A new family of distance-regular graphs with unbounded diameter

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Abstract

We construct distance-regular graphs with the same - classical - parameters as the Grassmann graphs on the e-dimensional subspaces of a $(2e+1)$-dimensional space over an arbitrary finite field. This provides the first known family of non-transitive distance-regular graphs with unbounded diameter.

1 Introduction

A simple connected graph is called distance-regular with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ if it has diameter d, and for any two vertices x, y at distance i the number of neighbours of y at distance $i+1$ from x equals b_i, and the number of neighbours of y at distance $i-1$ from x equals c_i, for all $i = 0, \ldots, d$ (where for convenience $c_0 = b_d = 0$). Distance-regular graphs play an important role in algebraic combinatorics because of the relation to design theory, coding theory, finite and Euclidean geometry, and group theory (cf. [2]).

Let q be a prime power, and let n and e be integers such that $\frac{n}{2} \geq e \geq 2$. The Grassmann graph $J_q(n, e)$ is the graph on the e-dimensional subspaces of an n-dimensional vector space over the finite field $GF(q)$, where two e-dimensional subspaces are adjacent whenever they intersect in an $(e-1)$-dimensional subspace. The Grassmann graphs are distance-regular graphs with diameter e and intersection array $\{b_0, b_1, \ldots, b_{e-1}; c_1, c_2, \ldots, c_e\}$ with

$$b_i = q^{2i+1} \begin{bmatrix} e-i \\ 1 \end{bmatrix} \begin{bmatrix} n-e-i \\ 1 \end{bmatrix} \text{ and } c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2,$$

where $\left[{n \atop m} \right] = \frac{(q^n-1)\cdots(q^{n-m+1}-1)}{(q^m-1)\cdots(q-1)}$ is the number of m-dimensional subspaces of an n-dimensional space over $GF(q)$ (cf. [2, pp. 268-269]).

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Metsch [5] showed that the Grassmann graphs are uniquely determined as distance-regular graphs with above parameters if $e \neq 2, \frac{n}{2}$, or $\frac{n-1}{2}$ (for all q) and $(e,q) \neq (\frac{n-2}{2}, 2), (\frac{n-2}{2}, 3)$, or $(\frac{n-3}{2}, 2)$. For $e = 2, n \geq 4$ the Grassmann graphs are in general not determined by their parameters, as the line graph of a 2-$(\left[{n \atop 1}\right], q+1, 1)$ design has the same parameters. Here we shall present new distance-regular graphs with the parameters of the Grassmann graphs for $n = 2e+1, e \geq 2$. This will provide the first known family of non-transitive distance-regular graphs with unbounded diameter. Indeed, besides the families of distance-transitive graphs (among which is the family of Grassmann graphs), the only known families of distance-regular graphs with unbounded diameter are the quadratic forms graphs, the Doob graphs (pseudo Hamming graphs), the Hemmeter graphs (pseudo $D_m(q)$ graphs), and the Ustimenko graphs (halved Hemmeter graphs) (cf. [2, p. 195]), and these are all transitive. It is specifically the non-transitivity which indicates that the solution of Bannai's problem (cf. [1, p. 293], [2, p. viii]) of determining all Q-polynomial distance-regular graphs with large diameter may be further away than expected. Another property of the new graphs is that the corresponding Terwilliger algebras (cf. [6] and the references therein) depend on the base points. No other known distance-regular graph with large diameter has this property (since these graphs are all transitive).

2 A partial linear space and its line graph

The origin of the discovery of the new distance-regular graphs lies in the construction of cospectral graphs of the Doubled Grassmann graphs (cf. [3]). For $n = 2e+1$, the incidence structure of the e-dimensional subspaces versus the $(e+1)$-dimensional subspaces, where incidence is symmetrized containment, is a partial linear space which has as point graph $J_q(2e+1, e)$, as line graph $J_q(2e+1, e+1)$ (which is isomorphic to $J_q(2e+1, e)$ through orthogonal complementing), and as incidence graph the Doubled Grassmann graph. By adjusting this partial linear space (in the right manner) in such a way that the point graph doesn't change, we are able to obtain as line graph a new distance-regular graph. The incidence graph of the partial linear space is cospectral with the Doubled Grassmann graph, but it is not distance-regular (cf. [3]).

The partial linear space is defined as follows. Let V be a $(2e+1)$-dimensional vector space over $GF(q)$, and fix a hyperplane H in V. The points are the e-dimensional subspaces of V, and there are two types of lines. First, we have the $(e+1)$-dimensional subspaces of V that are not contained in H; and a line of this type is incident to its e-dimensional subspaces. Second, we have the $(e-1)$-dimensional subspaces of H; a line of this type is incident to the e-dimensional subspaces of H that contain it. It is easy to check that this defines a partial linear space having as many ($\left[{2e+1 \atop e}\right]$) points as lines, such that each line is incident to $\left[{e+1 \atop 1}\right]$ points, and the other way around. Moreover, two points are collinear if and only if they intersect in an $(e-1)$-dimensional subspace. Thus the point graph of the partial linear space is the Grassmann graph $J_q(2e+1, e)$. If N is the point-line incidence matrix of the partial linear space, then $NN^T - \left[{e+1 \atop 1}\right]I$ is the adjacency matrix of the point graph. Since N^TN and NN^T have the same spectrum, and $N^TN - \left[{e+1 \atop 1}\right]I$ is the adjacency matrix of the line graph, it follows that the line graph has the same spectrum as the Grassmann graph. This leads to the following.

Theorem. *Let q be a prime power, and let $e \geq 2$ be an integer. Let V be a $(2e+1)$-dimensional vector space over $GF(q)$, and let H be a hyperplane in V. Let G be the graph whose vertices are the $(e+1)$-dimensional subspaces of V that are not contained in H and the $(e-1)$-dimensional subspaces of H; where two vertices of the first kind are adjacent if they intersect in an e-dimensional subspace; a vertex of the first kind is adjacent to a vertex of the second kind if the first contains the second; and two vertices of the second kind are adjacent if they intersect in an $(e-2)$-dimensional subspace. Then G is distance-regular, with the same parameters as the*

Grassmann graph $J_q(2e+1,e)$. Moreover, G is not transitive, and hence it is not isomorphic to the Grassmann graph.

Proof. It is clear that in G, two vertices of the first kind are at distance i if and only if they intersect in an $(e+1-i)$-dimensional subspace; a vertex of the first kind and a vertex of the second kind are at distance i if they intersect in an $(e-i)$-dimensional subspace; and two vertices of the second kind are at distance i if they intersect in an $(e-1-i)$-dimensional subspace. By using for example Lemma 9.3.2 in [2], it is straightforward now to check that each vertex has $k_i = q^{i^2}\begin{bmatrix} e+1 \\ i \end{bmatrix}\begin{bmatrix} e \\ i \end{bmatrix}$ vertices at distance i. One can also check that the intersection parameters b_i and c_i are well-defined and the same as in the Grassmann graph, but this is tedious and elaborate.

Alternatively, we note that G is indeed the line graph of the above described partial linear space, and thus has the same spectrum as $J_q(2e+1,e)$. We can then use a theorem by Fiol and Garriga [4, Thm. 4.4] which implies that a graph which is cospectral with a distance-regular graph Γ with diameter e is itself distance-regular if for every vertex the number of vertices at distance e is the same as in Γ. Since k_e in G is indeed the same as in the Grassmann graph, it thus follows that G is distance-regular. Since the parameters of a distance-regular graph follow from its spectrum, G has the same parameters as $J_q(2e+1,e)$.

To show that G is not transitive, we consider the maximal cliques in G. It is straightforward to check that there are four types of such cliques, and they can be described as follows. The first type consists of all $(e+1)$-dimensional subspaces not contained in H containing a fixed e-dimensional subspace S, and the $(e-1)$-dimensional subspaces contained in $S \cap H$. If $S \subset H$, then the size of such a maximal clique is $\begin{bmatrix} e+1 \\ 1 \end{bmatrix}$; otherwise it is one more. The second type consists of all $(e+1)$-dimensional subspaces not contained in H and contained in a fixed $(e+2)$-dimensional subspace S not contained in H. The size of such a maximal clique is $\begin{bmatrix} e+2 \\ 1 \end{bmatrix} - 1$. The third type consists of all $(e+1)$-dimensional subspaces not contained in H which are contained in a fixed $(e+2)$-dimensional subspace S not contained in H, which moreover are containing a fixed $(e-1)$-dimensional subspace S' of $S \cap H$, plus S' itself. The size of this type of maximal clique is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. The last type of clique consists of all $(e-1)$-dimensional subspaces of H containing a fixed $(e-2)$-dimensional subspace S of H. These largest cliques have size $\begin{bmatrix} e+2 \\ 1 \end{bmatrix}$. Since they consist of $(e-1)$-dimensional subspaces only, this shows that G is not transitive, and hence that G is not isomorphic to the Grassmann graph. □

It is clear that $P\Gamma L(2e+1,q)_{2e}$, the subgroup of $P\Gamma L(2e+1,q)$ that fixes H, is a group of automorphisms of the new graph. We expect that this is the full group of automorphisms. In any case, the graph has two orbits under the action of its automorphism group on the vertices, and four orbits of edges.

3 Further remarks

We claim that the Terwilliger algebra (cf. [6] and the references therein) with respect to an $(e-1)$-dimensional subspace is different from the Terwilliger algebra with respect to an $(e+1)$-dimensional subspace. We were able to show that this is true asymptotically; besides this we checked it for the case $q=2, e=3$. By counting the number of cliques of size four containing a fixed vertex one can show that the number of triangles in the local graph of an $(e-1)$-dimensional subspace is (in general) different from the number of triangles in the local graph of an $(e+1)$-dimensional subspace. This implies that the spectra of these two local graphs are different, and hence that the Terwilliger algebras with respect to these two vertices are different.

The Grassmann graphs $J_q(n,e)$ can be seen as q-analogues of the Johnson graphs $J(n,e)$. The latter are known to be determined as distance-regular graphs by the parameters, except for the case $n=8, e=2$ (cf. [2, p. 258]). For the Johnson graph $J(2e+1,e)$ one can define a partial linear space, and adjust it, in an analogous way as we did for the Grassmann graph $J_q(2e+1,e)$. However, the obtained distance-regular line graph is isomorphic to $J(2e+1,e)$. Indeed, the argument that the graph is not transitive by considering the maximal cliques fails in this case.

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