. In this case $e_i \in X$ for all $i \in \mathbb{N}$, where e_i denotes a standard unit vector, that is $e_i = (0, \dots, 0, 1, 0, \dots)$ with 1 as the ith component.

An element $f \in X$ is said to be *order continuous* if $\|f_n\| \downarrow 0$ for every sequence $\{f_n\}$ with $|f_n| \le |f|$ a.e. and $|f_n| \downarrow 0$ a.e. on Ω. A Banach function space X is said to be *order continuous* if every element of X is order continuous.

If X is a Banach function space on (Ω, μ), then the *associate space* X' of X is a Banach function space, which can be identified with the space of all functionals possessing an integral representation, that is,

$$X' = \{g \in L^0(\mu) : \|g\|_{X'} = \sup_{\|f\|\le 1} \int_\Omega |fg| d\mu < \infty\}.$$

It is well known that if X has the Fatou property, then $(X'', \|\cdot\|_{X''})$ coincides with $(X, \|\cdot\|)$. Moreover, if X is an order continuous Banach function space, then X^* is order isometric to X' ([5, 13, 15]).

A Banach function space X on (Ω, μ) is said to be *rearrangement invariant* (*r.i. or symmetric*) [5, 13, 14] if for every $f \in L^0(\mu)$ and $g \in X$ with $\mu_f = \mu_g$, we have $f \in X$ and $\|f\| = \|g\|$, where for any $h \in L^0(\mu)$, μ_h is a *distribution function* of h defined by

$$\mu_h(t) = \mu\{\omega \in \Omega : |h(\omega)| > t\}, \quad t \ge 0.$$

A *decreasing rearrangement* f^* of $f \in L^0(\mu)$ is then defined as

$$f^*(t) = \inf\{\theta > 0 : \mu_f(\theta) \le t\}, \quad t \in [0, \mu(\Omega)).$$

If $x = \{x(n)\} = \{x(n)\}_{n=1}^\infty$ is an $\mathbb{F}$-valued sequence, then considering the function $f(t) = \sum_{k=1}^\infty x(k)\chi_{[k-1,k)}(t)$ on $[0, \infty)$ equipped with Lebesgue measure, we define a *decreasing rearrangement* $x^* = \{x^*(n)\}$ *of* x as follows

$$x^*(n) = f^*(n-1), \quad n \in \mathbb{N}.$$

A closed subspace Y of a Banach space X is called an *M-ideal* in X if there is a bounded projection $\mathcal{P} : X^* \to X^*$ with range $Y^\perp$ such that for each $x^* \in X^*$,

$$\|x^*\| = \|\mathcal{P}x^*\| + \|(I - \mathcal{P})x^*\|.$$

We can write this decomposition as $X^* = Y^\perp \oplus_1 Y^*$.

2. Extensions of n-homogeneous polynomials

If X is a Banach space and Y is a closed M-ideal in X, then it is well known that a bounded linear functional on Y has a unique norm preserving extension to X [11]. With polynomials the situation is different. It depends on whether the space is real or complex. In [4] (see also [6] for some Marcinkiewicz sequence spaces), it has been shown that extension of n-homogeneous polynomials from c_0 to ℓ_∞ is not unique for $n \ge 2$ for real spaces and for $n \ge 3$ for complex spaces. It was also shown that in complex spaces and $n = 2$ some polynomials have unique extensions. Here we start by showing that the uniqueness of the extension of n-homogeneous polynomials, $n \ge 2$, never occurs in any real Banach spaces.

Theorem 2.1. *Let X be a real Banach space and Y a nontrivial proper closed subspace of X. Then for $n \ge 2$ there exists a norm-attaining n-homogeneous polynomial on Y which has infinitely many norm-preserving extensions to X.*

Proof. Let φ be a norm-one linear functional on X which vanishes on Y. Choose a norm-attaining linear functional ψ on Y with norm one and denote by $\widetilde{\psi}$ a Hahn-Banach extension of ψ to X. Then $P = \psi^n$ is a norm-attaining n-homogeneous polynomial on Y with norm one. Take $P_1 = \widetilde{\psi}^n$. Then, for every $0 < t < 1$, $P_t = \widetilde{\psi}^n - t^2\widetilde{\psi}^{n-2}\varphi^2$ are different norm preserving extensions of P on X since the completeness of X implies that $\ker \widetilde{\psi} \cup \ker \varphi \subsetneq X$. $\square$

In the next theorem we prove the lack of uniqueness of the extension of n-homogeneous polynomials for $n \ge 3$ in a large class of complex function spaces.

Theorem 2.2. *Let X be a complex Banach function space such that there exist two disjoint sets E_i, $i = 1, 2$, such that the projection*

$$\Phi f = \Big(\frac{1}{\mu E_1}\int_{E_1} f\Big)\chi_{E_1} + \Big(\frac{1}{\mu E_2}\int_{E_2} f\Big)\chi_{E_2}, \qquad f \in X,$$

is a contractive operator on X. Moreover, assume that Y is a proper closed subspace of X with $\chi_{E_i} \in Y$, $i = 1, 2$.

Then for $n \geq 3$ there exists a norm-attaining n-homogeneous polynomial P on Y which has at least two norm-preserving extensions to X.

Proof. Letting

$$\varphi_i(f) = \frac{1}{\mu E_i}\int_{E_i} f, \qquad f \in X, \quad i = 1, 2,$$

the operator

$$\Phi f = \varphi_1(f)\chi_{E_1} + \varphi_2(f)\chi_{E_2}$$

is a norm-one projection on X. Consider now the set

$$S = \{(z_1, z_2) \in \mathbb{C}^2 : \|z_1\chi_{E_1} + z_2\chi_{E_2}\| \leq 1\},$$

and the function

$$\psi(z_1, z_2) = |z_1|^2 + |z_2|^2, \qquad (z_1, z_2) \in S.$$

It is clear that ψ is continuous on the compact set S. Thus there exists $(u_1, u_2) \in S$ such that

$$\psi(u_1, u_2) = \max_{(z_1,z_2)\in S} \psi(z_1, z_2) = |u_1|^2 + |u_2|^2 = a^2 + b^2,$$

where $a = |u_1|$, $b = |u_2|$, $a^2 + b^2 \neq 0$, and $(a, b) \in S$.

In order to finish the proof we need the following lemma.

Lemma 2.3. *There exists $(a, b) \in S$ such that for $n \geq 2$ and for all $(z_1, z_2) \in S$,*

$$|az_1 + bz_2|^n + |bz_1 - az_2|^n \leq (a^2 + b^2)^n.$$

In particular for $n \geq 2$ and $f \in B_X$,

$$|a\varphi_1(f) + b\varphi_2(f)|^n + |b\varphi_1(f) - a\varphi_2(f)|^n \leq (a^2 + b^2)^n,$$

and so

$$|a\varphi_1(f) + b\varphi_2(f)| \leq a^2 + b^2 \quad \text{and} \quad |b\varphi_1(f) - a\varphi_2(f)| \leq a^2 + b^2.$$

Proof of lemma. For $n = 2$ and any $(z_1, z_2) \in S$ we have

$$\begin{aligned} |az_1 + bz_2|^2 + |bz_1 - az_2|^2 &= (az_1 + bz_2)(a\overline{z}_1 + b\overline{z}_2) + (bz_1 - az_2)(b\overline{z}_1 - a\overline{z}_2) \\ &= (a^2 + b^2)(|z_1|^2 + |z_2|^2) \leq (a^2 + b^2)^2. \end{aligned}$$

Hence $|az_1 + bz_2| \leq a^2 + b^2$ and $|bz_1 - az_2| \leq a^2 + b^2$ on S.

For $n > 2$ we apply induction. Assuming that the inequality is true for $n - 1 \geq 2$, we get for any $(z_1, z_2) \in S$,

$$|az_1 + bz_2|^n + |bz_1 - az_2|^n \leq (a^2 + b^2)\{|az_1 + bz_2|^{n-1} + |bz_1 - az_2|^{n-1}\} \leq (a^2 + b^2)^n.$$

Now, since Φ is a contraction, $\|\varphi_1(f)\chi_{E_1} + \varphi_2(f)\chi_{E_2}\| = \|\Phi f\| \leq 1$ for any $f \in B_X$. Thus $(\varphi_1(f), \varphi_2(f)) \in S$ and this completes the proof of the lemma. □

Given $n \geq 3$, define a polynomial P on Y as

$$P(f) = (a\varphi_1(f) + b\varphi_2(f))^n.$$

It is clear that P is an n-homogeneous polynomial on Y with $\|P\| = (a^2+b^2)^n$. In fact, it follows from Lemma 2.3, since we have $|P(f)| \leq (a^2+b^2)^n$ for $f \in B_X$, and also $P(a\chi_{E_1} + b\chi_{E_2}) = (a^2+b^2)^n$. Then the following polynomials

$$\begin{aligned} P_1(f) &= (a\varphi_1(f) + b\varphi_2(f))^n, \\ P_2(f) &= (a\varphi_1(f) + b\varphi_2(f))^n + (a^2+b^2)(b\varphi_1(f) - a\varphi_2(f))^{n-1}\varphi(f), \end{aligned}$$

are two distinct norm preserving extensions of P from Y to X, where $\varphi \in B_{X^*}$ is chosen in such a way that it vanishes on Y and $(b\varphi_1(f) - a\varphi_2(f))\varphi(f) \neq 0$ for some $f \in X$. In view of Lemma 2.3, it is clear that $\|P_1\| = (a^2+b^2)^n$. Moreover, again applying Lemma 2.3, we get for every $f \in B_X$,

$$\begin{aligned} |P_2(f)| &\leq |a\varphi_1(f) + b\varphi_2(f)|^n + |a^2+b^2||b\varphi_1(f) - a\varphi_2(f)|^{n-1} \\ &\leq (|a\varphi_1(f) + b\varphi_2(f)|^{n-1} + |b\varphi_1(f) - a\varphi_2(f)|^{n-1})(a^2+b^2) \leq (a^2+b^2)^n, \end{aligned}$$

since $n \geq 3$. Since we also have $P_2(a\chi_{E_1} + b\chi_{E_2}) = (a^2+b^2)^n$, it follows that $\|P_2\| = (a^2+b^2)^n$ and the proof is completed. □

If X is a r.i. space with the Fatou property over non-atomic or counting measure space then for any disjoint sets E_i, $i = 1, 2$, the projection Φ on X has norm one [5]. It is also clear by the lattice properties, that for a Banach sequence space X, for any distinct $i, j \in \mathbb{N}$, the projection $\Phi(x) = x(i)e_i + x(j)e_j$ on X also has norm one. Thus the following corollaries are immediate consequences of the previous result.

Corollary 2.4. *If X is a r.i. space with the Fatou property over non-atomic or counting measure space, then the conclusion of Theorem 2.2 is valid in X for any proper closed subspace Y in X with $\chi_{E_i} \in Y$, $i = 1, 2$.*

Corollary 2.5. *For any Banach sequence space X the conclusion of Theorem 2.2 is valid in X for any proper closed subspace Y with $e_i, e_j \in Y$.*

Example 2.6. In this example we show that there is a non-symmetric function space with a norm one projection Φ as in Theorem 2.2. Suppose that $p : \Omega \to [1, \infty)$ is a measurable function on a non-atomic σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ and define the following functional for each $f \in L^0$,

$$I(f) = \int_\Omega \frac{|f(t)|^{p(t)}}{p(t)}\, d\mu.$$

Then the Nakano space $L^{p(t)}$ is defined as the set of all $f \in L^0(\mu)$ such that $I(\lambda f) < \infty$ for some $\lambda > 0$. It is well known [16] that $L^{p(t)}$ is a Banach space equipped with the norm

$$\|f\| = \inf\{\lambda > 0 : I(f/\lambda) \leq 1\}.$$

Suppose now that $p(t)$ assumes constant values $a_i \geq 1$ on disjoint measurable sets E_i, $i = 1, 2$, respectively, with $0 < \mu E_1 = \mu E_2 < \infty$. Then the projection

$$\Phi f = \left(\frac{1}{\mu E_1}\int_{E_1} f\right)\chi_{E_1} + \left(\frac{1}{\mu E_2}\int_{E_2} f\right)\chi_{E_2}, \qquad f \in L^{p(t)},$$

is a contraction. Indeed, note that for any $\lambda > 0$,

$$\begin{aligned} I(\lambda \Phi f) &= \int_\Omega \frac{|\lambda \Phi f|^{p(t)}}{p(t)}\, d\mu \\ &\le \int_\Omega \left\{ \left(\frac{1}{\mu E_1} \int |\lambda f| \right)^{a_1} \frac{\chi_{E_1}}{a_1} + \left(\frac{1}{\mu E_2} \int |\lambda f| \right)^{a_2} \frac{\chi_{E_2}}{a_2} \right\} d\mu \\ &\le \int_{E_1} \frac{|\lambda f|^{a_1}}{a_1}\, d\mu + \int_{E_2} \frac{|\lambda f|^{a_2}}{a_2}\, d\mu \\ &\le \int_\Omega \frac{|\lambda f(t)|^{p(t)}}{p(t)}\, d\mu = I(\lambda f). \end{aligned}$$

This inequality yields that $\|\Phi f\| \le \|f\|$ for all $f \in L^{p(t)}$. Moreover, $\|\chi_{E_i}\| = \left(\frac{\mu E_i}{a_i}\right)^{\frac{1}{a_i}}$, $i = 1, 2$. So if we further assume that $\left(\frac{\mu E_1}{a_1}\right)^{\frac{1}{a_1}} \neq \left(\frac{\mu E_2}{a_2}\right)^{\frac{1}{a_2}}$, then the norms of χ_{E_i}, $i = 1, 2$, are different although they have the same distribution. Therefore we obtain a non-symmetric space $L^{p(t)}$ with a norm one projection Φ.

3. 2-homogeneous polynomials in r.i. sequence spaces

In view of the results of the previous section, our attention turns to 2-homogeneous polynomials on complex spaces. Let in this section X be a r.i. Banach sequence space. We will prove that under certain geometric assumption on the unit ball in r.i. Banach sequence space X, any 2-homogeneous norm-attaining polynomial on X has its unique extension to its bidual X^{**}. Before we state the main theorem we need some preliminary work.

An n-homogeneous polynomial P on X^{**} is said to be *finite* if there exists $m \in \mathbb{N}$ such that

$$P(x^{**}) = P\Big(\sum_{i=1}^m \langle x^{**}, e_i^* \rangle\, e_i \Big)$$

for all $x^{**} \in X^{**}$, where e_k^* are bounded linear functionals on X with $\langle e_k^*, x \rangle = x(k)$. By symmetry of X, each permutation σ of $\mathbb{N}$ induces an isometric isomorphism $T_\sigma : X \to X$ such that $T_\sigma x = (x(\sigma(1)), \cdots, x(\sigma(n)), \cdots)$ for every $x \in X$. Then $T_\sigma^{**} : X^{**} \to X^{**}$ is also an isometric isomorphism. Notice that the above definition of a finite polynomial is more general than the one used before (e.g. [4, 6]), since X^{**} itself does not need to be a sequence space.

Proposition 3.1. *Let P be an n-homogeneous polynomial on X^{**}. Then the following statements are equivalent:*

(1) *P is finite*
(2) *$P \circ T_\sigma^{**}$ is finite for every permutation σ.*
(3) *$P \circ T_\sigma^{**}$ is finite for some permutation σ.*

Proof. Suppose that P is a finite n-homogeneous polynomial and σ any fixed permutation of $\mathbb{N}$. Then clearly $P\mathcal{R}x^{**} = Px^{**}$, where

$$\mathcal{R}x^{**} = \sum_{j=1}^m \left\langle x^{**}, e_j^* \right\rangle e_j.$$

Let $Q = P \circ T_\sigma^{**}$. Note that for every $k \in \mathbb{N}$, $\langle T_\sigma^* e_k^*, x \rangle = \langle e_k^*, T_\sigma x \rangle = x(\sigma(k))$, and so $T_\sigma^* e_k^* = e_{\sigma(k)}^*$. Therefore

$$Q(x^{**}) = P(T_\sigma^{**}x^{**}) = P(\mathcal{R}T_\sigma^{**}x^{**}) = P\Big(\sum_{i=1}^{m} \langle T_\sigma^{**}x^{**}, e_i^*\rangle\, e_i\Big)$$
$$= P\Big(\sum_{i=1}^{m} \langle x^{**}, T_\sigma^{*}e_i^*\rangle\, e_i\Big) = P\Big(\sum_{i=1}^{m} \Big\langle x^{**}, e_{\sigma(i)}^*\Big\rangle\, e_i\Big).$$

Letting $s = \max\{\sigma(i) : i = 1, \cdots, m\}$, define

$$\mathcal{R}_s x^{**} = \sum_{j=1}^{s} \langle x^{**}, e_j^*\rangle\, e_j.$$

Clearly $s \geq m$ and in view of the above equations

$$Q(\mathcal{R}_s x^{**}) = P\Big(\sum_{i=1}^{m} \Big\langle \mathcal{R}_s x^{**}, e_{\sigma(i)}^*\Big\rangle\, e_i\Big) = P\Big(\sum_{i=1}^{m}\sum_{j=1}^{s} \langle x^{**}, e_j^*\rangle \Big\langle e_j, e_{\sigma(i)}^*\Big\rangle\, e_i\Big)$$
$$= P\Big(\sum_{i=1}^{m} \Big\langle x^{**}, e_{\sigma(i)}^*\Big\rangle\, e_i\Big) = Q(x^{**}).$$

Hence $Q = P \circ T_\sigma^{**}$ is finite for any permutation σ. Thus we showed that (1) implies (2). The implication (2) $\Rightarrow$ (3) is clear. Since $P = P \circ T_\sigma^{**} \circ T_{\sigma^{-1}}^{**}$, it is also clear that (3) $\Rightarrow$ (1) holds in view of (1) $\Rightarrow$ (2). ☐

Now we are ready to state the main result of this section.

Theorem 3.2. *Let X be a complex r.i. Banach sequence space. Suppose that for each $x \in B_X$ there exist $n \in \mathbb{N}$ and $\epsilon > 0$ such that $X^{**} = [e_1, \cdots, e_n] \oplus G$ and*

$$x + \epsilon B_G \subset B_{X^{**}}. \tag{3.1}$$

Then a 2-homogeneous polynomial P on X^{**} is norm-attaining on X, that is $P(x_0) = \|P\|$ for some $x_0 \in B_X$, if and only if P is finite.

Proof. Suppose P is finite. Then the values of P are completely determined by the elements of a finite dimensional subspace of X spanned by $\{e_1, \cdots, e_n\}$ for some $n \in \mathbb{N}$. But it clearly shows that P is norm-attaining on X.

Conversely, suppose that $P(x_0) = \|P\| = 1$ for some $x_0 \in B_X$. Let now $n \in \mathbb{N}$ and $\epsilon > 0$ be such that

$$x_0 + \epsilon B_G \subset B_{X^{**}}.$$

Then define on X^{**}

$$\mathcal{R}_n x^{**} = \sum_{i=1}^{n} \langle x^{**}, e_i^*\rangle\, e_i \quad \text{and} \quad \mathcal{S}_n = I - \mathcal{R}_n.$$

Hence

$$(\mathcal{R}_n|_X)^{**} = \mathcal{R}_n,\ (\mathcal{S}_n|_X)^{**} = \mathcal{S}_n,$$

and since both $\mathcal{R}_n|_X$ and $\mathcal{S}_n|_X$ are contractions by the monotonicity of the norm in X, so $\|\mathcal{R}_n\| = \|\mathcal{S}_n\| = 1$. Thus

$$|P(x_0 + \lambda\mathcal{S}_n x^{**})| = |1 + 2\lambda\check{P}(x_0, \mathcal{S}_n x^{**}) + \lambda^2 P(\mathcal{S}_n x^{**})| \leq |P(x_0)| = 1,$$

for all $x^{**} \in B_{X^{**}}$, and for all $|\lambda| < \epsilon$, where $\check{P}$ is the unique symmetric bilinear form associated to P ([2, 9]). By the Maximum Modulus Theorem,

$$\check{P}(x_0, \mathcal{S}_n x^{**}) = P(\mathcal{S}_n x^{**}) = 0 \text{ for } x^{**} \in B_{X^{**}}.$$

Taking $y_0 = (0, \cdots, 0, x_0(n+1), x_0(n+2), \cdots)$ we have $y_0 \in B_X$ and $\mathcal{S}_n(y_0) = y_0$. Hence $P(y_0) = \check{P}(x_0, y_0) = 0$, and so

$$P(x_0(1), \cdots, x_0(n), 0, \cdots) = P(x_0 - y_0) = P(x_0) + P(y_0) - 2\check{P}(x_0, y_0) = 1.$$

Letting $J(x) = \{i : x(i) \neq 0\}$, $x \in X$, denote

$$N = \min\{|J(x)| : P(x) = 1,\ x \in B_X\},$$

where $|J(x)|$ denotes cardinality of $J(x)$. It is clear that $N \leq n$. Suppose now that N is attained at some $x_1 \in B_X$ satisfying $P(x_1) = 1$. Then

$$|J(x_1)| = |\{i : x_1(i) \neq 0\}|.$$

Choose then a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sigma(\{1, \cdots, N\}) = J(x_1), \quad \sigma(\{N+1, \cdots\}) = \mathbb{N} \setminus J(x_1).$$

Let further

$$v = T_\sigma(x_1) = (x_1(\sigma(1)), \cdots, x_1(\sigma(N)), 0, \cdots)$$

and let

$$Q = P \circ T^{**}_{\sigma^{-1}}.$$

In view of Proposition 3.1 we need only to show that Q is finite.

It is clear that $Q(v) = 1$, $v \in B_X$ and $v(k) \neq 0$ for all $1 \leq k \leq N$. Thus by the assumption (3.1), there exist $m \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\begin{aligned} |Q(v + \lambda \mathcal{S}_m x^{**})| &= |Q(v) + 2\lambda \check{Q}(v, \mathcal{S}_m x^{**}) + \lambda^2 Q(\mathcal{S}_m x^{**})| \\ &\leq |Q(v)| = 1, \end{aligned} \tag{3.2}$$

for all $x^{**} \in B_{X^{**}}$ and for all $|\lambda| < \epsilon$. Again by the Maximum Modulus Theorem we have $\check{Q}(v, \mathcal{S}_m x^{**}) = Q(\mathcal{S}_m x^{**}) = 0$ for all $x^{**} \in B_{X^{**}}$. If $m < N$, then applying a similar argument as above we could show that $Q(v_0) = 1$ where $v_0 = (v(1), \ldots, v(m), 0, \ldots)$. The latter however is a contradiction to the choice of N since

$$1 = Q(v_0) = P \circ T^{**}_{\sigma^{-1}}(v_0) = P\Big(\sum_{i \in M_0} x_1(i) e_i\Big)$$

for some $M_0 \subset \mathbb{N}$ with $|M_0| < N$. So $m \geq N$. If $m > N$, then for every $x \in B_X$, $|\lambda| < \epsilon$,

$$\|v + \lambda \mathcal{S}_m x\| \leq 1.$$

Since X is a r.i. Banach sequence space, for all $x \in B_X$,

$$\|v + \lambda \mathcal{S}_N x\| \leq 1.$$

Note that $\mathcal{S}_N$ is weak*-to-weak* continuous. So weak*-lower semi-continuity of norm and density of B_X in $B_{X^{**}}$ in the weak* topology imply for all $x^{**} \in B_{X^{**}}$,

$$\|v + \lambda S_N x^{**}\| \leq 1. \tag{3.3}$$

So (3.2) holds for $m = N$. Therefore we may assume that $m = N$.

Now let $z_1 = (v(1), \cdots, v(m))$, $z_2 = (v(1), v(2) - mv(2), \cdots, v(m))$, $\cdots$, $z_m = (v(1), \cdots, v(m) - mv(m))$ be vectors in $\mathbb{C}^m$. Letting $\tilde{z}_j = (z_j, 0, \cdots)$ for $1 \leq j \leq m$ we have $\tilde{z}_1 = v$.

For any vectors $x = (x(1), \ldots, x(m)) \in \mathbb{C}^m$ we have the identity

$$\begin{aligned} (x(1), \cdots, x(m)) &= \frac{1}{m}\frac{x(1)}{v(1)}(z_1 + \cdots + z_m) + \sum_{j=2}^{m} \frac{1}{m}\frac{x(j)}{v(j)}(z_1 - z_j) \\ &= \frac{1}{m}\Big(\frac{x(1)}{v(1)} + \cdots + \frac{x(m)}{v(m)}\Big) z_1 + \frac{1}{m}\sum_{j=2}^{m}\Big(\frac{x(1)}{v(1)} - \frac{x(j)}{v(j)}\Big) z_j. \end{aligned}$$

Therefore for $x = (x(1), \cdots, x(m), 0\cdots)$ and each $x^{**} \in B_{X^{**}}$,

$$\begin{aligned} Q(x + \mathcal{S}_m x^{**}) &= Q(x) + \frac{2}{m}\sum_{j=2}^{m}\Big(\frac{x(1)}{v(1)} - \frac{x(j)}{v(j)}\Big)\breve{Q}(\tilde{z}_j, \mathcal{S}_m x^{**}) \\ &= Q(x) + \sum_{j=2}^{m}\Big(\frac{x(1)}{v(1)} - \frac{x(j)}{v(j)}\Big)\psi_j(\mathcal{S}_m x^{**}), \end{aligned}$$

where $\psi_j(\cdot) = \frac{2}{m}\breve{Q}(\tilde{z}_j, \cdot) \in X^{***}$.

For each $x^{**} \in B_{X^{**}}$ we will show that $\psi_j(\mathcal{S}_m x^{**}) = 0$. Let

$$v_\theta = (v(1), e^{i\theta}v(2), \cdots, v(m), 0, \cdots), \qquad \theta > 0.$$

Then for every $\|x^{**}\| \le 1$, $|\lambda| < \epsilon$ and any $\alpha > 0$, a similar argument as before (compare with (3.3)) shows

$$\big\|v_\theta + \lambda e^{i\alpha}\mathcal{S}_m x^{**}\big\| \le 1.$$

Thus, for each $\theta > 0$ there is a $\theta_1 > 0$ such that

$$\begin{aligned} |Q(v_\theta + \lambda e^{i\theta_1}\mathcal{S}_m x^{**})| &= |Q(v_\theta) + (1 - e^{i\theta})\psi_2(\lambda e^{i\theta_1}\mathcal{S}_m x^{**})| \\ &= |Q(v_\theta)| + |1 - e^{i\theta}||\psi_2(\lambda\mathcal{S}_m x^{**})| \\ &\le 1, \end{aligned}$$

Let now $f(\theta) = |Q(v_\theta)|$ and let $g(\theta) = |1 - e^{i\theta}| = 2\sin(\theta/2)$ for small $\theta > 0$. Then $|\psi_2(\lambda\mathcal{S}_m x^{**})| \le \frac{1-f(\theta)}{g(\theta)}$ for any $\lambda < \epsilon$. Therefore

$$\sup\{|\psi_2(\mathcal{S}_m x^{**})| : x^{**} \in \epsilon B_{X^{**}}\} \le \lim_{\theta\downarrow 0}\frac{1 - f(\theta)}{g(\theta)} = \lim_{\theta\downarrow 0}\frac{-f'(\theta)}{g'(\theta)} = 0.$$

This implies that for $x^{**} \in B_{X^{**}}$, $\psi_2(\mathcal{S}_m x^{**}) = 0$. Similar calculations show that $\psi_i(\mathcal{S}_m x^{**}) = 0$ for $i = 3, \ldots, m$. Thus for every $x = (x(1), \ldots, x(m), 0, \ldots)$ and every $\|x^{**}\| \le 1$ we get

$$Q(x + \mathcal{S}_m x^{**}) = Q(x).$$

Taking now $x = \mathcal{R}_m x^{**}$,

$$Q(x^{**}) = Q(\mathcal{R}_m x^{**} + \mathcal{S}_m x^{**}) = Q(\mathcal{R}_m x^{**}),$$

which shows that Q is finite and completes the proof. □

Remark 3.3. Observe that the geometric assumption (3.1) on X^{**} in the above theorem yields that no point of S_X is a complex extreme point of B_X. We will see later that in Marcinkiewicz sequence spaces the converse is also satisfied.

Notice that Theorem 3.2 does not hold for $n \ge 3$ as we can see in the following example.

Example 3.4. Let $n \ge 3$ and ℓ_∞ be a complex space. Consider the n-homogeneous polynomial P on ℓ_∞ given by the formula

$$P(x) = (x_1 + x_2)^n + 2(x_1 - x_2)^{n-1}\left(\sum_{k=3}^{\infty}\frac{x_k}{2^k}\right).$$

Then $P(e_1 + e_2) = 2^n = \|P\|$. It follows from Lemma 2.3 applied to ℓ_∞ and $E_i = e_i$, $i = 1, 2$. In this case $a = b = 1$. In fact P is of a similar form as P_2 in the proof of Theorem 2.2.

Corollary 3.5. *Suppose that X is a complex r.i. sequence space that satisfies the hypotheses of Theorem 3.2. Then a 2-homogeneous polynomial P on X attains its norm if and only if it is finite.*

Proof. Recall that every n-homogeneous polynomial on a Banach space X has a norm-preserving extension to its bidual X^{**} [8]. Let P attain its norm and Q be a norm-preserving extension of P to X^{**}. Then Q also attains its norm, and by Theorem 3.2, Q is finite. So there is $m \in \mathbb{N}$ such that for every $x \in X$,

$$P(x) = Q(x) = Q\left(\sum_{i=1}^{m} x(i)e_i\right) = P\left(\sum_{i=1}^{m} x(i)e_i\right),$$

which completes the proof. □

Corollary 3.6. *Suppose that X is a complex r.i. sequence space that satisfies the hypotheses of Theorem 3.2. Then every 2-homogeneous norm-attaining polynomial P on X has a unique norm-preserving extension to its bidual X^{**}.*

Proof. Let Q_1 and Q_2 be norm-preserving extensions of P from X to X^{**}. Then, by Theorem 3.2, Q_1 and Q_2 are finite. So there are $m_1, m_2 \in \mathbb{N}$ such that for each $x^{**} \in X^{**}$,

$$Q_1(x^{**}) = Q_1\left(\sum_{i=1}^{m_1} \langle x^{**}, e_i^*\rangle e_i\right) = \sum_{i=1}^{m_1}\sum_{j=1}^{i} a_{ij} \langle x^{**}, e_i^*\rangle \langle x^{**}, e_j^*\rangle,$$

$$Q_2(x^{**}) = Q_2\left(\sum_{s=1}^{m_2} \langle x^{**}, e_s^*\rangle e_s\right) = \sum_{s=1}^{m_2}\sum_{t=1}^{s} b_{st} \langle x^{**}, e_s^*\rangle \langle x^{**}, e_t^*\rangle,$$

for some complex numbers a_{ij}, b_{st}. They are equal on X so that there is $l \leq \min\{m_1, m_2\}$ such that $a_{ij} = b_{ij}$ for all $1 \leq j \leq i \leq l$ and $a_{ij} = 0 = b_{st}$ otherwise. So $Q_1(x^{**}) = Q_2(x^{**})$ for every $x^{**} \in X^{**}$, and the extension is unique. □

It is easy to show that c_0 satisfies the assumptions of Theorem 3.2, and thus we get immediately by Corollaries 3.5 and 3.6, the following result proved in [4].

Corollary 3.7. [4] *Every norm-attaining 2-homogeneous polynomial on a complex c_0 is finite and has a unique norm-preserving extension to ℓ_∞.*

We will see later (cf. Corollary 4.6) that we can obtain a stronger result for some renormings of c_0 and ℓ_∞.

The following simple example shows that the assumption of symmetry of X in Theorem 3.2 is essential.

Example 3.8. Consider the space ℓ_∞ with the equivalent norm

$$\|x\| = |x(1)| + |x(2)| + \sup\{|x(n)| : n \geq 3\}.$$

Clearly $(c_0, \|\cdot\|)^{**} = (\ell_\infty, \|\cdot\|)$. Define on ℓ_∞ the following 2-homogeneous polynomials

$$P(x) = x(1)^2, \quad Q(x) = x(1)^2 + x(2)\sum_{k=3}^{\infty} \frac{x(k)}{2^{k-2}}.$$

It is obvious that P is norm-attaining on c_0 and $\|P\| = 1$. Note also that for each $x \in \ell_\infty$ with $\|x\| \leq 1$ we have

$$Q(e_1) = 1 \quad \text{and} \quad |Q(x)| \leq 1.$$

It shows that Q is norm-attaining on c_0 but it is not finite. Choose now a norm one linear functional φ on ℓ_∞ which vanishes on c_0. Letting

$$P_1(x) = x(1)^2 \quad \text{and} \quad P_2(x) = x(1)^2 + x(2)\varphi(x),$$

we get two distinct norm-preserving extensions of P to ℓ_∞. Thus the conclusions of Theorem 3.2 and Corollary 3.6 are not valid for $(c_0, \|\cdot\|)$ and its bidual $(\ell_\infty, \|\cdot\|)$.

For norm-attaining linear functionals, we obtain analogous results as for 2-homogeneous polynomials.

Proposition 3.9. *Suppose X is a complex r.i. sequence space and X satisfies (3.1). Then a bounded linear functional φ on X attains its norm if and only if it is finite. Moreover, every norm-attaining bounded linear functional on X has a unique norm-preserving extension to X^{**}.*

Proof. If φ is finite, then it is clearly norm-attaining since its values depend only on a finite dimensional subspace of X.

Conversely, suppose that $\varphi(x_0) = \|\varphi\| = 1$ for some $x_0 \in B_X$. Then by the assumption (3.1) there are $n \in \mathbb{N}$ and $\epsilon > 0$ so that for every $|\lambda| < \epsilon$ and for every $y = (0, \cdots, 0, y(n+1), \cdots) \in B_X$,

$$|\varphi(x_0 + \lambda y)| = |\varphi(x_0) + \lambda\varphi(y)| \leq 1.$$

By the Maximum Modulus Theorem $\varphi(y) = 0$ for such a y. So for every $x \in X$, $\varphi(\mathcal{S}_n x) = 0$ and thus

$$\varphi(x) = \varphi(\mathcal{R}_n x) = \sum_{i=1}^{n} x_i \varphi(e_i) = \sum_{i=1}^{n} \langle e_i^*, x\rangle\, \varphi(e_i),$$

which shows that φ is finite. Moreover, it has a natural extension $\widetilde{\varphi}$ to X^{**}, defined by

$$\widetilde{\varphi}(x^{**}) = \sum_{i=1}^{n} \langle x^{**}, e_i^*\rangle\, \varphi(e_i).$$

Now, if φ has a norm-preserving extension ϕ to X^{**}, then similar arguments as above applied to ϕ show that ϕ is also finite. Since $\widetilde{\varphi}$ and ϕ are equal on X, so they must be equal on X^{**} too and the proof is completed. ☐

4. 2-homogeneous polynomials in Marcinkiewicz sequence spaces

In this section we will investigate uniqueness of norm-preserving extensions of 2-homogeneous polynomials in Marcinkiewicz sequence spaces.

Assume $\Psi = \{\Psi(n)\} = \{\Psi(n)\}_{n=0}^{\infty}$ is an increasing sequence such that $\Psi(0) = 0$ and $\Psi(n) > 0$ for $n \in \mathbb{N}$.

Definition 4.1. The *Marcinkiewicz sequence space* m_Ψ consists of all sequences $x = \{x(n)\} = \{x(n)\}_{n=1}^{\infty}$ such that

$$\|x\| = \|x\|_{\mathrm{m}_\Psi} = \sup_{n\geq 1} \frac{\sum_{k=1}^{n} x^*(k)}{\Psi(n)} < \infty,$$

where $x^* = \{x^*(n)\}$ is a decreasing rearrangement of $\{x(n)\}$. Let m_Ψ^0 be the subspace of m_Ψ, equipped with the same norm $\|\cdot\|_{m_\Psi}$ consisting of all $x \in \mathrm{m}_\Psi$ satisfying

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} x^*(k)}{\Psi(n)} = 0.$$

Without loss of generality we can add (and we will) in the above definition the assumption that the sequence $\{\Psi(n)/n\}$ is decreasing [12]. Notice that for a concave Ψ, m_Ψ^0 is a predual of a Lorentz space [12, 13].

Recall the following results on m_Ψ^0 and m_Ψ.

Theorem 4.2. [12]

The following hold true.

1. *If* $\lim_{n\to\infty} \Psi(n) = \infty$ *then m_Ψ^0 is the non-trivial proper subspace of m_Ψ consisting of all order continuous elements in m_Ψ.*

(2) m_Ψ *is the bidual of* m^0_Ψ *if and only if* $\lim_{n\to\infty}\Psi(n)=\infty$.
(3) *Assume that* $\lim_{n\to\infty}\Psi(n)=\infty$. *Then* m^0_Ψ *is an M-ideal in* m_Ψ *if*

$$\Psi(n)=n \quad or \quad \lim_{n\to\infty}\frac{\Psi(n)}{n}=0.$$

If m^0_Ψ is an M-ideal in m_Ψ, in particular when Ψ satisfies the conditions in the above theorem, then for any bounded linear functionals on m^0_Ψ there exists a unique Hahn-Banach extension to m_Ψ. We will see below, Theorem 4.4, that in the case of 2-homogeneous polynomials the crucial role in the extension problem is played by the geometric property (3.1), which is equivalent to the fact that no element of the unit sphere of m^0_Ψ is a complex extreme point of the unit ball of m_Ψ.

We first state the following lemma.

Lemma 4.3. *Assume that* $\lim_{n\to\infty}\Psi(n)=\infty$ *and* Ψ *is strictly increasing. Then for each* $x\in B_{\mathrm{m}^0_\Psi}$, *there exist* $n\in\mathbb{N}$ *and* $\epsilon>0$ *such that for each* $y\in B_{\mathrm{m}_\Psi}$, $y=(0,\cdots,0,y(n+1),y(n+2),\cdots)$, *and for each* $|\lambda|\le\epsilon$, $\|x+\lambda y\|\le 1$ *holds.*

Proof. We may assume that $\|x\|=1$. Since $\lim_{k\to\infty}\frac{\sum_{i=1}^k x^*(i)}{\Psi(k)}=0$, we can find the maximum integer $n_1\in\mathbb{N}$ such that

$$\|x\|=1=\frac{\sum_{i=1}^{n_1}x^*(i)}{\Psi(n_1)}.$$

Thus for every $k\ge n_1+1$,

$$\sum_{i=1}^{n_1}x^*(i)=\Psi(n_1) \ \text{ and } \sum_{i=1}^{k}x^*(i)<\Psi(k).$$

Take

$$a=1-\max\left\{\frac{\sum_{i=1}^k x^*(i)}{\Psi(k)}:k\ge n_1+1\right\}>0.$$

We note that $x^*(n_1)\neq 0$. Indeed, if we suppose that $x^*(n_1)=0$, then

$$\sum_{i=1}^{n_1}x^*(i)=\sum_{i=1}^{n_1-1}x^*(i)=\Psi(n_1)\le\Psi(n_1-1),$$

which is a contradiction to the fact that Ψ is strictly increasing.

Note that for $x\in\mathrm{m}^0_\Psi$,

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n x^*(k)}{\Psi(n)}=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{k=1}^n x^*(k)}{\frac{1}{n}\Psi(n)}=0,$$

which yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n x^*(k)=\lim_{n\to\infty}x^*(n)=\lim_{i\to\infty}|x(i)|=0.$$

Thus we can choose $n>n_1$ so that for all $i\ge n+1$,

$$|x(i)|<\frac{1}{2}x^*(n_1).$$

Take $\epsilon=\min\{\frac{x^*(n_1)\|e_1\|}{2},a\}>0$ and let $y=(0,\cdots,0,y(n+1),y(n+2),\cdots)$ be in B_{m_Ψ}. Fix λ with $|\lambda|<\epsilon$. Then for $i\ge n+1$, $\|e_i\|\,|y(i)|\le 1$, and so

$$|x(i)+\lambda y(i)|<\frac{x^*(n_1)}{2}+\frac{x^*(n_1)|y(i)|\,\|e_1\|}{2}\le x^*(n_1).$$

Thus for each $k \leq n_1$,

$$\sum_{i=1}^{k}(x+\lambda y)^*(i) = \sum_{i=1}^{k} x^*(i) \leq \Psi(k),$$

and for each $k > n_1$,

$$\sum_{i=1}^{k}(x+\lambda y)^*(i) \leq \sum_{i=1}^{k} x^*(i) + a\sum_{i=1}^{k} y^*(i) \leq (1-a)\Psi(k) + a\Psi(k) = \Psi(k).$$

Therefore $\|x+\lambda y\| \leq 1$ and the proof is complete. □

Theorem 4.4. *Let* $\lim_{n\to\infty}\Psi(n) = \infty$ *and* m_Ψ, m_Ψ^0 *be complex spaces. The following conditions are equivalent.*

1. Ψ *is strictly increasing.*
2. *For each* $x \in B_{m_\Psi^0}$, *there are* $n \in \mathbb{N}$ *and* $\epsilon > 0$ *such that for every* $y = (0, \cdots, 0, y(n+1), \cdots) \in B_{m_\Psi}$ *and for every* $|\lambda| < \epsilon$, $\|x+\lambda y\| \leq 1$.
3. *No element in* $S_{m_\Psi^0}$ *is a complex extreme point of* $B_{m_\Psi^0}$.
4. *No element in* $S_{m_\Psi^0}$ *is a complex extreme point of* B_{m_Ψ}.
5. *Every norm-attaining* 2*-homogeneous polynomial on* m_Ψ^0 *is finite.*
6. *Every norm-attaining* 2*-homogeneous polynomial on* m_Ψ^0 *has a unique norm-preserving extension to* m_Ψ.
7. *Every norm-attaining bounded linear functional on* m_Ψ^0 *is finite.*

Proof. In view of Theorem 4.2 and the assumption that $\lim_{n\to\infty}\Psi(n) = \infty$, m_Ψ^0 is a non-trivial and proper subspace of m_Ψ. Moreover, m_Ψ is the bidual of m_Ψ^0. The implications (2) ⇒ (3) ⇒ (4) are clear by definition. In view of Lemma 4.3, (1) implies (2), and by Corollaries 3.5 and 3.6 we have that (2) implies (5) and (6). We also have that (2) yields (7) by Proposition 3.9. We will complete the proof if we show that each condition (4)-(7) is not satisfied whenever (1) is not satisfied.

Suppose for the rest of the proof that (1) is not satisfied, that is Ψ is not strictly increasing. Then there is $n \in \mathbb{N}$ such that $\Psi(n) = \Psi(n+1)$. Set

$$x_0 = \sum_{i=1}^{n} \frac{\Psi(n)}{n}\, e_i.$$

Since $\frac{\Psi(n)}{n} \leq \frac{\Psi(k)}{k}$ for each k, $1 \leq k \leq n$, so

$$\sup_{k\geq 1} \frac{\sum_{i=1}^{k} x_0^*(i)}{\Psi(k)} = \max_{1\leq k\leq n} \frac{k\Psi(n)}{n\Psi(k)} = 1.$$

Thus $x_0 \in S_{\mathrm{m}_\Psi^0}$. We shall show that x_0 is a complex extreme point of B_{m_Ψ}. Take $y \in \mathrm{m}_\Psi$ such that $\|x_0 + \zeta y\| \leq 1$ for all $|\zeta| \leq 1$. Then

$$\frac{1}{\Psi(n)}\sum_{i=1}^{n}\left|\frac{\Psi(n)}{n} + \zeta y(i)\right| \leq \sum_{i=1}^{n}\frac{(x_0+\zeta y)^*(i)}{\Psi(n)} \leq 1, \quad \text{for all} \quad |\zeta| \leq 1.$$

Consider the analytic function $f : B_{\mathbb{C}} \to \ell_1$, defined by

$$f(\zeta) = \frac{1}{\Psi(n)}\sum_{i=1}^{n}\left(\frac{\Psi(n)}{n} + \zeta y(i)\right) e_i.$$

Then $\|f(\zeta)\|_1$ has maximum 1 at $\zeta = 0$. Since S_{ℓ_1} consists entirely of complex extreme points, the strong form of the Maximum Modulus Theorem holds true (cf. Theorem 3.1

in [17]), and thus f is constant. Therefore $y(i) = 0$ for $1 \le i \le n$. For each $y(k)$, $k > n$ and for all $|\zeta| \le 1$,

$$\frac{1}{\Psi(n+1)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| + \frac{|\zeta y(k)|}{\Psi(n+1)} \le \sum_{i=1}^{n+1} \frac{(x_0 + \zeta y)^*(i)}{\Psi(n+1)} \le 1.$$

This implies for all $|\zeta| \le 1$,

$$\begin{aligned} &\frac{1}{\Psi(n+1)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| + \frac{|\zeta y(k)|}{\Psi(n+1)} \\ &\qquad = \frac{1}{\Psi(n)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| + \frac{|\zeta y(k)|}{\Psi(n)} = 1 + \frac{|\zeta y(k)|}{\Psi(n)} \le 1. \end{aligned}$$

So we have that $y(k) = 0$ for any $k > n$. Therefore $y = 0$ and x_0 is a complex extreme point of $B_{\mathrm{m}^0_\Psi}$, that is (4) is not satisfied. Now take the following 2-homogeneous polynomials on m^0_Ψ

$$\begin{aligned} P(x) &= \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2}, \\ Q(x) &= \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2} + \frac{x(n+1)}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x(k+n+1)}{\Psi(1)2^k}. \end{aligned}$$

Observe that $P(x_0) = Q(x_0) = 1$. So P is a norm-attaining 2-homogeneous polynomial. We can see that Q is also norm-attaining. Indeed, for each $\|x\| \le 1$,

$$\begin{aligned} |Q(x)| &\le \left(\frac{|x(1)| + \cdots + |x(n)|}{\Psi(n)} \right)^2 + \frac{|x(n+1)|}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x^*(1)}{2^k \Psi(1)} \\ &\le \frac{|x(1)| + \cdots + |x(n)|}{\Psi(n)} + \frac{|x(n+1)|}{\Psi(n)} \\ &\le \frac{x^*(1) + \cdots + x^*(n+1)}{\Psi(n+1)} \le 1, \end{aligned}$$

in view of the assumption that $\Psi(n) = \Psi(n+1)$. Hence, we get a norm-attaining 2-homogeneous polynomial on m^0_Ψ which is not finite. So (5) $\Rightarrow$ (1) is proved. Choose further a norm one linear functional ϕ on m_Ψ which vanishes on m^0_Ψ. Letting for $x \in \mathrm{m}_\Psi$,

$$P_1(x) = \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2},$$

$$P_2(x) = \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2} + \frac{x(n+1)}{\Psi(n+1)} \phi(x),$$

we can easily see that they are two distinct norm-preserving extensions of P to m_Ψ. This proves (6) $\Rightarrow$ (1). Finally, we will construct a norm-attaining bounded linear functional which is not finite. Define a linear functional φ on m^0_Ψ as follows

$$\varphi(x) = \frac{x(1) + \cdots + x(n)}{\Psi(n)} + \frac{1}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x(n+k)}{2^k}.$$

Then $\varphi(x_0) = 1$, $\|\varphi\| = 1$, and φ is not finite. Indeed, for each $\|x\| \le 1$, by the Hardy-Littlewood inequality [5],

$$\begin{aligned}|\varphi(x)| &\le \frac{x^*(1) + \cdots + x^*(n)}{\Psi(n)} + \frac{1}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x^*(n+k)}{2^k} \\ &\le \frac{x^*(1) + \cdots + x^*(n)}{\Psi(n)} + \frac{1}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x^*(n+1)}{2^k} \\ &\le \frac{x^*(1) + \cdots + x^*(n) + x^*(n+1)}{\Psi(n+1)} \le 1.\end{aligned}$$

This shows that (7) yields (1) and completes the proof.

□

Remark 4.5. (*i*) In view of Proposition 3.9, property (7) above yields that every norm attaining linear functional on m^0_Ψ has a unique norm-preserving extension from m^0_Ψ to m_Ψ. The converse implication however does not need to hold. Indeed, take Ψ such that $\lim_{n\to\infty} \Psi(n) = \infty$, $\lim_{n\to\infty} \Psi(n)/n = 0$ and Ψ is not strictly increasing. Then a norm-attaining functional does not need to be finite (since (1) is equivalent to (7)), but still its norm-preserving extension is unique since m^0_Ψ is an M-ideal in m_Ψ by Theorem 4.2.

(*ii*) For Ψ strictly concave the equivalence of (5) and (6) in Theorem 4.4 has been shown in [6].

As a corollary of the above theorem we obtain that there exists a renorming of c_0 and ℓ_∞ such that the results in [4] (Propositions 2 and 3) still hold true, despite the fact that under this renorming c_0 does not need to be an M-ideal in ℓ_∞ (cf. Example 4.7). In fact, let Ψ be such that $\lim_{n\to\infty} \Psi(n)/n > 0$. Then it is easy to show (cf. [12]) that $m^0_\Psi = c_0$ and $m_\Psi = \ell_\infty$ as sets and the norms $\|\cdot\|_{m_\Psi}$ and $\|\cdot\|_\infty$ are equivalent. Thus we get the following result.

Corollary 4.6. *Let* $\lim_{n\to\infty} \Psi(n)/n > 0$*,* $\lim_{n\to\infty} \Psi(n) = \infty$ *and* Ψ *be strictly increasing. Then for* $k = 1, 2$ *every* k*-homogeneous norm-attaining polynomial on complex* $(c_0, \|\cdot\|_{\mathrm{m}_\Psi})$ *is finite and has a unique extension to its bidual* $(\ell_\infty, \|\cdot\|_{\mathrm{m}_\Psi})$*.*

Example 4.7. Let $\Psi(0) = 0$, $\Psi(n) = \max\{\frac{2n}{3}, 1\}$ for $n \in \mathbb{N}$. Then $m_\Psi = \ell_\infty$ with the norm

$$\|x\|_\Psi = \sup\left\{x^*(1), \frac{3(x^*(1) + x^*(2))}{4}, \dots, \frac{3\sum_{k=1}^n x^*(k)}{2n}, \dots\right\}$$

that is equivalent to $\|\cdot\|_\infty$-norm. Then $(c_0, \|\cdot\|_{\mathrm{m}_\Psi})$ is not an M-ideal of $(\ell_\infty, \|\cdot\|_{\mathrm{m}_\Psi})$, but the conclusion of Corollary 4.6 still holds.

Proof. It is clear that Ψ satisfies the assumptions of Corollary 4.6. In order to show that $(c_0, \|\cdot\|_{\mathrm{m}_\Psi})$ is not an M-ideal of $(\ell_\infty, \|\cdot\|_{\mathrm{m}_\Psi})$, we will use the so called 3-ball property [11], which states that a closed subspace Y is an M-ideal in a Banach space X if and only if for all $y_1, y_2, y_3 \in B_Y$, all $x \in B_X$ and $\epsilon > 0$ there is $y \in Y$ satisfying

$$\|x + y_i - y\| \le 1 + \epsilon \quad \text{for all } i = 1, 2, 3.$$

Let now $x_1 = e_1 + \frac{1}{3}e_2$, $x_2 = e_1 - \frac{1}{3}e_2$, $x_3 = -e_1 + \frac{1}{3}e_2$, and let $x \equiv 2/3$. Note that $\|x_i\| = \|x\| = 1$. Then there is no $y \in c_0$ such that $\|x_i + x - y\|_\Psi < \frac{5}{4}$. Observe the following formulas for any $y \in c_0$,

$$\begin{aligned}|x_1 + x - y| &= (|5/3 - y(1)|, |1 - y(2)|, |2/3 - y(3)|, \dots), \\ |x_2 + x - y| &= (|5/3 - y(1)|, |0 - y(2)|, |2/3 - y(3)|, \dots), \\ |x_3 + x - y| &= (|1/3 + y(1)|, |1 - y(2)|, |2/3 - y(3)|, \dots).\end{aligned}$$

Then $\max\{|5/3 - y(1)|, |1/3 + y(1)|\} \geq 1$ for all scalars $y(1)$. Therefore for each $y \in c_0$ there is i such that $(x_i + x - y)^*(1) \geq 1$ and note that $\lim_{n\to\infty} |2/3 - y(n)| = 2/3$, so that $(x_i + x - y)^*(2) \geq 2/3$ for all $i = 1, 2, 3$. This proves that for every $y \in c_0$ there is some i such that $\|x_i + x - y\|_\Psi \geq 3/4(1 + 2/3) = 5/4$, which shows that 3-ball property is not satisfied. □

We also see that it is not true that every renorming of c_0 and ℓ_∞ guarantees the hypothesis of Corollary 4.6. In fact, in Example 3.8 we constructed a non-symmetric norm $\|\cdot\|$ equivalent to $\|\cdot\|_\infty$ such that the conclusion of Corollary 4.6 failed. However we can ask another question, whether or not, in c_0 equipped with an equivalent symmetric norm, every 2-homogeneous norm-attaining polynomial is finite and has a unique extension to its bidual ℓ_∞? But, as we see below, both answers are negative.

Example 4.8. Let $\Psi(0) = 0$, $\Psi(n) = \max\{n, 2\}$ for $n \in \mathbb{N}$. Then $\mathrm{m}_\Psi = \ell_\infty$ and $\mathrm{m}_\Psi^0 = c_0$ with the norm $\|x\| = \frac{x^*(1)+x^*(2)}{2}$, which is equivalent to $\|\cdot\|_\infty$-norm. Since $\lim_{n\to\infty} \Psi(n) = \infty$ and Ψ is not strictly increasing, Theorem 4.4 shows that there is a norm-attaining polynomial on $\mathrm{m}_\Psi^0 = c_0$ which has at least two different norm preserving extensions to $\mathrm{m}_\Psi = \ell_\infty$. Note also that m_Ψ is a symmetric space not satisfying the condition (3.1) of Theorem 3.2.

Example 4.9. Let $\Psi(0) = 0$, $\Psi(n) = \max\{\sqrt{n}, 2\}$ for $n \in \mathbb{N}$. Then m_Ψ^0 is an M-ideal of its bidual m_Ψ (see Theorem 4.2) with the norm

$$\|x\| = \|x\|_\Psi = \max\left\{ \max_{k\in\{1,2,3,4\}} \frac{\sum_{i=1}^k x^*(i)}{2}, \ \sup_{k\geq 5} \frac{\sum_{i=1}^k x^*(i)}{\sqrt{k}} \right\}.$$

Then Theorem 4.4 can be used to show that there are two distinct norm-preserving extensions of a 2-homogeneous polynomial from m_Ψ^0 to m_Ψ, and also that there exists a norm-attaining polynomial on m_Ψ^0 which is not finite.

So even though m_Ψ^0 is an M-ideal in m_Ψ, we cannot obtain the results similar to Corollaries 3.5 and 3.6, without the assumption (3.1) of Theorem 3.2.

5. Applications to r.i. sequence spaces

Suppose now that X is a complex r.i. sequence space with the Fatou property. We will apply the results of Theorem 4.4 to X. Let Φ be a *fundamental function* of X, that is $\Phi(0) = 0$ and for each $n \in \mathbb{N}$,

$$\Phi(n) = \|e_1 + \cdots + e_n\|_X .$$

It is well known [5] that $\{\Phi(n)/n\}$ is decreasing and the associated space X' is an r.i. space with the fundamental function Ψ satisfying for every $n \in \mathbb{N} \cup \{0\}$,

$$\Phi(n)\Psi(n) = n.$$

Given X with the fundamental function Φ, define the Marcinkiewicz sequence space m_Ψ as the set of all $x = \{x(n)\}$ such that

$$\|x\|_{\mathrm{m}_\Psi} = \sup_{n\in\mathbb{N}} \left\{ \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} \right\} = \sup_{n\in\mathbb{N}} \left\{ \frac{\Phi(n)}{n} \sum_{k=1}^n x^*(k) \right\} < \infty.$$

Then obviously the fundamental function of m_Ψ is Φ. Moreover, it is well known [5] that m_Ψ is the smallest r.i. space 1-embedded in X with the same fundamental function as X. Thus we have

$$\|x\|_{\mathrm{m}_\Psi} \leq \|x\|_X , \quad x \in X.$$

This implies that if $x \in S_X$ is a complex extreme point of B_{m_Ψ}, then x is a complex extreme point of B_X.

In the proof of Theorem 4.4, we showed that if Ψ is not strictly increasing then there is an $n \in \mathbb{N}$ such that

$$x_0 = \sum_{i=1}^{n} \frac{\Psi(n)}{n} e_i$$

is a complex extreme point of B_{m_Ψ}. Note that

$$\|x_0\|_X = \frac{\Psi(n)}{n} \|e_1 + \cdots + e_n\|_X = \frac{\Psi(n)\Phi(n)}{n} = 1.$$

Hence if Ψ is not strictly increasing, then x_0 is a complex extreme point of B_X. Note also that if Ψ is not strictly increasing, then we can take Q and φ as in the proof of Theorem 4.4. Since $\|x\|_{\mathrm{m}_\Psi} \le \|x\|_X$, Q is a 2-homogeneous norm-attaining polynomial on X and φ is a norm-attaining bounded linear functional on X. Moreover, they are not finite. Thus we proved the following proposition.

Proposition 5.1. *Suppose a complex r.i. sequence space X with the Fatou property has a fundamental function Φ such that $\{\Phi(n)/n\}$ is not strictly decreasing. Then B_X has a complex extreme point. Moreover, for $k = 1, 2$ there is a norm-attaining k-homogeneous polynomial on X which is not finite.*

Assume now that X is not reflexive. Then we can choose a norm one linear functional ϕ on X^{**} which vanishes on X. So if Ψ is not strictly increasing then we can use P_1 and P_2 from the proof of Theorem 4.4 as two different norm-preserving extensions of P from X to X^{**}. Hence we get the following result.

Proposition 5.2. *Suppose a complex r.i. sequence space X with the Fatou property has a fundamental function Φ such that $\{\Phi(n)/n\}$ is not strictly decreasing and that X is not reflexive. Then there is a norm-attaining 2-homogeneous polynomial which has at least two norm-preserving extensions from X to X^{**}.*

Corollary 5.3. *Let X be a complex r.i. sequence space with the Fatou property. Assume no point of S_X is a complex extreme point of B_X. Then the fundamental function of its associate space X' is strictly increasing.*

We shall show that the converse of Corollary 5.3 does not hold in general, even though X is an order continuous symmetric sequence space. Before we present an example contradicting the converse of Corollary 5.3 we will need the following simple but useful fact about complex extreme points of a unit ball in a r.i. sequence space.

Proposition 5.4. *Let X be a complex r.i. sequence space. Then an order continuous element $x_0 \in S_X$ is a complex extreme point of B_X if and only if its decreasing rearrangement x_0^* is a complex extreme point of B_X.*

Proof. Observe that if $T : X \to X$ is an isometric isomorphism, then T preserves the complex extreme points of B_X.

Let $x_0 \in S_X$ and x_0 be an order continuous element. Then $\lim_{n\to\infty} x_0^*(n) = 0$. So there is a permutation σ of $\mathbb{N}$ such that $|x_0(\sigma(n))| = x_0^*(n)$ for each $n \in \mathbb{N}$. Let $\lambda_n = \operatorname{sign}(x_0(\sigma(n))$ for $n \in \mathbb{N}$, where for $z \in \mathbb{C}$, $\operatorname{sign} z = \bar{z}/|z|$ if $z \neq 0$ and $\operatorname{sign} z = 1$ if $z = 0$. Define an isometric isomorphism T on X as follows

$$Tx = \{\lambda_n x(\sigma(n))\}, \quad x \in X.$$

Then $Tx_0 = x_0^*$, and so x_0 is a complex extreme point of B_X if and only if x_0^* is a complex extreme point of B_X. ∎

Example 5.5. Let X be the set of all complex sequences $x = \{x(n)\}$ such that

$$\|x\| = \sum_{k=1}^{\infty} (\sqrt{n} - \sqrt{n-1}) x^*(n) < \infty.$$

Since the sequence $\{\sqrt{n} - \sqrt{n-1}\}$ is decreasing, $(X, \|\cdot\|)$ is a Lorentz space and it is order continuous [13, 14]. It is clear that the fundamental functions Φ and Ψ of X and X', respectively, are equal and $\Phi(n) = \sqrt{n} = \Psi(n)$ for all $n \in \mathbb{N}$.

We shall show that every point of S_X is a complex extreme point of B_X. By Proposition 5.4, we have only to show that every point $x^* \in S_X$ is a complex extreme point of B_X. Let $x^* \in S_X$ and $y \in X$ be such that $\|x^* + \zeta y\| \leq 1$ for all $|\zeta| < 1$. Then by the Hardy-Littlewood inequality [5],

$$\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n-1}) \, |x^*(n) + \zeta y(n)| \leq \sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n-1}) (x^* + \zeta y)^*(n) \leq 1.$$

The function $f : B_{\mathbb{C}} \to \ell_1$ defined by

$$f(\zeta) = \sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n-1}) \, (x^*(n) + \zeta y(n)) \, e_n,$$

is analytic and $\|f(\zeta)\|_1$ attains its maximum at $\zeta = 0$. By a strong version of the Maximum Modulus Theorem (cf. Theorem 3.1 in [17]), f is constant. Hence $y = 0$ and x^* is a complex extreme point of B_X.

Note that even though both Φ and Ψ are strictly increasing concave functions and X is order continuous, we cannot obtain the converse of Corollary 5.3.

Note also that although m_Ψ^0 is order continuous and it has the same fundamental function as X, no point of $S_{\mathrm{m}_\Psi^0}$ is a complex extreme point of B_{m_Ψ} since Ψ is strictly increasing. Therefore we cannot completely determine the complex extreme points of an r.i. space X by its fundamental function.

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