Improving diameter bounds for distance-regular graphs

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Abstract

In this paper we study the sequence $(c_i)_{0 \le i \le d}$ for a distance-regular graph. In particular we show that if $d \ge 2j$ and $c_j = c > 1$ then $c_{2j} > c$ holds. Using this we give improvements on diameter bounds by Hiraki and Koolen [5], and Pyber [8], respectively, by applying this inequality.

1 Introduction

Let Γ be a distance-regular graph of diameter $d \ge 2$, valency $k \ge 2$ and intersection numbers c_i, a_i, b_i $(0 \le i \le d)$. (For definitions, see next section.) We define

$$\mathtt{h} = \mathtt{h}(\Gamma) := |\{i \mid 1 \le i \le d-1 \text{ and } (c_i, a_i, b_i) = (c_1, a_1, b_1)\}|\,.$$

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For each $1 \leq c \leq c_d$ we define

$$\begin{aligned} \xi_c &:= \min\{i \mid c_i \geq c\} \\ \eta_c &:= |\{i \mid c_i = c\}| \,. \end{aligned}$$

In this paper we study for a given integer c, the number of η_c for a distance-regular graph. We obtain the following result:

Theorem 1.1 *Let Γ be a distance-regular graph of diameter $d \geq 2$. Let c be an integer with $2 \leq c \leq c_d$. Then $\eta_c \leq \xi_c - 1$.*

Using Theorem 1.1, we will give improvements on the diameter bounds of distance-regular graphs found by Hiraki and Koolen [5] and Pyber [8], respectively.

In [5] it was shown that the diameter of a distance-regular graph of valency k is bounded by $\frac{1}{2}k^2\eta_1$. In the next result we show we can interchange the power 2 by α, where

$$\alpha := \min\{x > 0 \mid 4^{\frac{1}{x}} - 2^{\frac{1}{x}} \leq 1\}.$$

Note 1.2 *We remark that $1.44 < \alpha < 1.441$.*

Theorem 1.3 *Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $C := \{c_i \mid i = 1, \ldots, d\}$. Then*

$$\xi_c \leq \frac{1}{2}(c^\alpha \eta_1 + 1) \tag{1}$$

and

$$\xi_c + \eta_c \leq c^\alpha \eta_1 + 1 \tag{2}$$

hold for all $c \in C$.
In particular if $\hbar := \hbar(\Gamma) \geq 2$ then

$$\xi_c \leq \frac{1}{2}\left\{2c^\alpha(\hbar + 1) + 1\right\} \tag{3}$$

and

$$\xi_c + \eta_c \leq 2c^\alpha(\hbar + 1) + 1 \tag{4}$$

hold for all $c \in C$.

Corollary 1.4 *Let Γ be a distance-regular graph of diameter $d \geq 2$, valency $k \geq 3$ and $\hbar := \hbar(\Gamma)$. Then*

$$d \leq \frac{1}{2}\left(k^\alpha \eta_1 + 1\right).$$

In particular if $\hbar \geq 2$ then

$$d \leq k^\alpha(\hbar + 1) + \frac{1}{2}$$

holds.

In [8] Pyber showed that the diameter of distance-regular graphs is at most 5 times the 2-logarithm of the number of vertices. The following gives an improvement of this bound.

Theorem 1.5 *Let Γ be a distance-regular graph with v vertices. Let d be the diameter of Γ. Then*

$$d < \frac{8}{3} \log_2 v.$$

The paper is organized as follows: In Section 2 we give definitions, in Section 3 we give the proof of Theorem 1.1, in Section 4 we give the proofs of Theorem 1.3 and Corollary 1.4, and in the last section we give the proof of Theorem 1.5.

2 Defintions

All graphs in this paper are undirected graphs without loops and multiple edges. Suppose that Γ is a finite connected graph with vertex set $V\Gamma$. We define the distance between any two vertices x and y, $d(x, y)$, as to be the length of any shortest path in Γ between x and y, and the diameter d of Γ to be the largest distance between any pair of vertices in $V\Gamma$. For a vertex $x \in V\Gamma$ and any non-negative integer i not exceeding d, let $\Gamma_i(x)$ denote the subset of vertices in $V\Gamma$ that are at distance i from x and put $\Gamma(x) := \Gamma_1(x)$ and $\Gamma_{-1}(x) = \Gamma_{d+1}(x) := \emptyset$. For any two vertices x and y in $V\Gamma$ at distance i, let

$$\begin{aligned} C_i(x, y) &:= \Gamma_{i-1}(x) \cap \Gamma_1(y) \\ A_i(x, y) &:= \Gamma_i(x) \cap \Gamma_1(y) \\ B_i(x, y) &:= \Gamma_{i+1}(x) \cap \Gamma_1(y). \end{aligned}$$

A graph Γ is called *distance-regular* if there are integers b_i, c_i $(0 \le i \le d)$ which satisfy $c_i = |C_i(x, y)|$ and $b_i = |B_i(x, y)|$ for any two vertices x and y in $V\Gamma$ at distance i. Clearly such a graph is regular of valency $k := b_0$. The numbers c_i, b_i, and a_i, where

$$a_i := k - b_i - c_i \quad (i = 0, \ldots, d)$$

is the number of neighbors of y in $\Gamma_i(x)$ for $x, y \in V\Gamma$ at distance i, are called the *intersection numbers* of Γ. The array

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & \cdots & b_i & \cdots & b_{d-1} & b_d \end{bmatrix}$$

is called the *intersection array* of Γ.

Now, suppose that Γ is a distance-regular graph of diameter $d \ge 2$, valency $k \ge 2$ and intersection numbers c_i, a_i, b_i, $0 \le i \le d$. Clearly, $b_d = c_0 = a_0 = 0$ and $c_1 = 1$. In [2, Section 4.1], it is shown that $\Gamma_i(x)$ contains k_i elements where

$$k_0 := 1, \;\; k_1 := k, \;\; k_{i+1} := k_i b_i / c_{i+1}, \quad i = 1, \ldots, d-1,$$

and in [2, Proposition 4.1.6] it is shown that

$$
\begin{aligned}
& k = b_0 > b_1 \geq b_2 \geq \cdots \geq b_{d-1} > b_d = 0, \\
& 1 = c_1 \leq c_2 \leq \cdots \leq c_d \leq k \\
\text{and} \quad & c_i \leq b_j \ \text{ if } \ i + j \leq d. \qquad (5)
\end{aligned}
$$

For more information on distance-regular graphs, see [2].

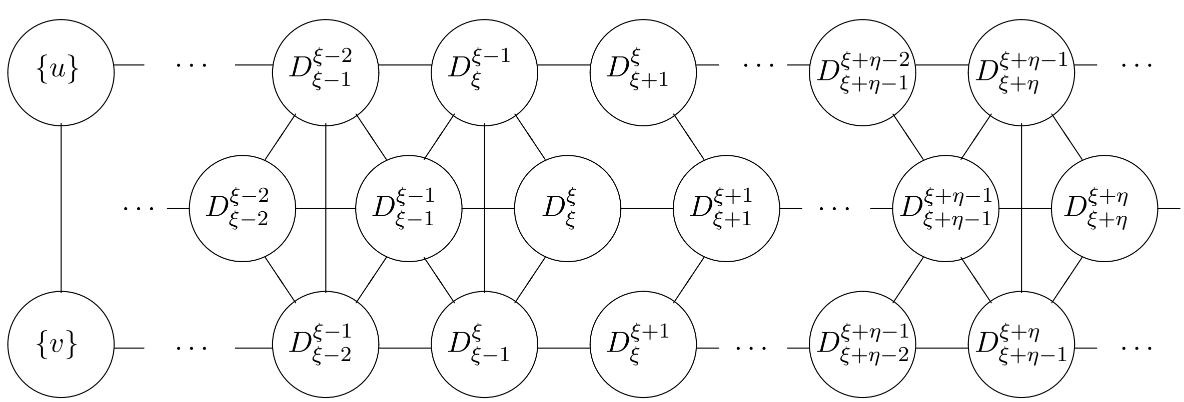
3 Proof of Theorem 1.1

In this section we show Theorem 1.1. In order to show the theorem we use the intersection diagram with respect to an edge. Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let (u, v) be an edge in Γ. For each $0 \leq i, j \leq d$ set $D^i_j = D^i_j(u, v) := \Gamma_i(u) \cap \Gamma_j(v)$. The *intersection diagram with respect to* (u, v) is the collection $\{D^i_j\}_{0 \leq i,j \leq d}$ with lines between them such that if there is no line between D^i_j and D^s_t, then it means that there is no edge (x, y) for any $x \in D^i_j$ and $y \in D^s_t$. Also if we know that D^i_j is the empty set then we erase it.

Remark 3.1 *If $c_i = c_{i+1}$ then there are no edges between any two of $\{D^{i+1}_i, D^i_i, D^i_{i+1}\}$.*

Let c be an integer with $2 \leq c \leq c_d$. Let $\eta := \eta_c$ and $\xi := \xi_c$. Then for any edge (u, v) in Γ the intersection diagram with respect to (u, v) has the shape as in Figure 1.

Figure 1.



Lemma 3.2 *If $c_\eta + c_{\xi-1} > c_\xi$ then the following hold:*
(i) *There are no edges between $D^{\xi}_{\xi-1}$ and $D^{\xi-1}_{\xi}$.*
(ii) *There exists an edge (x_0, x_1) such that $x_0 \in D^{\xi-1}_{\xi-1}$ and $x_1 \in D^{\xi}_{\xi-1}$.*
(iii) *$b_\eta \geq c_{\xi-1} + c_\eta$.*

Proof: (i) Assume that there is an edge between $D^{\xi-1}_{\xi}$ and $D^{\xi}_{\xi-1}$. Then there exists a path $(y_0, y_1, \ldots, y_\eta)$ such that $y_0 \in D^{\xi-1}_{\xi}$ and $y_i \in D^{\xi-1+i}_{\xi-2+i}$ for $1 \leq i \leq \eta$. Clearly $d(y_0, y_\eta) = \eta$. It follows that

$$C_\xi(v, y_0) \supseteq \left[\Gamma(y_0) \cap D^{\xi}_{\xi-1}\right] \cup \left[\Gamma(y_0) \cap D^{\xi-2}_{\xi-1}\right] \supseteq C_\eta(y_\eta, y_0) \cup C_{\xi-1}(u, y_0).$$

This contradicts to $c_\eta + c_{\xi-1} > c_\xi$. This shows (i).

(ii) Let $x_1 \in D^{\xi}_{\xi-1}$. As $c_\xi > c_{\xi-1}$ and there are no edges between $D^{\xi}_{\xi-1}$ and $D^{\xi-1}_{\xi}$, there must be a neighbour $x_0 \in D^{\xi-1}_{\xi-1}$ of x_1. This shows (ii).

(iii) It follows, from (ii), that there exists a path $(x_0, x_1, \ldots, x_\eta)$ such that $x_0 \in D^{\xi-1}_{\xi-1}$ and $x_i \in D^{\xi-1+i}_{\xi-2+i}$ for $1 \leq i \leq \eta$. Now $C_\eta(x_\eta, x_0) \subseteq \Gamma(x_0) \cap D^{\xi}_{\xi-1}$ and by symmetry we find that

$$c_\eta \leq \left|\Gamma(x_0) \cap D^{\xi}_{\xi-1}\right| = \left|\Gamma(x_0) \cap D^{\xi-1}_{\xi}\right|.$$

We also have

$$C_{\xi-1}(u, x_0) \cup \left[\Gamma(x_0) \cap D^{\xi-1}_{\xi}\right] \subseteq B_\eta(x_\eta, x_0),$$
$$C_{\xi-1}(u, x_0) \cap \left[\Gamma(x_0) \cap D^{\xi-1}_{\xi}\right] = \emptyset,$$

which implies

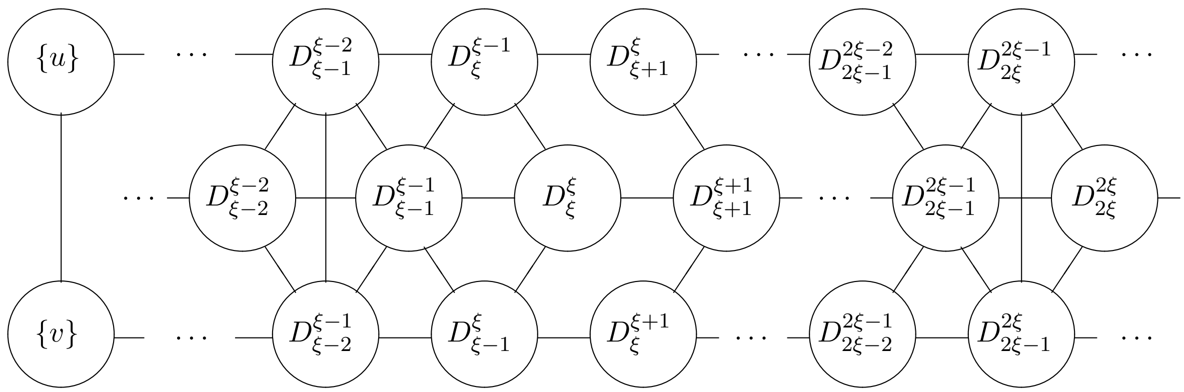
$$c_{\xi-1} + c_\eta \leq b_\eta.$$

This shows (iii). ∎

Proof of Theorem 1.1: Suppose $\xi \leq \eta$. We will derive a contradiction and this shows the theorem. In order to do this we will show several claims.

Let (u, v) be any edge in Γ and $D^i_j = D^i_j(u, v) := \Gamma_i(u) \cap \Gamma_j(v)$. Then Remark 3.1 and Lemma 3.2 (i) imply that the intersection diagram with respect to (u, v) has the shape as in Figure 2, and Lemma 3.2 (ii) implies that there exists a path $(x_0, x_1, \ldots, x_\xi)$ of length ξ such that $x_0 \in D^{\xi-1}_{\xi-1}$ and $x_i \in D^{\xi+i-1}_{\xi+i-2}$ for all $1 \leq i \leq \xi$.

Figure 2.



Claim 1. (i) The set $D^{\xi}_{\xi} \cap A_{\xi}(x_{\xi}, x_0)$ is non-empty.
(ii) Let $z \in D^{\xi}_{\xi} \cap A_{\xi}(x_{\xi}, x_0)$. Then $C_{\xi}(z, x_{\xi}) \subseteq D^{2\xi-2}_{2\xi-3} \cap \Gamma_1(x_{\xi})$.
(iii) Let $z \in D^{\xi}_{\xi} \cap A_{\xi}(x_{\xi}, x_0)$. Then $D^{\xi+1}_{\xi+1} \cap C_{\xi}(x_{\xi}, z)$ is non-empty.

Proof: (i) The set $D^{\xi}_{\xi} \cap A_{\xi}(x_{\xi}, x_0)$ is non-empty, as otherwise

$$\begin{aligned} C_{\xi-1}(u, x_0) \cup B_{\xi-1}(v, x_0) &\subseteq B_{\xi}(x_{\xi}, x_0), \\ C_{\xi-1}(u, x_0) \cap B_{\xi-1}(v, x_0) &= \emptyset \end{aligned}$$

hold and thus $c_{\xi-1} + b_{\xi-1} \leq b_{\xi}$ which is a contradiction as $\xi \geq 2$. This shows (i).
(ii) It is clear that $C_{\xi}(z, x_{\xi}) \subseteq D^{2\xi-2}_{2\xi-3} \cup D^{2\xi-1}_{2\xi-1}$. If there exists $z' \in D^{2\xi-1}_{2\xi-1} \cap C_{\xi}(z, x_{\xi})$ then

$$C_{\xi}(x_0, z') \cup \{x_{\xi}\} \subseteq C_{2\xi-1}(v, z').$$

This contradicts to $c_{\xi} = c_{2\xi-1}$ as $\eta \geq \xi$. Thus $D^{2\xi-1}_{2\xi-1} \cap C_{\xi}(z, x_{\xi}) = \emptyset$ and this shows (ii).
(iii) It is clear that $C_{\xi}(x_{\xi}, z) \subseteq D^{\xi}_{\xi-1} \cup D^{\xi+1}_{\xi+1}$. If $D^{\xi+1}_{\xi+1} \cap C_{\xi}(x_{\xi}, z) = \emptyset$ then

$$C_{\xi}(x_{\xi}, z) \cup \{x_0\} \subseteq C_{\xi}(v, z).$$

which is a contradiction. Thus we have (iii). ∎

Define $P := C_{\xi}(x_{\xi}, x_0)$ and $Q := B_{\xi-1}(x_{\xi-1}, x_0) - B_{\xi}(x_{\xi}, x_0)$.

Claim 2. The set $P \cap Q$ is empty and $P \cup Q \subseteq D^{\xi}_{\xi-1}$.

Proof: It is clear that $P \subseteq D^{\xi}_{\xi-1}$. Let $z \in Q$. If $z \in P$, then $x_0 \in D^{\xi}_{\xi-1}(x_\xi, x_{\xi-1})$ and $z \in D^{\xi-1}_{\xi}(x_\xi, x_{\xi-1})$ hold. As $\eta \geq \xi \geq 2$, it follows by Lemma 3.2 (i) that there are no edges between $D^{\xi}_{\xi-1}$ and $D^{\xi-1}_{\xi}$. Therefore $P \cap Q = \emptyset$ and $Q \subseteq A_\xi(x_\xi, x_0)$ hold. As

$$\left[D^{\xi-2}_{\xi-2} \cup D^{\xi-2}_{\xi-1} \cup D^{\xi-1}_{\xi}\right] \cap \Gamma_1(x_0) \subseteq B(x_\xi, x_0),$$

it follows that $Q \subseteq D^{\xi}_{\xi} \cup D^{\xi-1}_{\xi-1} \cup D^{\xi-1}_{\xi-2} \cup D^{\xi}_{\xi-1}$. Suppose $z \in \left[D^{\xi}_{\xi} \cup D^{\xi-1}_{\xi-1} \cup D^{\xi-1}_{\xi-2}\right]$. Then

$$C_\xi(z, x_\xi) \subseteq D^{2\xi-2}_{2\xi-3} \cap \Gamma_1(x_\xi) = C_{2\xi-1}(u, x_\xi). \tag{6}$$

(The Equation (6) is clear when $z \in D^{\xi-1}_{\xi-1} \cup D^{\xi-1}_{\xi-2}$. If $z \in D^{\xi}_{\xi}$, then it follows from Claim 1 (ii).) By comparing both sides of (6) we have $x_{\xi-1} \in C_{2\xi-1}(u, x_\xi) = C_\xi(z, x_\xi)$ which contradicts to $z \in B_{\xi-1}(x_{\xi-1}, x_0)$. Hence $Q \subseteq D^{\xi}_{\xi-1}$. The claim is proved. ∎

Claim 3. There exists $u' \in B_{\xi-1}(x_0, v) - B_{\xi-1}(x_1, v)$. Define $R := B_\xi(u', x_0)$. Then $R \subseteq D^{\xi}_{\xi}$.

Proof: Since $u \in B_{\xi-1}(x_1, v) - B_{\xi-1}(x_0, v)$, there exists $u' \in B_{\xi-1}(x_0, v) - B_{\xi-1}(x_1, v)$.

Consider the intersection diagram with respect to (u', v). Then we have $x_0 \in D^{\xi}_{\xi-1}(u', v)$ and $x_1 \in D^{\xi-1}_{\xi-1}(u', v)$. Take any $w_2 \in R$. Then $w_2 \in D^{\xi+1}_{\xi}(u', v)$ and we can take a path $(w_2, w_3, \ldots, w_\xi)$ such that $w_i \in D^{\xi+i-1}_{\xi+i-2}(u', v)$ for $2 \leq i \leq \xi$. By Claim 1 (i),(iii) there exists $z_0 \in D^{\xi}_{\xi}(u', v) \cap A_\xi(w_\xi, x_1)$ and $z_1 \in D^{\xi+1}_{\xi+1}(u', v) \cap C_\xi(w_\xi, z_0)$ Let $(z_1, \ldots, z_\xi = w_\xi)$ be a shortest path connecting z_1 and w_ξ. Since $z_1 \in D^{\xi+1}_{\xi+1}(u', v)$ and $z_\xi \in D^{2\xi-1}_{2\xi-2}(u', v)$ such that $d(z_1, z_\xi) = \xi - 1$ there exists an integer t with $2 \leq t \leq \xi - 1$ such that $z_i \in D^{\xi+i}_{\xi+i}(u', v)$ for all $0 \leq i \leq t-1$ and $z_i \in D^{\xi+i-1}_{\xi+i-2}(u', v)$ for all $t \leq i \leq \xi$ by Claim 1 (ii). Next we return to the intersection diagram with respect to (u, v). Since $x_0 \in D^{\xi-1}_{\xi-1}$ and $w_2 \in B_{\xi-1}(v, x_0)$, we have $w_2 \in D^{\xi-1}_{\xi} \cup D^{\xi}_{\xi}$.

Suppose $w_2 \in D^{\xi-1}_{\xi}$. Then we have $w_i \in D^{\xi+i-3}_{\xi+i-2}$ for all $2 \leq i \leq \xi$ as the former diagram gives us the distance between v and w_i. Similarly we have $z_i \in D^{\xi+i-3}_{\xi+i-2}$ for all $t \leq i \leq \xi$ and $z_i \in D^{\xi+i-1}_{\xi+i}$ for all $0 \leq i \leq t-1$. Hence (x_1, z_0) is an edge such that $x_1 \in D^{\xi}_{\xi-1}$ and $z_0 \in D^{\xi-1}_{\xi}$. This contradicts Lemma 3.2 (i) as $c_\eta + c_{\xi-1} > c_\xi$ holds by the assumption $\eta \geq \xi$. Hence we have $w_2 \in D^{\xi}_{\xi}$ and our claim is proved. ∎

Claims 2 and 3 show that the sets P, Q and R are disjoint and

$$P \cup Q \cup R \subseteq B_{\xi-1}(u, x_0).$$

As $|P| = c_\xi$, $|Q| = b_{\xi-1} - b_\xi$ and $|R| = b_\xi$ hold, this implies $c_\xi + (b_{\xi-1} - b_\xi) + b_\xi \leq b_{\xi-1}$ which is a contradiction. The theorem is proved. ∎

4 Proofs of Theorem 1.3 and Corollary 1.4

In this section we prove Theorem 1.3 and Corollary 1.4. Recall $C := \{c_i \mid i = 1, \ldots, d\}$.

Lemma 4.1 *Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $c \in C \setminus \{1\}$. Then*

(i) $c_i + c_{\xi_c - i} \leq c$ *for all* $1 \leq i \leq \xi_c - 1$.

(ii) $\displaystyle\prod_{j=1}^{\xi_c - 1} c_j \leq \left(\frac{c}{2}\right)^{\xi_c - 1}$.

Proof: (i) This is [7, Proposition 1 (ii)].
(ii) Let $\xi := \xi_c$. Then (i) implies that

$$2\sqrt{c_j c_{\xi-j}} \leq c_j + c_{\xi-j} \leq c$$

holds for all $1 \leq j \leq \xi - 1$. Hence we have

$$\prod_{j=1}^{\xi-1} c_j = \prod_{j=1}^{\xi-1} \sqrt{c_j c_{\xi-j}} \leq \left(\frac{c}{2}\right)^{\xi-1}.$$

∎

Lemma 4.2 *Let $\beta := \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}$. For a real number x with $\beta \leq x \leq 1$*

$$x^\alpha + 2(1-x)^\alpha \leq 1$$

holds.

Proof: Define $f : [\beta, 1] \to \mathbf{R}$ by

$$f(x) := x^\alpha + 2(1-x)^\alpha.$$

By definition of β (and α), it follows easily that $f(\beta) = 1$. Also $f(1) = 1$. By straightforward calculation one sees that on $[\beta, 1]$, the function f has maxima at $x = 1$ and $x = \beta$. This shows the lemma. ∎

Proof of Theorem 1.3: Let $C = \{\gamma_1, \gamma_2, \ldots, \gamma_q\}$ such that

$$1 = \gamma_1 < \gamma_2 < \cdots < \gamma_q = c_d$$

holds. We will prove

$$\xi_{\gamma_j} \leq \frac{1}{2}\{(\gamma_j)^\alpha \eta_1 + 1\} \tag{7}$$

and

$$\xi_{\gamma_j} + \eta_{\gamma_j} \leq (\gamma_j)^\alpha \eta_1 + 1 \tag{8}$$

hold for all $j = 1, 2, \ldots, q$ by induction on j.
As $1 = \xi_1 \leq \frac{1}{2}(\eta_1 + 1)$ and $\xi_1 + \eta_1 = \eta_1 + 1$ hold, so (7) and (8) hold for $j = 1$. Now let $s \geq 2$ and assume that (7) and (8) hold for all γ_i with $1 \leq i < s$. Let $c := \gamma_s$, $c' := \gamma_{(s-1)}$ and $c' = bc$ for some $0 < b < 1$.

First we will prove Equation (7) holds for $j = s$. In order to show this we need to consider two cases, namely the case $0 < b \leq \beta$ and the case $\beta < b < 1$, respectively.

Case-1: $0 < b \leq \beta$
Proof: As $c' = bc \leq \beta c$ and $b^\alpha \leq \beta^\alpha = \frac{1}{2}$ hold, we find

$$\xi_c = \xi_{c'} + \eta_{c'} - 1 \leq (bc)^\alpha \eta_1 \leq \frac{1}{2} c^\alpha \eta_1$$

hold by our induction hypothesis. ∎

Case-2: $\beta < b < 1$
Proof: Let $c'' := c_{\xi_c - \xi_{c'}}$. Then Lemma 4.1 (i) implies that $c = c_{\xi_c} \geq c_{\xi_{c'}} + c_{\xi_c - \xi_{c'}} = c' + c''$ holds and thus $c'' \leq (1-b)c$ holds. Therefore we find that

$$\begin{aligned}
\xi_c &\leq \xi_{c'} + (\xi_{c''} + \eta_{c''} - 1) \\
&\leq \frac{1}{2}\left\{(c')^\alpha \eta_1 + 1\right\} + (c'')^\alpha \eta_1 \\
&\leq \frac{1}{2}\left\{(b^\alpha + 2(1-b)^\alpha)\, c^\alpha \eta_1 + 1\right\} \\
&\leq \frac{1}{2}\left(c^\alpha \eta_1 + 1\right)
\end{aligned}$$

hold by Lemma 4.2 and our induction hypothesis. ∎

Hence Equation (7) holds for $j = s$. The fact that Equation (8) holds for $j = s$ follows from Theorem 1.1. Therefore we have shown that Equations (7) and (8) hold for all $1 \leq j \leq q$.

Now assume $\mathtt{h} \geq 2$. In [6, Theorem 2] it is shown that $\eta_1 \leq 2(\mathtt{h}+1)$ holds. Equations (3) and (4) follow now immediately from Equations (1) and (2), respectively. ∎

Proof of Corollary 1.4: If $\xi_{c_d} = d$ then by Equation (1) the following holds:

$$d = \xi_{c_d} \leq \frac{1}{2}\left\{(c_d)^{\alpha}\eta_1 + 1\right\} \leq \frac{1}{2}\left(k^{\alpha}\eta_1 + 1\right).$$

Now we assume that $\xi_{c_d} < d$. If $c_d \leq \beta k$ then the result holds as

$$d \leq \xi_{c_d} + \eta_{c_d} - 1 \leq (c_d)^{\alpha}\eta_1 \leq (\beta k)^{\alpha}\eta_1 = \frac{1}{2}k^{\alpha}\eta_1$$

holds by (2). To complete the proof we need to consider $\beta k < c_d$. Let $c_d = \epsilon k$ for some $\beta < \epsilon < 1$ and $c := c_{(d-\xi_{c_d})}$. Then $c = c_{(d-\xi_{c_d})} \leq b_{\xi_{c_d}} \leq k - c_d = (1-\epsilon)k$ holds by (5). Therefore the result follows by Theorem 1.3 and Lemma 4.2:

$$\begin{aligned} d &\leq \xi_{c_d} + (\xi_c + \eta_c - 1) \\ &\leq \frac{1}{2}\left\{(c_d)^{\alpha}\eta_1 + 1\right\} + c^{\alpha}\eta_1 \\ &\leq \frac{1}{2}\left[\left\{\epsilon^{\alpha} + 2(1-\epsilon)^{\alpha}\right\}k^{\alpha}\eta_1 + 1\right] \\ &\leq \frac{1}{2}\left(k^{\alpha}\eta_1 + 1\right). \end{aligned}$$

∎

5 Proof of Theorem 1.5

In the proof of Theorem 1.5 we will use the following results.

Lemma 5.1 *Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $\mathtt{h} = \mathtt{h}(\Gamma)$. Then for any integer $1 \leq i \leq d$ the following hold :*
(i) If $k_{i+1} \leq k_i$, then $d \leq 3i$.
(ii) $a_1 + 2 \leq c_i + b_{i-1}$.
(iii) If $b_{i-1} = 2$ and $c_i = 1$ then $i \leq 2\mathtt{h} + 2$.

Proof: (i),(ii): These follow from [2, Corollary 5.9.7] and [2, Proposition 5.5.1], respectively.
(iii) We only need to consider the case $\mathtt{h} = 1$ as for the case $\mathtt{h} \geq 2$ the assertion follows

from [6, Theorem 2]. Note that by (ii) $a_1 \leq 1$ holds. Now [4, Theorem 1.1–1.2] imply that $c_4 \neq 1$. Hence $\eta_1 \leq 3 < 2\mathtt{h} + 2$. The lemma is shown. ∎

Lemma 5.2 *Let Γ be a distance-regular graph of diameter $d \geq 2$ and valency $k \geq 3$. Let $\ell \leq d$ be maximal such that $2c_\ell \leq b_{\ell-1}$. Then*

$$d \leq 4\ell.$$

Proof: By definition of ℓ, $2c_\ell \leq b_{\ell-1}$ and $2c_{\ell+1} > b_\ell$ hold. We may assume that $3\ell + 1 \leq d$. Let $c := c_{2\ell+1}$, $\xi := \xi_c$ and $\eta := \eta_c$. Then $2c_{\ell+1} > b_\ell \geq c_{2\ell+1} = c$ holds by (5). We have $c \geq 2$ as otherwise $c_{\ell+1} = 1$ and $b_\ell = 1$ hold by the definition of ℓ, and thus $k_{\ell+1} = k_\ell$ holds which contradicts to Lemma 5.1 (i).

If $\eta \geq \ell + 1$, then $c_\eta + c_{\xi-1} \geq 2c_{\ell+1} > c = c_\xi$ as $\xi - 1 \geq \eta \geq \ell + 1$ by Theorem 1.1. By Lemma 3.2(iii) we have

$$b_\ell \geq b_\eta \geq c_{\xi-1} + c_\eta \geq 2c_{\ell+1}.$$

But that is a contradiction with the definition of ℓ. Hence we find $\eta \leq \ell$. As $2(\ell+1) \leq \eta + \xi \leq 3\ell + 1$, $c_{3\ell+1} \geq c_{\eta+\xi} \geq 2c_{\ell+1} > b_\ell$ holds by Lemma 4.1 (i). So $d \leq 4\ell$ holds. ∎

Proof of Theorem 1.5: Let $\ell \leq d$ be maximal such that $2c_\ell \leq b_{\ell-1}$. Let $c := c_\ell$, $\xi := \xi_c$ and $\eta := \eta_c$. In order to prove the theorem we need to consider three cases, namely $c_\ell \geq 2$, ($c_\ell = 1$ and $b_{\ell-1} \geq 3$) and ($c_\ell = 1$ and $b_{\ell-1} = 2$).

Case-1: $c_\ell \geq 2$
Proof: We have

$$\prod_{j=1}^{\xi-1} b_{j-1} \geq (2c)^{\xi-1} \quad \text{and} \quad \prod_{j=1}^{\xi-1} c_j \leq \left(\frac{c}{2}\right)^{\xi-1}$$

from Lemma 4.1 (ii). Theorem 1.1 implies that $\ell \leq \xi + \eta - 1 \leq 2\xi - 2$. Hence we have

$$k_\ell = \prod_{j=1}^{\xi-1} \frac{b_{j-1}}{c_j} \prod_{j=\xi}^{\ell} \frac{b_{j-1}}{c_j} \geq 4^{\xi-1} 2^{\ell-\xi+1} = 2^{\ell+\xi-1}$$

and thus $\log_2 v > \log_2 k_\ell \geq \ell + \xi - 1 \geq \frac{3}{2}\ell$ holds. Therefore by Lemma 5.2

$$d \leq 4\ell < \frac{8}{3}\log_2 v$$

holds. ∎

Case-2: $c_\ell = 1$ and $b_{\ell-1} \geq 3$
Proof: We have $\frac{b_{i-1}}{c_i} \geq 3$ for all $1 \leq i \leq \ell$. Hence $v > k_\ell \geq 3^\ell$ holds. By Lemma 5.2,

$$d \leq 4\ell < 4\log_3 v = 4(\log_3 2)(\log_2 v) < \frac{8}{3}\log_2 v$$

holds. ∎

Case-3: $c_\ell = 1$ and $b_{\ell-1} = 2$
Proof: There are two possibilities, namely $c_{\ell+1} = 1$ or $c_{\ell+1} \geq 2$, but in each case $k_{\ell+1} \leq k_\ell$ holds by $b_\ell \leq b_{\ell-1} = 2$ and $k_{\ell+1}c_{\ell+1} = k_\ell b_\ell$. Hence Lemma 5.1 (i) and (iii) implies that $d \leq 3\ell$ and $\ell \leq 2\mathtt{h} + 2$. If $b_1 \leq 2$ then $k \leq 4$ as $a_1 \leq 1$ by Lemma 5.1 (ii). Hence the result is proved by [1] and [3].

Now we may assume $b_1 \geq 3$. Then

$$v > k_\ell = \prod_{j=1}^{\ell} b_{j-1} \geq k b_1^{\mathtt{h}} 2^{\ell-1-\mathtt{h}} \geq 3^{\mathtt{h}+1} 2^{\ell-\mathtt{h}-1} \geq 6^{\frac{\ell}{2}}.$$

Therefore we have $\log_2 v > \log_2 6^{\frac{\ell}{2}} > \frac{9}{8}\ell$ and thus

$$d \leq 3\ell < \frac{8}{3}\log_2 v$$

holds. ∎

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