A Generalization of an Inequality of Brouwer-Wilbrink

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Abstract

Brouwer and Wilbrink showed that $t + 1 \leq (s^2 + 1)c_{d-1}$ holds for a regular near $2d$-gon of order (s, t) with $s \geq 2$ and d is even.

In this note we generalize their inequality to all diameter.

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1 Introduction

The reader is referred to next section for the definitions.

Generalized n-gons of order (s, t) were introduced by Tits in [14]. Although formally n is unbounded, a famous theorem of Feit-G. Higman asserts that, apart from the ordinary polygons, finite examples can exist only for $n = 3, 4, 6, 8$ or 12. (See [6] and [3, Theorem 6.5.1].) Moreover, if $n = 12$ holds, then $s = 1$ or $t = 1$. In the case of $n = 4, 6, 8$, D.G. Higman [8, 9] and Haemers [7] showed that s and t are bounded from above by functions in t and s, respectively. To show this they used the Krein condition. (See [3, Theorem 6.5.1].)

Regular near polygons were introduced by Shult and Yanushka [15] as point-line geometries satisfying certain axioms. It is well known that (the collinearity graph of) a regular near $2d$-gon of order (s, t) is a distance-regular graph of valency $s(t + 1)$, diameter d and $a_i = c_i(s - 1)$ for all $1 \le i \le d$ such that for any vertex x the subgraph induced by the neighbors of x is the disjoint union of $t + 1$ complete graphs of size s.

Let Γ be a regular near $2d$-gon of order (s, t) with $s > 1$ and let $t_i := c_i - 1$ for all $1 \le i \le d$. Brouwer and Wilbrink [5] showed, by using the Krein condition $q^d_{dd} \ge 0$, that

$$\sum_{i=0}^{d-1} \left(\frac{-1}{s^2}\right)^i \prod_{j=1}^{i} \left(\frac{t - t_j}{1 + t_j}\right) \ge 0.$$

In particular, this implies that $1 + t \le (s^2 + 1)(1 + t_{d-1})$ holds if d is even. This means that if d is even, then $\frac{t+1}{c_{d-1}}$ is bounded from above by a function of s, namely by $(s^2 + 1)$.

We remark that a generalized $2d$-gon of order (s, t) is a regular near $2d$-gon of order (s, t) with $c_{d-1} = 1$. So the result of Brouwer-Wilbrink can be regarded as a generalization of Higman-Haemers result for generalized $2d$-gons to regular near $2d$-gons.

In this note we generalize the Brouwer-Wilbrink inequality to all diameters.

The following is our result.

Theorem 1 *Let Γ be a regular near $2d$-gon of order (s, t) with $t \ge 2$ and $s \ge 2$. Let $r = r(\Gamma) := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. Then the following hold.*
(1) Let $\tau := \frac{2d+1}{2d-2}$. Then

$$\frac{t+1}{c_{d-1}} < s\left(2s^\tau + \frac{1}{2}\right)^2 + 1.$$

(2) Let $\rho := \frac{3d+r}{d+r-1}$. Suppose $\theta_1 \le \frac{(t+1)(s-1)}{s^\rho + 1}$. Then $t \le 2s^{2\rho-1}$.

We remark that $\lim_{d\to\infty} \tau = 1$ and that $2 < \rho < 3$ as $r + 1 \leq d$.

We also remark that the dual polar graph on $[^2A_{2d-1}(s)]$ is a regular near $2d$-gon of order (s,t) with $t+1 = \dfrac{s^{2d}-1}{s-1}$ and $c_{d-1} = \dfrac{s^{2d-2}-1}{s-1}$. In this case we have $s^2 < \dfrac{t+1}{c_{d-1}} \leq s^2 + 1$. This example shows that the Brouwer-Wilbrink inequality is quite sharp for even diameter.

Note that for fixed s and d much larger then s the bound in Theorem 1 looks like

$$\frac{t+1}{c_{d-1}} < 5s^3.$$

There are generalized hexagons with $t = s^3$. In this light we wonder whether the bound

$$\frac{t+1}{c_{d-1}} \leq s^3 + 1.$$

would be true for all regular near $2d$-gons

The results of Brouwer-Wilbrink and Higman-Haemers were shown by using the fact that the Krein parameters are non-negative. In our proof we use the so-called absolute bound instead. This bound relates the multplicities of eigenvalues.

This note is organized as follows. In Section 2 we give definitions and prove a basic result. And we prove main result in Section 3.

2 Preliminaries

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected undirected graph without loops or multiple edges. For vertices x and y in Γ we denote by $\partial_\Gamma(x,y)$ the usual shortest path distance between x and y in Γ. The *diameter* of Γ, denoted by d, is the maximal distance of two vertices in Γ. For a vertex x in Γ we denote by $\Gamma_i(x)$ the set of vertices which are at distance i from x, and put $\Gamma_{-1}(x) = \Gamma_{d+1}(x) := \emptyset$.

A connected graph Γ with diameter d is called *distance-regular* if for all $0 \leq i \leq d$ there are numbers c_i, a_i and b_i such that for any two vertices x and y in Γ at distance i the sets

$$\Gamma_{i-1}(x) \cap \Gamma_1(y), \Gamma_i(x) \cap \Gamma_1(y) \text{ and } \Gamma_{i+1}(x) \cap \Gamma_1(y)$$

have cardinalities c_i, a_i and b_i, respectively. Then Γ is regular with valency $k := b_0$. The numbers c_i, a_i and b_i are called the *intersection numbers* of Γ.

Now suppose Γ is a distance-regular graph of diameter $d \geq 2$, valency $k \geq 3$ with the intersection numbers c_i, a_i and b_i. Define $r = r(\Gamma) := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$.

Let $k_i := |\Gamma_i(x)|$ for all $0 \leq i \leq d$ which does not depend on the choice of x.

By an eigenvalue of Γ we will mean an eigenvalue of its adjacency matrix A. Its multiplicity is its multiplicity as eigenvalue of A. Define the polynomials $u_i(x)$ by

$$\begin{array}{rl} u_0(x) := 1, u_1(x) := x/k, & \text{and} \\ c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) = x u_i(x), & i = 1, 2, \ldots, d-1. \end{array}$$

Let θ be an eigenvalue of Γ with multiplicity $m(\theta)$. The sequence $(u_0(\theta), u_1(\theta), \ldots, u_d(\theta))$ is called the *standard sequence corresponding to* θ.

The following is well-known basic result. (See [3, Corollary 4.1.2, Theorem 4.1.4].)

Proposition 2 *Let Γ be a distance-regular graph with diameter $d \geq 2$ and valency $k \geq 3$. Then Γ has exactly $d+1$ distinct eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_{d-1} > \theta_d$. Then*
(1) The standard sequence corresponding to θ_j has exactly j singe changes.
(2)

$$m(\theta_j) = \frac{|V\Gamma|}{\sum_{i=0}^{d} k_i u_i(\theta_j)^2}.$$

We would like to refer to the books [1, 2, 3, 4] for more information on distance-regular graphs.

A graph Γ is said to be *of order* (s, t) if $\Gamma_1(x)$ is a disjoint union of $t+1$ cliques of size s for every vertex x in Γ. In this case, Γ is a regular graph of valency $k = s(t+1)$ and every edge lies on a clique of size $s+1$. A clique of size $s+1$ is called a *singular line* of Γ.

A graph Γ is called (the collinearity graph of) *a regular near $2d$-gon of order* (s, t) if it is a distance-regular graph of order (s, t) with diameter d and $a_i = c_i(s-1)$ for all $1 \leq i \leq d$.

More information on regular near $2d$-gons will be found in [3, §6.4–6.6].

To close this section we recall the following result.

Lemma 3 *Let Γ be a distance-regular graph with diameter d such that $b_{d-1} \geq c_{d-1}$, then the second largest eigenvalue θ_1 satisfies*

$$\theta_1 \leq a_{d-1} + 2\sqrt{c_{d-1} b_{d-1}}.$$

Proof. Let $u_i := u_i(\theta_1)$ for all $0 \leq i \leq d$. If $\theta_1 > a_{d-1} + 2\sqrt{b_{d-1} c_{d-1}}$, then it is easy to show by induction that $u_i > 0$ for all i. This is a contradiction as the standard sequence corresponding to θ_1 has to have exactly one sign change. ∎

3 Proof of the Theorem

The rest of this paper $\Gamma = (V\Gamma, E\Gamma)$ denotes a regular near $2d$-gon of order (s,t) with $s \geq 2$. Let θ_d be the smallest eigenvalues of Γ, and let θ_1 be the second largest eigenvalue of Γ with the standard sequence $(u_0, u_1, \ldots, u_d)$. Let $r := \max\{i \mid (c_i, a_i, b_i) = (1, s-1, st)\}$.

Lemma 4 *(1)* $a_{d-1} \leq \theta_1$ *and* $m(\theta_d) < s^{2d}$.
(2) If $\theta_1 < a_d$, *then* $0 < u_i < \left(\frac{\theta_1}{st}\right)^i$ *for* $1 \leq i \leq d-1$ *and* $u_d < 0$ *with* $|u_d| \leq \frac{c_{d-1}}{b_{d-1}} u_{d-2}$.
(3) If $\theta_1 \leq a_d - b_{d-1}s^{-2}$, *then the Krein parameter* $q^1_{d,d}$ *is not zero.*
(4) If the Krein parameter $q^1_{d,d}$ *is not zero, then* $m(\theta_1) < \frac{s^{4d}}{2}$.

Proof. (1) These are proved in [13, Lemma 8] and [12, Lemma 4 (4)].
(2) Since $c_d u_{d-1} = (\theta_1 - a_d)u_d$, u_{d-1} and u_d have different signs. Note that the standard sequence corresponding to θ_1 has only one sign change. Hence we have $0 < u_i$ for $1 \leq i \leq d-1$ and $u_d < 0$.

For $1 \leq i \leq d-2$ we have

$$u_{i+1} \leq \frac{\theta_1 - a_i}{b_i} u_i < \frac{\theta_1}{st} u_i.$$

The first assertion is proved by induction on i.

Since $a_{d-1} \leq \theta_1$ and $b_{d-1}u_d = (\theta_1 - a_{d-1})u_{d-1} - c_{d-1}u_{d-2}$. The desired result is proved.
(3) Note that $q^1_{d,d} = 0$ if and only if

$$\sum_{i=0}^{d} k_i u_i s^{-2i} = 0.$$

Since $c_d k_d = b_{d-1}k_{d-1}$, $c_d u_{d-1} = (\theta_1 - a_d)u_d$ and $u_d < 0$, we have

$$k_{d-1}u_{d-1} + k_d u_d s^{-2} = \frac{k_d u_d}{b_{d-1}}(\theta_1 - a_d + b_{d-1}s^{-2}) \geq 0.$$

This implies $q^1_{d,d} \neq 0$, as $u_i > 0$ for all $0 \leq i \leq d-1$ and $u_d < 0$.
(4) Suppose $q^1_{d,d} \neq 0$. Then we have

$$m(\theta_1) \leq \sum_{q^j_{d,d} \neq 0} m(\theta_j) \leq \frac{m(\theta_d)\{m(\theta_d)+1\}}{2}$$

by the absolute bound. The assertion follows from (1). The lemma is proved. ∎

Proposition 5 *Suppose $\theta_1 \le a_d - b_{d-1}s^{-2}$. Then we have*

$$|V\Gamma| < \frac{s^{4d}}{2}\sum_{i=0}^{d} k_i u_i^2.$$

Proof. We have $q_{d,d}^1 \ne 0$ from Lemma 4 (3). It follows, by Lemma 4 (4), that

$$\frac{|V\Gamma|}{\sum_{i=0}^d k_i u_i^2} = m(\theta_1) < \frac{s^{4d}}{2}.$$

∎

Let ρ be as in Theorem 1. Since $r+1 \le d$, we have $2 < \rho < 3$.

Lemma 6 *Suppose $\theta_1 \le \dfrac{(t+1)(s-1)}{s^\rho + 1}$. Then the following hold.*

(1) $\displaystyle\sum_{i=r+1}^{d} k_i u_i^2 \le \frac{2k_{d-1}}{s^{4d}}$.

(2) $k_i u_i^2 \le \left(\dfrac{t}{s^{2\rho-1}}\right)^i$ *for all* $0 \le i \le r$.

(3) If $2s^{2\rho-1} \le t$, *then* $\displaystyle\sum_{i=0}^{r} k_i u_i^2 \le 2\left(\frac{t}{s^{2\rho-1}}\right)^r$.

Proof. Since $c_{d-1}(s-1) = a_{d-1} \le \theta_1$, we have $(s^\rho+1)c_{d-1} \le t+1$. It follows that $b_{d-1} = s(t+1-c_{d-1}) \ge s^{\rho+1}c_{d-1}$ and

$$\frac{\theta_1}{st} = \frac{(t+1)(s-1)}{st(s^\rho+1)} < \frac{1}{s^\rho}.$$

(1) For all $r+1 \le i \le d-1$ we have

$$k_i = k_{d-1}\frac{c_{d-1}\cdots c_{i+1}}{b_i \cdots b_{d-2}} \le k_{d-1}\left(\frac{c_{d-1}}{b_{d-1}}\right)^{d-1-i} \le k_{d-1}\left(\frac{1}{s^{\rho+1}}\right)^{d-1-i}.$$

Hence Lemma 4 (2) implies that

$$k_i u_i^2 \le k_{d-1}\left(\frac{1}{s^{\rho+1}}\right)^{d-1-i}\left(\frac{1}{s^\rho}\right)^{2i} \le \left(\frac{k_{d-1}}{s^{(\rho+1)(d-1)}}\right)\left(\frac{1}{s^{\rho-1}}\right)^i$$

and

$$k_d u_d^2 \le \frac{k_{d-1}b_{d-1}}{c_d}\left(\frac{c_{d-1}}{b_{d-1}}\right)^2\left(\frac{1}{s^\rho}\right)^{2(d-2)} \le \left(\frac{k_{d-1}}{s^{(\rho+1)(d-1)}}\right)\left(\frac{1}{s^{\rho-1}}\right)^d.$$

It follows that

$$\sum_{i=r+1}^{d} k_i u_i^2 \le 2\left(\frac{k_{d-1}}{s^{(\rho+1)(d-1)}}\right)\left(\frac{1}{s^{\rho-1}}\right)^{r+1} \le \frac{2k_{d-1}}{s^{4d}}.$$

(2) We have $k_0 u_0^2 = 1$ and $k_1 u_1^2 = \dfrac{{\theta_1}^2}{s(t+1)} \le \left(\dfrac{t}{s^{2\rho-1}}\right)$. For all $1 \le i \le r-1$ we have

$$u_{i+1} = \frac{1}{st}(\theta_1 - s + 1)u_i - u_{i-1} < \frac{\theta_1}{st}u_i.$$

It follows, by using induction on i, that

$$k_{i+1}{u_{i+1}}^2 \le k_i u_i^2 \left(\frac{t}{s^{2\rho-1}}\right) \le \left(\frac{t}{s^{2\rho-1}}\right)^{i+1}.$$

The desired result is proved.
(3) This follows from (2). The lemma is proved. ∎

Proof of Theorem 1. (1) Let $\gamma := s\left(2s^\tau + \dfrac{1}{2}\right)^2$. Suppose $c_{d-1}(\gamma+1) \le (t+1)$. Then

$$c_{d-1}b_{d-1} = c_{d-1}(t+1-c_{d-1})s \le \frac{t+1}{\gamma+1}\left\{t+1-\frac{t+1}{\gamma+1}\right\}s = \frac{(t+1)^2 s\gamma}{(\gamma+1)^2}.$$

It follows, by Lemma 3, that

$$\theta_1 \le \frac{(t+1)(s-1)}{\gamma+1} + \frac{2(t+1)\sqrt{s\gamma}}{\gamma+1} = \frac{(t+1)(4s^{\tau+1}+2s-1)}{\gamma+1} < \frac{(t+1)}{s^\tau}.$$

For all $1 \le i \le d-2$ we have $\theta_1 - a_i \le \dfrac{1}{s^\tau}(t+1-c_i)$ and thus

$$k_{i+1}{u_{i+1}}^2 < \frac{b_i k_i}{c_{i+1}}\left(\frac{\theta_1 - a_i}{b_i}\right)^2 u_i^2 \le \frac{b_i}{c_{i+1}}\left(\frac{1}{s^{\tau+1}}\right)^2 k_i u_i^2$$

It follow that

$$k_{i+1}{u_{i+1}}^2 \le \prod_{j=0}^{i}\left(\frac{b_j}{c_{j+1}s^{2\tau+2}}\right)$$

By Lemma 4 (2) we have

$$k_d {u_d}^2 \le \frac{k_{d-2}b_{d-2}b_{d-1}}{c_d c_{d-1}}\left(\frac{c_{d-1}}{b_{d-1}}\right)^2 {u_{d-2}}^2 < \frac{s}{\gamma}k_{d-2}{u_{d-2}}^2 < \frac{1}{4s^2}k_{d-2}{u_{d-2}}^2.$$

Since

$$\frac{b_j}{c_{j+1}s^{2\tau+2}} \ge \frac{b_{d-1}}{c_{d-1}s^{2\tau+2}} \ge \frac{\gamma}{s^{2\tau+1}} \ge 4,$$

we have

$$\sum_{i=0}^{d} k_i {u_i}^2 \leq \prod_{j=0}^{d-2} \left(\frac{b_j}{c_{j+1} s^{2\tau+2}} \right) \{4^{1-d} + \cdots + 4^{-1} + 1 + 4^{-2} s^{-2}\} < \frac{2k_{d-1}}{s^{4d-1}}.$$

It follows, by Proposition 5, that

$$k_{d-1} \left(1 + \frac{b_{d-1}}{c_d} \right) = k_{d-1} + k_d < |V\Gamma| \leq \frac{s^{4d}}{2} \sum_{i=0}^{d} k_i {u_i}^2 < s k_{d-1}.$$

Hence we have

$$t + 1 < s(t+1) - b_{d-1} = c_{d-1} s.$$

which is a contradiction.
(2) Suppose $2s^{2\rho-1} \leq t$ to derive a contradiction. Proposition 5 and Lemma 6 imply that

$$k_{d-1} + k_d < |V\Gamma| \leq \frac{s^{4d}}{2} \sum_{i=0}^{d} k_i u_i^2 \leq \frac{s^{4d}}{2} \left\{ 2 \left(\frac{t}{s^{2\rho-1}} \right)^r + \frac{2k_{d-1}}{s^{4d}} \right\}.$$

Since

$$k_d = \frac{b_0 \cdots b_{d-1}}{c_1 \cdots c_d} \geq s(st)^r \left(\frac{b_{d-1}}{c_{d-1}} \right)^{d-r-1} \geq t^r s^{r+1} (s^{\rho+1})^{d-r-1}.$$

we have

$$s^d (s^\rho)^{d-r-1} < s^{4d} s^{r(1-2\rho)}.$$

This is a contradiction. The theorem is proved. ∎

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