OH-TYPE AND OH-COTYPE OF OPERATOR SPACES AND COMPLETELY SUMMING MAPS

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Abstract. The definition and basic properties of OH-type and OH-cotype of operator spaces are given. We prove that every bounded linear map from $C(K)$ into OH-cotype q $(2 \leq q \leq \infty)$ space for a compact set K satisfies completely $(q,2)$-summing property, a noncommutative analogue of absolutely $(q,2)$-summing property. At the end of this paper, we observe that "OH-cotype 2" is equivalent to the previous definition of "OH-cotype 2" of G. Pisier.

1. Introduction

Type and cotype of Banach spaces plays an important role to extend classical results concerning L_p-spaces to more general spaces. For example, we have the "little Grothendieck's theorem":

Every bounded linear map from $C(K)$ into a Hilbert space is 2-summing

for a compact set K, which can be generalized as follows.

Every bounded linear map from $C(K)$ into cotype q space is $(q,2)$-summing

for $2 \leq q < \infty$. The aim of this paper is to give an appropriate definition of an operator space version of type and cotype and use them to prove an operator space version of the above theorem. For general information on p-summing operators and (q,p)-summing operators, see [1] and [15].

As a noncommutative analogue of type and cotype of Banach spaces, the notion of type and cotype of operator spaces was considered in many versions. G. Pisier introduced the notion of "OH-cotype 2" in [14]. This definition is based on the following equivalent formulation of cotype 2 of Banach spaces.
A Banach space X has cotype 2 if and only if there is a constant $C > 0$ such that for all $u : l_2^n \to X$, we have

$$\pi_2(u) \leq C \cdot l(u),$$

where $l(u)$ is the l-norm of u defined by

$$l(u) := \Big[\int_\Omega \Big\| \sum_{k=1}^n g_k(\omega) u e_k \Big\|_X^2 dP(\omega) \Big]^{\frac{1}{2}}$$

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for an i.i.d. gaussian variables $\{g_k\}$ *on a probability space* (Ω, P).
By trace duality, it is equivalent to

$$l^*(v) \leq C\pi_2(v)$$

for all $v : X \to l_2^n$*, where* $l^*(v)$ *is the adjoint* l*-norm of* v *defined by*

$$l^*(v) := \sup\{\mathrm{tr}(vu) : l(u) \leq 1\}.$$

G. Pisier defined that *an operator space* E *has "OH-cotype 2" if there exist a constant* $C > 0$ *such that for all* $v : E \to l_2^n$*, we have*

$$l^*(v) \leq C\pi_{2,oh}(v),$$

where $\pi_{2,oh}(v)$ *is the* $(2, oh)$*-summing norm defined by the infimum of* $C > 0$ *satisfying*

$$\sum_k \|ux_k\|^2 \leq C^2 \left\|\sum_k x_k \otimes \overline{x}_k\right\|_{E\otimes_{\min}\overline{E}}.$$

For more information about $(2, oh)$-summing operators, see [12].

Later, M. Junge ([4]) introduced "cotype $(2, R + C)$" by replacing $(2, oh)$-summing norm in the above definition into $(2, S_{R+C})$-summing norm in order to prove the following.

Every bounded linear map from $C(K)$ *into* $S_p(l_2)$ *is completely bounded*

for $1 \leq p \leq 2$.
He showed the above by proving that $S_p = S_p(l_2)$ $(1 \leq p \leq 2)$ have cotype $(2, R+C)$. Since S_p $(1 \leq p \leq 2)$ does not have OH-cotype 2 of G. Pisier, cotype $(2, R+C)$ is strictly weaker than OH-cotype 2.

The definition of G. Pisier and M. Junge deal $q = 2$ case only. For wider range of p and q, J. Garcia-Cuerva and J. Parcet introduced the notion of type p $(1 \leq p \leq 2)$ and cotype q $(2 \leq q \leq \infty)$ of operator spaces with respect to quantized orthonormal systems in [3] recently. The starting point of this definition is totally different from the previous ones.

Let (Ω, P) be a probability space and (Σ, d_Σ) be a pair of an index set Σ and a collection of natural numbers indexed by Σ, $d_\Sigma = \{d_\sigma \in \mathbb{N} : \sigma \in \Sigma\}$. The quantized Rademacher system $\mathcal{R}_\Sigma$ with parameter (Σ, d_Σ) is the collection of independent random matrices $r^\sigma = (r_{ij}^\sigma) : \Omega \to \mathcal{O}(d_\sigma)$ indexed by Σ, and the distribution of r^σ is exactly the normalized Haar measure on the orthogonal group $\mathcal{O}(d_\sigma)$. The quantized gaussian system $\mathcal{G}_\Sigma$ with parameter (Σ, d_Σ) is the collection of random matrices $g^\sigma = \frac{1}{\sqrt{d_\sigma}}(g_{ij}^\sigma) : \Omega \to M_{d_\sigma}$ indexed by Σ, where g_{ij}^σ's are i.i.d. gaussian random variables. We consider the following transforms as in the Banach space case:

$$\mathcal{F}_{\mathcal{R}_\Sigma}(f)(\sigma) = \int_\Omega f(\omega) r^\sigma(\omega)^* dP(\omega) \quad \text{and} \quad \mathcal{F}_{\mathcal{R}_\Sigma}^{-1}(A)(\omega) = \sum_{\sigma\in\Sigma} d_\sigma \mathrm{tr}(A^\sigma r^\sigma(\omega))$$

for appropriate $f : \Omega \to \mathbb{C}$ and $A \in \prod_{\sigma\in\Sigma} M_{d_\sigma}$.

For $1 \leq p \leq 2$ and $2 \leq q \leq \infty$, we say that an operator space E has $\mathcal{R}_\Sigma$-type p (resp. Banach $\mathcal{R}_\Sigma$-type p) if

$$\sup_{\text{finite}\,\Gamma\subseteq\Sigma} \left\|\mathcal{F}^{-1}_{\mathcal{R}_\Sigma} \otimes I_E\right\|_{cb(\mathcal{L}_p(\Gamma,E),L_{p'}(\Omega,E))} < \infty$$

$$(\text{resp. } \sup_{\text{finite}\,\Gamma\subseteq\Sigma} \left\|\mathcal{F}^{-1}_{\mathcal{R}_\Sigma} \otimes I_E\right\|_{\mathcal{L}_p(\Gamma,E)\to L_{p'}(\Omega,E)} < \infty)$$

and that E has $\mathcal{R}_\Sigma$-cotype q (resp. Banach $\mathcal{R}_\Sigma$-cotype q) if

$$\sup_{\text{finite}\,\Gamma\subseteq\Sigma} \left\|\mathcal{F}_{\mathcal{R}_\Sigma} \otimes I_E\right\|_{cb(L^\Gamma_{q'}(\Omega,E),\mathcal{L}_q(\Gamma,E))} < \infty,$$

$$(\text{resp. } \sup_{\text{finite}\,\Gamma\subseteq\Sigma} \left\|\mathcal{F}_{\mathcal{R}_\Sigma} \otimes I_E\right\|_{L^\Gamma_{q'}(\Omega,E)\to\mathcal{L}_q(\Gamma,E)} < \infty)$$

where $L^\Gamma_{q'}(\Omega, E)$ is the closed linear span of $\{r^\sigma_{ij} : \sigma \in \Gamma\} \otimes E$ in $L_{q'}(\Omega, E)$ and

$$\mathcal{L}_r(\Gamma, E) = \{A \in \prod_{\sigma\in\Gamma} M_{d_\sigma} \otimes E : \|A\|_{\mathcal{L}_r(\Gamma,E)} = \Big(\sum_{\sigma\in\Gamma} d_\sigma \|A^\sigma\|^r_{S^{d_\sigma}_r(E)}\Big)^{\frac{1}{r}} < \infty\}$$

for $1 \leq r < \infty$ and

$$\mathcal{L}_\infty(\Gamma, E) = \{A \in \prod_{\sigma\in\Gamma} M_{d_\sigma} \otimes E : \|A\|_{\mathcal{L}_\infty(\Gamma,E)} = \sup_{\sigma\in\Gamma} \|A^\sigma\|_{S^{d_\sigma}_\infty(E)} < \infty\}.$$

Note that $S^n_r(E)$ is a vector-valued Schatten class defined in [13]. We define $\mathcal{G}_\Sigma$-type p and cotype q similarly. In [2, 3] and [5], it is shown that l_p and S_p $(1 \leq p \leq \infty)$ has $\mathcal{R}_\Sigma$(resp. $\mathcal{G}_\Sigma$)-type $\min\{p, p'\}$ and $\mathcal{R}_\Sigma$(resp. $\mathcal{G}_\Sigma$)-cotype $\max\{p, p'\}$ and unlike in the Banach space case, these are 'sharp' in the sense that l_p and S_p do not have better (closer to 2) type and cotype. In particular, l_p and S_p $(1 \leq p \leq 2)$ does not have cotype 2 in this sense. This lack of cotype 2 spaces leads us to a weaker notion of type and cotype.

In this paper, we take a similar approach as in [3] to define OH-type and OH-cotype of operator spaces but the requirement will be somewhat weakened enough to include commutative L_1 spaces as OH-cotype 2 spaces. The reason why we use the same terminology 'OH-' with the one of G. Pisier is that we have the equivalence when $q = 2$. In Section 2, we will define OH-type and OH-cotype of operator spaces and develop basic theory. We will see how OH-type and OH-cotype is related to the type and cotype in [3]. In Section 3, we compute type and cotype of several concrete spaces. In Section 4, applications to completely (q, p)-summing maps are presented. This new class of mappings is defined by the same way in [13]. At the end of this paper, we observe that that our "OH-cotype 2" is equivalent to "OH-cotype 2" of G. Pisier.

Note that all Lebesgue spaces (commutative or noncommutative) are endowed with their natural operator space structure in the sense of G. Pisier ([13]).

2. Complete type and cotype of operator spaces

We use the Rademacher system (resp. Gaussian system) to define type and cotype (resp. Gaussian type and cotype) in the Banach space setting. In this paper, we also use the same system but with different indices. Let $\{r_{ij}\}$ (resp. $\{g_{ij}\}$) be an enumeration of the classical Rademacher system $\{r_i\}$ on $[0,1]$. (resp. a gaussian system $\{g_i\}$ on a probability space (Ω, P).) Now we define OH-type and OH-cotype using the following transforms:

$$\mathcal{F}_{\mathcal{R}} : f \mapsto \Big(\int_0^1 f(t) r_{ij}(t) dt \Big)_{i,j=1}^{\infty} \quad \text{and} \quad \mathcal{F}_{\mathcal{R}}^{-1} : (x_{ij}) \mapsto \sum_{i,j=1}^{\infty} r_{ij}(t) x_{ij}$$

for appropriate $f : [0,1] \to \mathbb{C}$ and $(x_{ij}) \in M_\infty$. We define $\mathcal{F}_{\mathcal{G}}$ and $\mathcal{F}_{\mathcal{G}}^{-1}$ similarly by replacing $\{r_{ij}\}$ into $\{g_{ij}\}$. Note that $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{G}}$ are complete isometries from S_2 onto Rad_2 (resp. $\mathcal{G}_2$), the closed linear span of $\{r_{ij}\}$ (resp. $\{g_{ij}\}$) in $L_2[0,1]$.

Definition 2.1. *Let E be an operator space.*

(1) *E is said to have* ***OH-type*** *p $(1 \le p \le 2)$ if*

$$\mathcal{F}_{\mathcal{R}}^{-1} \otimes I_E : S_p \otimes E \to L_2[0,1] \otimes E$$

extends to a bounded linear map from $S_p(E)$ into $L_2([0,1], E)$;

(2) *E is said to have* ***OH-cotype*** *q $(2 \le q \le \infty)$ if*

$$\mathcal{F}_{\mathcal{R}} \otimes I_E : Rad_2 \otimes E \subseteq L_2[0,1] \otimes E \to S_q \otimes E$$

extends to a bounded linear map from $Rad_2(E)$ into $S_q(E)$,

where $Rad_r(E)$ refers to the closed linear span of $\{r_{ij}\} \otimes E$ in $L_r([0,1], E)$ for $1 \le r < \infty$. We denote $T_p^o(E)$ and $C_q^o(E)$ for operator norms of $\mathcal{F}_{\mathcal{R}}^{-1} \otimes I_E$ and $\mathcal{F}_{\mathcal{R}} \otimes I_E$ respectively. Here, 'o' means operator space setting as in [13]. *The definition for* ***gaussian OH-type*** *and* ***gaussian OH-cotype*** *is similar. We use notations $GT_p^o(E)$ and $GC_q^o(E)$ in this case.*

Remark 2.2. (1) Note that we do not require $\mathcal{F}_{\mathcal{R}}^{-1} \otimes I_E$ and $\mathcal{F}_{\mathcal{R}} \otimes I_E$ to be completely bounded.

(2) Considering diagonals, it is trivial that every OH-type p (resp. OH-cotype q) space has type p (resp. cotype q) as a Banach space.

(3) By the classical Khinchine's inequality, we get equivalent definitions if we replace $L_2[0,1]$ into $L_r[0,1]$ and Rad_2 into Rad_r for any $1 \le r < \infty$.

We first check the trivial cases.

Proposition 2.3. *Every operator space has OH-type 1 and OH-cotype ∞.*

Proof. Let E be an arbitrary operator space. Observe that $\mathcal{F}_{\mathcal{R}}^{-1}$ factorizes as follows:

$$\mathcal{F}_{\mathcal{R}}^{-1} : S_1 \xrightarrow{j_{1,2}} S_2 \xrightarrow{\mathcal{F}_{\mathcal{R}}^{-1}} Rad_2 \subseteq L_2[0,1] \xrightarrow{i_{2,1}} L_1[0,1],$$

where $j_{1,2}$ and $i_{2,1}$ are the corresponding formal identities. Thus, $\mathcal{F}_{\mathcal{R}}^{-1}$ is a complete contraction from S_1 into $L_1[0,1]$, and so is

$$\mathcal{F}_{\mathcal{R}}^{-1} \otimes I_E : S_1(E) = S_1 \otimes_{\max} E \to L_1[0,1] \otimes_{\max} E = L_1([0,1], E).$$

This implies that E has OH-type 1 by (3) of Remark 2.2.

For the cotype case, we consider the following factorization:

$$\mathcal{F}_{\mathcal{R}} : Rad_1 \xrightarrow{\varphi_{1,2}} Rad_2 \xrightarrow{\mathcal{F}_{\mathcal{R}}} S_2 \xrightarrow{j_{2,\infty}} S_\infty,$$

where $\varphi_{1,2}$ and $j_{2,\infty}$ are the corresponding formal identities. By the classical Khinchine's inequality $\varphi_{1,2}$ is bounded, so that we can consider its extension $\tilde{\varphi}_{1,2}$ to $L_1[0,1]$ by the Hahn-Banach theorem. Then the maximal operator space structure of $L_1[0,1]$ assures that $\tilde{\varphi}_{1,2}$ is completely bounded, which means that $\varphi_{1,2}$ is completely bounded. Thus, $\mathcal{F}_{\mathcal{R}}$ is a completely bounded map from Rad_1 into S_∞, and so is

$$\mathcal{F}_{\mathcal{R}} \otimes I_E : Rad_1 \otimes_{\min} E \to S_\infty \otimes_{\min} E = S_\infty(E).$$

Since $L_1(E) = L_1 \otimes_{\max} E$ is embedded into $L_1 \otimes_{\min} E$ by a natural complete contraction, we have $\mathcal{F}_{\mathcal{R}} \otimes I_E : Rad_1(E) \to S_\infty(E)$ is (completely) bounded. □

As in the Banach space case, gaussian OH-type and gaussian OH-cotype are equivalent to OH-type and OH-cotype respectively.

Proposition 2.4. *Let $1 \le p \le 2$, $2 \le q \le \infty$ and E be an operator space.*

(1) *E has OH-type p if and only if it has gaussian OH-type p;*
(2) *E has OH-cotype q if and only if it has gaussian OH-cotype q.*

Proof. By the estimations in Proposition 12.11 and Theorem 12.27 in [1], they are equivalent when E has finite cotype as a Banach space. When E has no finite cotype, E contains isomorphic copies of l_∞^n's uniformly, which means that E has only trivial OH-type, OH-cotype, gaussian OH-type and gaussian OH-cotype. □

We can reformulate gaussian OH-type 2 and gaussian OH-cotype 2 in the style of [3], which leads to a operator space version of Kwapien's theorem.

Proposition 2.5. *Let E be an operator space and $\mathcal{G}_\Sigma$ be the quantized gaussian system with parameter (Σ, d_Σ). Suppose that d_Σ is unbounded.*

(1) *E has gaussian OH-type 2 if and only if it has Banach $\mathcal{G}_\Sigma$-type 2;*
(2) *E has gaussian OH-cotype 2 if and only if it has Banach $\mathcal{G}_\Sigma$-cotype 2.*

Proof. Let Γ be a finite subset of Σ and $A(= (A^\sigma)) \in \Pi_{\sigma\in\Gamma} M_{d_\sigma} \otimes E$. If we set $B = \oplus_{\sigma\in\Gamma}\sqrt{d_\sigma}A^\sigma$, then we get

$$\mathcal{F}_{\mathcal{G}_\Sigma}^{-1}(A)(\omega) = \sum_{\sigma\in\Gamma} d_\sigma \mathrm{tr}(A^\sigma g^\sigma(\omega)) = \sum_{\sigma\in\Gamma} \sqrt{d_\sigma}\mathrm{tr}(A^\sigma(g_{ij}^\sigma(\omega))) = \mathcal{F}_{\mathcal{G}}^{-1}(B)(\omega)$$

and

$$\|A\|_{\mathcal{L}_2(\Gamma,E)} = \Big[\sum_{\sigma\in\Gamma} d_\sigma \|A^\sigma\|_{S_2^\sigma(E)}^2\Big]^{\frac{1}{2}} = \|B\|_{S_2(E)}.$$

Thus, we get the desired result. □

By a slight modification of the proof of Theorem 5.6 in [3], we get the following operator space version of Kwapien's theorem. Recall that $\Gamma_{oh}(E, F)$ refers to the collection of linear maps from E into F factorizing through a operator Hilbert space ([13]).

Corollary 2.6. *Let E be an operator space with OH-type 2 and F be an operator space with OH-cotype 2. Then $\mathcal{L}(E, F) = \Gamma_{oh}(E, F)$. In particular, every operator space with OH-type 2 and OH-cotype 2 is completely isomorphic to an operator Hilbert space.*

OH-type and OH-cotype have a duality property as in the Banach space case. The proof is almost the same in the Banach space case so that we omit it. See Proposition 11.10 and 13.17 in [1].

Proposition 2.7. *Let E be an operator space.*

(1) *If E has OH-type p, then E^* has OH-cotype p';*
(2) *If E has OH-cotype p' and is K-convex as a Banach space, then E^* has OH-type p.*

It is interesting that we can guarantee a full duality by a Banach space property only. The next proposition is another example that we can obtain a nice formulation with a Banach space property only. See p.332 in [1] for the proof of Banach space case. With slight modification of it, we can get the following.

Proposition 2.8. *Let E be an operator space and F be a closed subspace of E.*

(1) *If E has OH-type p, then E/F has OH-type p;*
(2) *If E has OH-cotype q and F is K-convex as a Banach space, then E^* has OH-cotype q.*

3. Examples

First, we compute OH-type and OH-cotype of Lebesgue spaces. We have the same results as in the Banach space case for commutative Lebesgue spaces. However, for noncommutative spaces, we only obtain a partial similarity.

Proposition 3.1. *Let $(\mathcal{M}, \mu)$ be a measure space, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.*

(1) *$L_p(\mu)$ has OH-type $\min\{p, 2\}$ and OH-cotype $\max\{2, p'\}$ and cannot have better (closer to 2) OH-type and OH-cotype.*
(2) *S_p has OH-type $\min\{p, p'\}$ and OH-cotype $\max\{p, p'\}$. S_p ($1 \leq p \leq 2$) cannot have better OH-type and S_q ($2 \leq q < \infty$) cannot have better OH-type.*
(3) *$L_\infty(\mu)$ and S_∞ cannot have nontrivial OH-type and finite OH-cotype.*

Proof. It is trivial that $L_2(\mu)$ and S_2 have OH-type 2 and OH-cotype 2. Also, $L_1(\mu)$, L_∞, S_1 and S_∞ have OH-type 1 and OH-cotype ∞. Then by the complex interpolation, $L_p(\mu)$ and S_p has OH-type $\min\{p,p'\}$ and OH-cotype $\max\{p,p'\}$.

For $1 \leq p \leq 2$ and $(x_{ij}) \in S_2(L_p(\mu))$, we have

$$\begin{aligned}
\Big\|\sum_{i,j} x_{ij} r_{ij}(\cdot)\Big\|_{L_2([0,1],L_p(\mu))} &\leq \Big\|\sum_{i,j} x_{ij} r_{ij}(\cdot)\Big\|_{L_p([0,1],L_p(\mu))} \\
&= \Big[\int_0^1 \int_{\mathcal{M}} \Big|\sum_{i,j} x_{ij}(s) r_{ij}(t)\Big|^p d\mu(s)dt\Big]^{\frac{1}{p}} \\
&= \Big[\int_{\mathcal{M}} \int_0^1 \Big|\sum_{i,j} x_{ij}(s) r_{ij}(t)\Big|^p dtd\mu(s)\Big]^{\frac{1}{p}} \\
&\leq B_p\Big[\int_{\mathcal{M}} \Big(\int_0^1 \Big|\sum_{i,j} x_{ij}(s) r_{ij}(t)\Big|^2 dt\Big)^{\frac{p}{2}} d\mu(s)\Big]^{\frac{1}{p}} \\
&= B_p\Big[\int_{\mathcal{M}} \big\|(x_{ij}^k)\big\|_{S_2}^p d\mu(s)\Big]^{\frac{1}{p}} \\
&= B_p \left\|(x_{ij}(s))\right\|_{L_p(\mu,S_2)} \leq B_p \left\|(x_{ij}(s))\right\|_{S_2(L_p(\mu))},
\end{aligned}$$

where B_p is the constant in Khinchine's inequality such that

$$\Big\|\sum \alpha_n r_n(\cdot)\Big\|_{L_p[0,1]} \leq B_p \left\|(\alpha_n)\right\|_{l_2}.$$

The last line is by Corollary 1.10 in [13]. Thus $L_p(\mu)$ ($1 \leq p \leq 2$) has OH-cotype 2 , and similarly we can show that $L_p(\mu)$ ($2 \leq p < \infty$) has OH-type 2.

(3) and the remaining statements in (1) and (2) are trivial by Remark 2.2.

□

Next we consider OH-type and OH-cotype of several homogeneous Hilbertian spaces.

Theorem 3.2. *R and C do not have OH-type p nor OH-cotype q for $\frac{4}{3} < p \leq 2 \leq q < 4$.*

Proof. Note that R and R_n, n-dimensional subspace of R spanned by first n bases, are isometric to OH and OH_n respectively. Thus, we have that

$$\begin{aligned}
R \text{ has OH-cotype } q &\Leftrightarrow \mathcal{F}_{\mathcal{R}} \otimes I_R : Rad_2(R) \to S_q(R) \text{ is bounded} \\
&\Leftrightarrow \mathcal{F}_{\mathcal{R}} \otimes I_R : Rad_2(OH) \to S_q(R) \text{ is bounded} \\
&\Leftrightarrow I_{S_2} \otimes id : S_2(OH) \to S_q(R) \text{ is bounded}
\end{aligned}$$

$$\Rightarrow I_{S_2^n} \otimes id_n : S_2^n(OH_n) \to S_q^n(R_n) \text{ is uniformly bounded for all } n \in \mathbb{N}.$$

By the way, we have $\|id_n : OH_n \to R_n\|_{cb} = n^{\frac{1}{4}}$. Furthermore, we can obtain cb-norm of id_n at the n-th matrix level, that is,

$$\|I_{M_n} \otimes id_n : M_n(OH_n) \to M_n(R_n)\| = n^{\frac{1}{4}}.$$

Then by Lemma 1.7 of [13], we have

$$\left\|I_{S_2^n} \otimes id_n : S_2^n(OH_n) \to S_2^n(R_n)\right\| = n^{\frac{1}{4}}.$$

Now, by Lemma 2.9 in [8], we have

$$\|(x_{ij})\|_{S_2^n(R_n)} \leq n^{\frac{1}{2}-\frac{1}{q}} \|(x_{ij})\|_{S_q^n(R_n)} \leq n^{\frac{1}{2}-\frac{1}{q}} C_q^o(R) \|(x_{ij})\|_{S_2^n(R_n)}. \tag{3.1}$$

Thus, we get $C_q^o(R) \geq n^{\frac{1}{q}-\frac{1}{4}}$, which means R does not have OH-cotype q for $2 \leq q < 4$. We can show that C does not have OH-cotype q for $2 \leq q < 4$ similarly. OH-type case is obtained by the duality (Proposition 2.7). □

Since we have completely bounded embeddings $\max(l_2) \hookrightarrow \Phi$, $\Phi \hookrightarrow R \cap C$, $R \cap C \hookrightarrow R$ and $R \cap C \hookrightarrow OH$, we can estimate OH-type and OH-cotype of $\max(l_2)$, Φ and $R \cap C$ by the above theorem, where Φ is the operator space spanned the operators satisfying 'CAR' conditions (See Section 9.3 in [10]).

Corollary 3.3. *$\max(l_2)$, Φ and $R\cap C$ have OH-type 2 but do not have OH-cotype q for $2 \leq q < 4$.*

By the same observation as in Theorem 3.2, we can estimate OH-cotype of $R[p] = [R, C]_{\frac{1}{p}}$ and $C[p] = [C, R]_{\frac{1}{p}}$.

Theorem 3.4. *For $p \neq 2$, $R[p]$ and $C[p]$ do not have OH-type r nor OH-cotype q for $\min\{\frac{4p}{p+2}, \frac{4p}{3p-2}\} < r \leq 2 \leq q < \max\{\frac{4p}{p+2}, \frac{4p}{3p-2}\}$.*

Proof. Since we have

$$\left\|\sum_i e_{1i} \otimes e_{1i}\right\|_{\mathcal{K}\otimes_{\min} R[p]} = \sup\left\{\left\|\sum_i e_{1i} z e_{1i}^*\right\|^{\frac{1}{2}} : \|z\|_{S_p} \leq 1\right\},$$

$z = n^{-\frac{1}{p}} I_n$ gives the following estimation:

$$\|I_{M_n} \otimes id_n : M_n(OH_n) \to M_n(R_n[p])\| \geq n^{\frac{1}{4}-\frac{1}{2p}}.$$

Combining this with (3.1), we get $C_q^o(R[p]) \geq n^{\frac{1}{q}-\frac{1}{4}-\frac{1}{2p}} = n^{\frac{1}{q}-\frac{p+2}{4p}}$. Similarly, we can get $C_q^o(C[p]) \geq n^{\frac{1}{q}-\frac{p+2}{4p}}$. Since $R[p] = C[p']$ for the conjugate exponent p' of p, we also have $C_q^o(R[p]) \geq n^{\frac{1}{q}-\frac{p'+2}{4p'}} = n^{\frac{1}{q}-\frac{3p-2}{4p}}$. OH-type case is obtained by the duality. □

Since $R[p]$ is a subspace of S_p, we have the following as a corollary.

Corollary 3.5. *For $1 \leq p < 2$, S_p does not have OH-cotype q for $2 \leq q < \frac{4p}{3p-2}$, and for $2 \leq p < \infty$, S_p does not have OH-type r for $\frac{4p}{p+2} < r \leq 2$.*

4. Completely (q,p)-summing maps and OH-cotype

As an operator space version of "absolutely p-summing operators", G. Pisier introduced "completely p-summing maps" in [13] as follows:
A linear map between operator spaces $u : E \to F$ is called "completely p-summing" for $1 \leq p < \infty$ if $I_{S_p} \otimes u : S_p \otimes_{\min} E \to S_p(F)$ is a bounded map. We denote $\pi_p^o(u)$ for the operator norm of $I_{S_p} \otimes u$ and $\Pi_p^o(E,F)$ for the collection of all such operators from E into F. Now we define an operator space version of "absolutely (q,p)-summing operators" as follows.

Definition 4.1. *A linear map between operator spaces $u : E \to F$ is called "completely (q,p)-summing" for $1 \leq p \leq q < \infty$ if*

$$I_{p,q} \otimes u : S_p \otimes_{\min} E \to S_q(F)$$

is a bounded map, where $I_{p,q}$ is the formal identity from S_p into S_q. We denote $\pi_{q,p}^o(u)$ for the operator norm of $I_{p,q} \otimes u$ and $\Pi_{q,p}^o(E,F)$ for the collection of all such operators from E into F.

Remark 4.2. Note that being "completely (q,p)-summing" does not guarantee complete boundedness if $p < q$ unlike "completely p-summing" property.

Completely summing properties of a linear map between operator spaces are affected by OH-cotype of the target space.

Theorem 4.3. *Let E and F be operator spaces. Suppose that F has OH-cotype q $(2 \leq q < \infty)$. Then we have*

$$\Pi_r^o(E,F) \subseteq \Pi_{q,2}^o(E,F) \text{ and } \Pi_r(E,F) \subseteq \Pi_{q,2}^o(E,F)$$

for $q < r < \infty$.

Proof. Let $E \subseteq \mathcal{L}(H)$ for a Hilbert space H. For $u \in \Pi_r^o(E,F)$ and $(x_{ij}) \in M_\infty(E)$, we have by Theorem 5.1 of [13] that

$$\begin{aligned}\|(ux_{ij})\|_{S_q(F)} &\leq C_q^o(F)\Big[\int_0^1 \Big\|u(\sum_{i,j} r_{ij}(t)x_{ij})\Big\|_F^2 dt\Big]^{\frac{1}{2}} \\ &\leq C_q^o(F)\pi_r^o(u)\Big[\int_0^1 \lim_{\mathcal{U}} \Big\|a_\alpha(\sum_{i,j} r_{ij}(t)x_{ij})b_\alpha\Big\|_{S_r(H)}^2 dt\Big]^{\frac{1}{2}}\end{aligned}$$

for some ultrafilter $\mathcal{U}$ over I and $(a_\alpha)_{\alpha\in I}, (b_\alpha)_{\alpha \in I} \subseteq B_{S_{2p}(H)}$.
By Fatou's lemma and the fact that $S_2(H)$ embeds into $S_r(H)$, we have that

$$\begin{aligned}\|(ux_{ij})\|_{S_q(F)} &\leq C_q^o(F)\pi_r^o(u) \lim_{\mathcal{U}} \|(a_\alpha x_{ij} b_\alpha)\|_{S_2(l_2\otimes H)} \\ &\leq C_q^o(F)\pi_r^o(u)\,\|(x_{ij})\|_{S_2\otimes_{\min} E}.\end{aligned}$$

The last line is by Theorem 5.3 in [13]. This means $\Pi_r^o(E,F) \subseteq \Pi_{q,2}^o(E,F)$.
For $u \in \Pi_r(E,F)$, we have by the same calculation in the proof of Theorem 11.13 in [1] that

$$\|(ux_{ij})\|_{S_q(F)} \leq C^o_q(F)\Big[\int_0^1 \Big\|u(\sum_{i,j} r_{ij}(t)x_{ij})\Big\|^2_F dt\Big]^{\frac{1}{2}}$$
$$\leq B_r C^o_q(F)\pi_r(u)\,\|(x_{ij})\|_{S_2\otimes_\lambda E}$$
$$\leq B_r C^o_q(F)\pi_r(u)\,\|(x_{ij})\|_{S_2\otimes_{\min} E},$$

where $\otimes_\lambda$ refers to the injective tensor product in the category of Banach spaces.

□

Much more can be said when $E = C(K)$ for some compact set K.

Theorem 4.4. *Let F be operator spaces with OH-cotype q $(2 \leq q < \infty)$. Then we have*

$$\mathcal{L}(C(K), F) \subseteq \Pi^o_{q,2}(C(K), F).$$

Proof. Since $C(K)$ has the minimal operator space structure, $S_2 \otimes_{\min} C(K) \simeq C(K)(S_2)$ isometrically. Thus, by considering diagonals, we have

$$\Pi^o_{q,2}(C(K), F) \subseteq \Pi_{q,2}(C(K), F).$$

Let $r > q$, then by Theorem 10.9 in [1] we have

$$\Pi^o_{q,2}(C(K), F) \subseteq \Pi_{q,1}(C(K), F) \subseteq \Pi_r(C(K), F),$$

which means by Lemma 11.15 in [1] that

$$\pi_{2r} \leq \|u\|^{\frac{1}{2}}\,\pi_r(u)^{\frac{1}{2}} \leq \|u\|^{\frac{1}{2}}\,\pi^o_{q,2}(u)^{\frac{1}{2}}. \tag{4.1}$$

Furthermore, by Theorem 4.3, we have

$$\pi^o_{q,2}(u) \leq B_4 C^o_q(F)\pi_{2r}(u). \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\pi^o_{q,2}(u) \leq B_4^2 C^o_q(F)^2\,\|u\|.$$

□

OH-cotype conditions in Theorem 4.4 are essential. In order to give an counterexample of Theorem 4.4, we consider the following factorization theorem. The next one is one of the special cases of Theorem 5.1 in [13] and the proof takes intermediate way between the Banach space case and [13].

Lemma 1. *Let F be an operator space and $u \in \Pi^o_p(C(K), F)$ with $1 \leq p < \infty$. Denote $\pi^{o,n}_p(u)$ be the infimum of constants $C > 0$ satisfying*

$$\|(ux_{ij})\|_{S^n_p(F)} \leq C\,\|(x_{ij})\|_{S^n_p\otimes_{\min} C(K)}$$

for all $(x_{ij}) \in S^n_p \otimes C(K)$. Then we have

$$\|(ux_{ij})\|_{S^n_p(F)} \leq \pi^{o,n}_p(u)\,\|(x_{ij})\|_{L_p(\mu, S^n_p)}$$

for some probability μ on K. Consequently, we have the following factorization of u.

$$u : C(K) \xrightarrow{j_p} L_p(\mu) \xrightarrow{\tilde{u}} F,$$

where j_p is the formal identity and $\|I_{M_n} \otimes \tilde{u} : M_n(L_p(\mu)) \to M_n(F)\| = \pi_p^{o,n}(u)$.

Proof. Consider the set $\mathcal{F}$ of all function's on K of the form

$$f(a) = \sum_m \Big[\pi_p^{o,n}(u)^p \left\|(x_{ij}^m(a))\right\|_{S_p^n}^p - \left\|(ux_{ij}^m)\right\|_{S_p^n(F)}^p\Big],$$

for finite collection of $\{(x_{ij}^m)\} \subseteq S_p^n \otimes C(K)$. Since

$$\left\|(x_{ij}^m)\right\|_{S_p^n \otimes_{\min} C(K)} = \sup_{a\in K} \|x_{ij}(a)\|_{S_p^n},$$

we have $\sup_{a\in K} f(a) \geq 0$. Thus, $\mathcal{F}$ is a convex cone in $C(K)$, which is disjoint with an open convex set $\mathcal{A} = \{\varphi \in C(K) : \sup_{a\in K} \varphi(a) < 0\}$. Then by the Hahn-Banach theorem there is a probability μ on K such that $\langle \mu, f\rangle \geq 0 > \langle \mu, \varphi\rangle$ for all $f \in \mathcal{F}$ and $\varphi \in \mathcal{A}$. Thus, we have for any $(x_{ij}^m) \in S_p^n \otimes C(K)$

$$\begin{aligned} \|(ux_{ij})\|_{S_q^n(F)} &\leq \pi_p^{o,n}(u)\Big[\int_K \|(x_{ij}(a))\|_{S_p^n}^p \, d\mu(a)\Big]^{\frac{1}{p}} \\ &= \pi_p^{o,n}(u) \left\|(x_{ij})\right\|_{L_p(\mu, S_p^n)}. \end{aligned}$$

□

Theorem 4.5.

$$\mathcal{L}(l_\infty, R) = CB(l_\infty, R) \subsetneqq \Pi_{q,2}^o(l_\infty, R)$$

for $2 \leq q < 4$.

Proof. The first equality comes from Corollary 4.2.8 of [4] and the fact that R is a subspace of S_1. Let $u_n : c_o \to R, e_i \mapsto \lambda_i e_{1i}$, $\lambda_i = 1$ for $1 \leq i \leq n$ and $\lambda_i = 0$ for $i > n$. Then $u_n \in \Pi_2^o(l_\infty^n, R_n)$ and $\|u_n\| = \sqrt{n}$. By Lemma 1, there is a probability μ on $\{1, 2, \cdots, n\}$ such that

$$u_n : l_\infty^n \xrightarrow{i_2} l_2^n(\mu) \xrightarrow{\tilde{u}_n} R_n$$

and $\pi_2^{o,n}(u_n) = \|I_{M_n} \otimes \tilde{u}_n : M_n(l_2(\mu)) \to M_n(R_n)\|$ for the formal identity i_2. Note that $\mu_i = \mu(\{i\}) > 0$ for all i. Now observe that

$$\left\|\sum_{i=1}^n \mu_i^{-\frac{1}{2}} e_{1i} \otimes e_{1i}\right\|_{M_n \otimes_{\min} R_n} = \left\|\sum_{i=1}^n \mu_i^{-1} e_{1i} e_{1i}^*\right\|^{\frac{1}{2}} = \Big(\sum_{i=1}^n \mu_i^{-1}\Big)^{\frac{1}{2}}$$

and

$$\left\| \sum_{i=1}^n \mu_i^{-\frac{1}{2}} e_{1i} \otimes e_{1i} \right\|_{M_n \otimes_{\min} l_2^n(\mu)} = \left\| \sum_{i=1}^n e_{1i} \otimes e_{1i} \right\|_{M_n \otimes_{\min} OH_n} = \left\| \sum_{i=1}^n e_{1i} \otimes \overline{e}_{1i} \right\|_{\min}^{\frac{1}{2}} = n^{\frac{1}{4}}.$$

Since $\left(\sum_{i=1}^n \mu_i^{-1} \right)^{\frac{1}{2}}$ has minimum with the constraint $\sum_{i=1}^n \mu_i = 1, u_i > 0$ at $\mu_1 = \cdots = \mu_n = n^{-1}$ we have by combining the above two

$$\frac{\pi_2^{o,n}(u_n)}{\|u_n\|} \geq n^{\frac{1}{4}}. \tag{4.3}$$

By the way, for any $(y_{ij}) \in S_2^n \otimes l_\infty^n$ we have

$$\begin{aligned} \|(u_n y_{ij})\|_{S_2^n(R_n)} &\leq n^{\frac{1}{2}-\frac{1}{q}} \|(u_n y_{ij})\|_{S_2^n(R_n)} \\ & n^{\frac{1}{2}-\frac{1}{q}} \pi_{q,2}^o(u_n) \|(y_{ij})\|_{S_2^n \otimes_{\min} l_\infty^n}, \end{aligned}$$

which means

$$\pi_2^{o,n}(u_n) \leq n^{\frac{1}{2}-\frac{1}{q}} \pi_{q,2}^o(u_n). \tag{4.4}$$

By combining (4.3) and (4.4), $\frac{\pi_{q,2}^o(u_n)}{\|u_n\|}$ cannot be bounded for $2 \leq q < 4$, which leads us to our desired result. □

Remark 4.6. Actually, we have $\pi_2^o(u_n) = n^{\frac{3}{4}}$ for u_n in the proof of Theorem 4.5. By (4.3), we have

$$n^{\frac{3}{4}} \leq \pi_2^{o,n}(u_n) \leq \pi_2^o(u_n).$$

For the converse inequality, consider the following factorization:

$$u_n : l_\infty^n \xrightarrow{i_2} l_2^n(\mu) \xrightarrow{v} OH_n \xrightarrow{\sqrt{n}\cdot id_n} R_n$$

for $\mu_1 = \cdots = \mu_n = n^{-1}$ and $v(e_i) = n^{-\frac{1}{2}} e_i$. Since v is a complete isometry, we have

$$\pi_2^o(u_n) \leq \pi_2^o(i_2) \|v\|_{cb} \left\| \sqrt{n} \cdot id_n \right\|_{cb} = n^{\frac{3}{4}}.$$

An equivalent formulation of OH-cotype can be drawn with the help of (q,p)-summing property.

Theorem 4.7. *Let E be an operator space. E has (gaussian) OH-cotype q ($2 \leq q < \infty$) if and only if there exist a constant $C > 0$ such that*

$$\pi_{q,2}^o(u) \leq C \cdot l(u)$$

for every $u : l_2^n \to E$ and $n \in \mathbb{N}$. When E has (gaussian) OH-cotype q, $GC_q^o(E)$ is equal to the infimum of such C.

Proof. (⇒) Let $u : l_2^n \to E$ and $(x_{ij}) \in S_q(l_2^n)$. If we set $v : S_2 \to l_2^n, e_{ij} \mapsto x_{ij}$, then we have $\|v\| = \|v\|_{cb} = \|(x_{ij})\|_{S_2 \otimes_{\min} l_2^n}$. Now we have by (12.5) in [15]

$$\begin{aligned} \|(ux_{ij})\|_{S_q(E)} = \|(uve_{ij})\|_{S_q(E)} &\leq GC_q^o(E) \left\| \sum_{i,j} g_{ij}(\cdot) uve_{ij} \right\|_{L_2(\Omega,E)} \\ &= GC_q^o(E) l(uv) \\ &\leq GC_q^o(E) l(u) \|v\| \\ &= GC_q^o(E) l(u) \|(x_{ij})\|_{S_2 \otimes_{\min} l_2^n} . \end{aligned}$$

(⇐) For any $(x_{ij}) \in S_q(E)$, we consider $u : S_2 \to E, e_{ij} \mapsto x_{ij}$. Then we have

$$\begin{aligned} \|(x_{ij})\|_{S_q(E)} &\leq \pi_{q,2}^o \|(e_{ij})\|_{S_2 \otimes_{\min} E} \\ &\leq C \cdot l(u) = C \left\| \sum_{i,j} g_{ij}(\cdot) uve_{ij} \right\|_{L_2(\Omega,E)} . \end{aligned}$$

Thus, E has gaussian OH-cotype q with $GC_q^o(E) \leq C$.

□

We close this paper by checking that the OH-cotype 2 in this paper coincide with the OH-cotype 2 of G. Pisier. It can be achieved by the following lemma about trace duality of π_2^o-norm.

Lemma 2. *For any operator space E and operators $u : l_2^n \to E$ and $v : E \to l_2^n$, we have*

$$|\mathrm{tr}(vu)| \leq \pi_2^o(v) \pi_2^o(u).$$

Proof. Let $l_2^n \subseteq \mathcal{L}(H)$ for a Hilbert space H. By Proposition 6.1 in [13], there are $V : \mathcal{L}(H) \to OH(I)$ and $T : OH(I) \to E$ such that $u = TV|_{l_2^n}$ with $\pi_2^o(V) \leq \pi_2^o(u)$ and $\|T\|_{cb} \leq 1$. Then we have

$$vu : l_2^n \xrightarrow{V|_{l_2^n}} OH(I) \xrightarrow{T} E \xrightarrow{v} l_2^n$$

and by Proposition 6.3 in [13]

$$\|vT\|_{HS} = \pi_2^o(vT) \leq \|T\|_{cb} \pi_2^o(v).$$

Thus, we have

$$\begin{aligned} |\mathrm{tr}(vu)| &\leq \|vT\|_{HS} \left\| V|_{l_2^n} \right\|_{HS} \\ &= \|vT\|_{HS} \pi_2^o(V|_{l_2^n}) \\ &\leq \|vT\|_{HS} \pi_2^o(V) \leq \pi_2^o(v) \pi_2^o(u). \end{aligned}$$

□

Corollary 4.8. *The OH-cotype 2 in this paper coincide with the OH-cotype 2 of G. Pisier.*

Proof. By Proposition 6.2 in [13], we have $\pi_2^o(v) = \pi_{2,oh}(v)$ for any $v : E \to l_2^n$. Thus we get the desired conclusion by Lemma 2 and Theorem 4.7. □

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