Perturbation of frame sequences in shift-invariant spaces

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Abstract

We prove a new perturbation criteria for frame sequences, which generalizes previous results and is easier to apply. In the special case of frames in finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ the condition can be formulated in terms of the norm of a finite Gram matrix and a corresponding rank condition.

1 Introduction

Perturbation questions for frames in Hilbert spaces have been studied intensively in the last decade, with satisfying results for general frames [5] as well as for Gabor frames and wavelet frames [9, 13, 14]. Some results have also been obtained for frame sequences, i.e., frames which only span a closed subspace of a given Hilbert space [6, 7]; however, all of these are based on calculation of the gap between two subspaces, and are quite complicated to apply. In the present paper, we prove that if the so-called infimum cosine angle between the relevant subspaces is strictly positive, the convenient perturbation condition for frames for the entire Hilbert space can also be applied to the subspace case. Furthermore, we show that for frames in finitely-generated shift-invariant subspaces, the condition on the angle can be expressed as a rank condition on finite-dimensional matrices.

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In the rest of the introduction, we state the necessary definitions, as well as some known perturbation results.

Given a separable Hilbert space $\mathcal{H}$, a family of elements $\{f_k\}_{k=1}^{\infty}$ is a *frame* for $\mathcal{H}$ if there exist two constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B||f||^2, \quad \forall f \in \mathcal{H}. \tag{1}$$

The numbers A, B are called *frame bounds*. If at least the right-hand inequality of (1) holds, then $\{f_k\}_{k=1}^{\infty}$ is called a *Bessel sequence*.

The frame condition implies that $\{f_k\}_{k=1}^{\infty}$ is total in $\mathcal{H}$. If the frame condition only holds on $V := \overline{\text{span}}\{f_k\}_{k=1}^{\infty}$ (which, in general, is a subspace of $\mathcal{H}$) we say that $\{f_k\}_{k=1}^{\infty}$ is a *frame sequence* with bound B. It is well-known that this is the case if and only if

$$\left|\left|\sum c_k f_k\right|\right| \leq \sqrt{B}\left(\sum |c_k|^2\right)^{1/2}$$

for all finite scalar sequences $\{c_k\}$.

When studying perturbation questions for frame sequences we need to consider the *gap* between the spaces spanned by the given frame $\{f_k\}_{k=1}^{\infty}$ and its perturbation $\{g_k\}_{k=1}^{\infty}$. In general, letting V, W be two subspaces of $\mathcal{H}$ and assuming that $V \neq \{0\}$, we define the gap from V to W by

$$\delta(V, W) = \sup_{x \in V, ||x||=1} \text{dist}(x, W) = \sup_{x \in V, ||x||=1} \inf_{y \in W} ||x - y||.$$

Let us state the versions we need of the general perturbation results for frames; (i) appeared in [5], and (ii) appeared in [7]. Note that the crucial condition (2) is a Bessel condition on the sequence $\{f_k - g_k\}_{k=1}^{\infty}$.

Theorem 1.1 *Let $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}$ be sequences in a Hilbert space $\mathcal{H}$ and assume that there exists a constant $\mu \geq 0$ such that*

$$\left|\left|\sum c_k (f_k - g_k)\right|\right| \leq \mu \left(\sum |c_k|^2\right)^{1/2} \tag{2}$$

for all finite scalar sequences $\{c_k\}$. Then the following holds:

(i) If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds A, B and $\frac{\mu}{\sqrt{A}} < 1$, then $\{g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$.

(ii) If $\{f_k\}_{k=1}^{\infty}$ *is a frame sequence with bounds* A, B *and*

$$\frac{\mu}{\sqrt{A}} < \sqrt{1 - \delta(\overline{span}\,\{g_k\}_{k=1}^{\infty}\,, \overline{span}\{f_k\}_{k=1}^{\infty})^2}, \tag{3}$$

then $\{g_k\}_{k=1}^{\infty}$ *is a frame sequence.*

In both cases $\{g_k\}_{k=1}^{\infty}$ *has the frame bounds*

$$A\left(1 - \mu/\sqrt{A}\right)^2, \quad B\left(1 + \mu/\sqrt{B}\right)^2.$$

Observe that, depending on the size of the gap $\delta(\overline{\text{span}}\,\{g_k\}_{k=1}^{\infty}\,, \overline{\text{span}}\{f_k\}_{k=1}^{\infty})$, the condition (ii) is more restrictive than (i). We refer to [7] (or Example 3.5 below) which shows that the condition in (i), i.e., $\mu/\sqrt{A} < 1$, is not enough to guarantee the frame property if a frame sequence is perturbed.

One of our findings (Theorem 2.1) is that if the gap is strictly smaller than one, the condition $\frac{\mu}{\sqrt{A}} < 1$ is sufficient also for the case of perturbation of a frame sequence. Thus, exact calculation of the gap is not needed.

In order to proceed, we need some more concepts related to subspaces V, W of $\mathcal{H}$. Assuming again that $V \neq \{0\}$, the *angle* from V to W is defined as the unique number $\theta(V, W) \in [0, \pi/2]$ for which

$$\cos\theta(V, W) = \inf_{f \in V, ||f||=1} ||P_W f||. \tag{4}$$

We will frequently use the notation

$$R(V, W) := \cos\theta(V, W), \tag{5}$$

which is called the *infimum cosine angle.* If we take the supremum, instead of the infimum, of the right-hand side of (4), we obtain the *supremum cosine angle* $S(V, W)$ of V and W, that is,

$$S(V, W) = \sup_{f \in V, ||f||=1} ||P_W f||.$$

See [1, 10, 11, 15]. As proved in [15], the two angles are related by

$$R(V, W) = (1 - S(V, W^{\perp})^2)^{1/2}.$$

We recall that, in general, $R(W, V) \neq R(V, W)$.

In case $V = \{0\}$, we use the convention

$$\delta(V, W) = 0, \;\; \theta(V, W) = 0.$$

Thus, for $V = \{0\}$ we have $R(V, W) = 1$ and $S(V, W) = 0$ by convention. We also note that if $V \neq \{0\}$ and $W = \{0\}$, then $R(V, W) = 0$ and $S(V, W) = 0$. An elementary calculation (cf. [8]) shows that the gap and the angle are related via

$$\delta(V, W) = \sin \theta(V, W). \tag{6}$$

2 Perturbation results in Hilbert spaces

In this section, we improve Theorem 1.1(ii).

Theorem 2.1 *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $W \subset \mathcal{H}$, with bounds A, B. For a sequence $\{g_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, let $V := \overline{span}\{g_k\}_{k=1}^{\infty}$, and assume that there exists a constant $\mu > 0$ such that*

$$\left|\left|\sum c_k(f_k - g_k)\right|\right| \leq \mu \left(\sum |c_k|^2\right)^{1/2} \tag{7}$$

for all finite scalar sequences $\{c_k\}$. Then the following holds:

(a) *$\{g_k\}_{k=1}^{\infty}$ is a Bessel sequence with bound $B(S(V, W) + \mu/\sqrt{B})^2$;*

(b) *If $\mu < \sqrt{A}$, then $R(W, V) > 0$.*

If, in addition to (b), $R(V, W) > 0$, then

(c) *$\{g_k\}_{k=1}^{\infty}$ is a frame for V with bounds*

$$A(1 - \mu/\sqrt{A})^2, B(S(V, W) + \mu/\sqrt{B})^2.$$

(d) *V is isomorphic to W.*

Proof. Let P_V be the orthogonal projection onto V and $P_V|_W$ its restriction to W. P_W and $P_W|_V$ are defined similarly. Then, a direct calculation shows that $(P_V|_W)^* = P_W|_V$.

(a): Let $g \in V$. Then we have

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle g, g_k\rangle|^2 &= \sum_{k=1}^{\infty} |\langle g, f_k\rangle - \langle g, f_k - g_k\rangle|^2 \\
&= \sum_{k=1}^{\infty} |\langle g, f_k\rangle|^2 + \sum_{k=1}^{\infty} |\langle g, f_k - g_k\rangle|^2 \\
&- \sum_{k=1}^{\infty} \langle g, f_k\rangle \overline{\langle g, f_k - g_k\rangle} - \sum_{k=1}^{\infty} \overline{\langle g, f_k\rangle} \langle g, f_k - g_k\rangle \\
&\leq \sum_{k=1}^{\infty} |\langle g, f_k\rangle|^2 + \sum_{k=1}^{\infty} |\langle g, f_k - g_k\rangle|^2 \\
&+ 2\sqrt{\sum_{k=1}^{\infty} |\langle g, f_k\rangle|^2} \sqrt{\sum_{k=1}^{\infty} |\langle g, f_k - g_k\rangle|^2} \\
&= \left(\sqrt{\sum_{k=1}^{\infty} |\langle g, f_k\rangle|^2} + \sqrt{\sum_{k=1}^{\infty} |\langle g, f_k - g_k)\rangle|^2} \right)^2 \\
&= \left(\sqrt{\sum_{k=1}^{\infty} |\langle P_W g, f_k\rangle|^2} + \sqrt{\sum_{k=1}^{\infty} |\langle g, f_k - g_k)\rangle|^2} \right)^2 \\
&\leq \left(\sqrt{B} ||P_W g|| + \mu ||g|| \right)^2 \\
&\leq \left(\sqrt{B}\ S(V, W) + \mu \right)^2 ||g||^2.
\end{aligned}$$

Now, letting g be an arbitrary element in $\mathcal{H}$, this calculation yields that

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle g, g_k\rangle|^2 &= \sum_{k=1}^{\infty} |\langle P_V g, g_k\rangle|^2 \\
&\leq \left(\sqrt{B}\ S(V, W) + \mu \right)^2 ||P_V g||^2 \\
&\leq \left(\sqrt{B}\ S(V, W) + \mu \right)^2 ||g||^2.
\end{aligned}$$

(b): Let $f \in W$. By (a),

$$
\begin{aligned}
(\sqrt{B}S(V,W)+\mu)^2||P_V f||^2 &\geq \sum_{k=1}^{\infty} |\langle P_V f, g_k\rangle|^2 && (8)\\
&= \sum |\langle f, g_k\rangle|^2 \\
&\geq \left(\sqrt{\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2} - \sqrt{\sum_{k=1}^{\infty} |\langle f, f_k - g_k)\rangle|^2}\right)^2 \\
&\geq (\sqrt{A}-\mu)^2||f||^2. && (9)
\end{aligned}
$$

Hence,

$$R(W,V) \geq \frac{\sqrt{A}-\mu}{\sqrt{B}S(V,W)+\mu} > 0.$$

(c): Let $f \in W$. As in the calculation (8)–(9), we have

$$\sum_{k=1}^{\infty} |\langle P_V f, g_k\rangle|^2 \geq \left(\sqrt{A}-\mu\right)^2 ||f||^2 \geq \left(\sqrt{A}-\mu\right)^2 ||P_V f||^2.$$

Since $||P_W|_V g|| = ||P_W g|| \geq R(V,W)||g||$ for every $g \in V$, $P_W|_V$ is bounded below. Thus, $(P_W|_V)^* = P_V|_W$ is onto. Moreover, $R(W,V) > 0$ by (b) and so $P_V|_W$ is one-to-one. Hence, for each $g \in V$, there exists a unique $f \in W$ such that $P_V|_W f = g$. Therefore,

$$\sum_{k=1}^{\infty} |\langle g, g_k\rangle|^2 \geq \left(\sqrt{A}-\mu\right)^2 ||g||^2, \text{ for any } g \in V.$$

(d): $P_V|_W$ is an isomorphism from W to V. □

Note that in case $S(V,W) < 1$ the upper frame bound for $\{g_k\}_{k=1}^{\infty}$ in Theorem 2.1 is smaller than the upper bound in Theorem 1.1(ii). This implies that we obtain a better estimate of the condition number of the frame operator, which is important in order to estimate the speed of convergence in algorithms involving frames. In the case where V and W are finitely generated shift-invariant spaces, a useful formula for computing of $S(V,W)$ was presented in [11], see Example 3.4 below.

3 Perturbation in shift-invariant subspaces

Given two subsets $\Phi := \{\phi_1, \phi_2, \cdots, \phi_n\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$ with the same number of functions, we let

$$W = \overline{\text{span}}\{T_k\phi_j : k \in \mathbb{Z}^d, 1 \le j \le n\},$$
$$V = \overline{\text{span}}\{T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}.$$

Assuming that $\{T_k\phi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ is a frame sequence, our purpose is to present a perturbation condition assuring that another sequence $\{T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ is a frame sequence as well. Our approach uses the Gramian analysis originally introduced by Ron and Shen [12] (see also [2]).

For $f \in L^2(\mathbb{R}^d)$, $\gamma \in \mathbb{T}^d$, let

$$\hat{f}_{||\gamma} = \{\hat{f}(\gamma + k)\}_{k\in\mathbb{Z}^d}$$

be the *fiber of* f *at* γ. The fiber belongs to $\ell^2(\mathbb{Z}^d)$ for almost all $\gamma \in \mathbb{T}^d$. Here $\wedge$ denotes the Fourier transform defined by

$$\hat{f}(\gamma) := \int_{\mathbb{R}^d} f(t)^{-2\pi i\gamma\cdot t}\, dt$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and extended to be a unitary operator on $L^2(\mathbb{R}^d)$ by the Plancherel theorem. The $n \times n$ matrix

$$G_\Phi(\gamma) := (\langle \hat{\phi}_{j_{||\gamma}}, \hat{\phi}_{i||\gamma}\rangle)_{1\le i,j\le n}$$

is called the *Gramian of* Φ *at* $\gamma \in \mathbb{T}^d$. $G_{\Psi-\Phi}(\gamma)$ denotes the Gramian of $\Phi - \Psi := \{\phi_j - \psi_j : 1 \le j \le n\}$ at γ. The $n \times n$ matrix

$$G_{\Phi,\Psi}(\gamma) := (\langle \hat{\phi}_{j_{||\gamma}}, \hat{\psi}_{i||\gamma}\rangle)_{1\le i,j\le n},\ \gamma \in \mathbb{T}^d,$$

is called the *mixed Gramian of* Φ *and* Ψ *at* γ. In the following we will consider these matrices as operators on $\mathbb{C}^n$.

We need the following result. The implication (a) $\Rightarrow$ (c) was first proved in [3].

Proposition 3.1 ([10]) *Suppose that* $R(W, V) > 0$. *Then the following are equivalent:*

(a) $\text{rank}\, G_\Phi(\gamma) = \text{rank}\, G_\Psi(\gamma)$ *for a.e.* $\gamma \in \mathbb{T}^d$*;*

(b) $R(V,W) > 0$;

(c) $R(V,W) = R(W,V)$.

Proposition 3.1 allows us to express the crucial angle condition in Theorem 2.1 by a rank condition on the finite-dimensional matrices $G_\Phi(\gamma)$ and $G_\Psi(\gamma)$:

Theorem 3.2 *Assume that $\{T_k\phi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ is a frame for W with bounds A, B. Let*

$$\mu := \text{ess-sup}_{\gamma\in\mathbb{T}^d} ||G_{\Phi-\Psi}(\gamma)||^{1/2}. \tag{10}$$

Then the following holds:

(i) If $\mu < \infty$, then $\{T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ is a Bessel sequence with bound

$$B\left(S(V,W) + \frac{\mu}{\sqrt{B}}\right)^2.$$

(ii) If $\mu < \sqrt{A}$ and $\operatorname{rank} G_\Phi(\gamma) = \operatorname{rank} G_\Psi(\gamma)$ for a.e. $\gamma \in \mathbb{T}^d$, then $\{T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ is a frame for V with bounds

$$A\left(1 - \frac{\mu}{\sqrt{A}}\right)^2, B\left(S(V,W) + \frac{\mu}{\sqrt{B}}\right)^2.$$

(iii) In the single-generator cased, i.e., $\Phi = \{\phi\}, \Psi = \{\psi\}$, the rank condition in (ii) can be replaced by the condition

$$\left\{\gamma : \sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma + k)|^2 = 0\right\} \subseteq \left\{\gamma : \sum_{k\in\mathbb{Z}} |\hat{\psi}(\gamma + k)|^2 = 0\right\}. \tag{11}$$

Proof. By [12, Theorem 2.3.6], μ^2 with μ as in (10) is the optimal Bessel bound of the sequence $\{T_k\phi_j - T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$. Now (i) follows from Theorem 2.1 (a).

Suppose that $\mu < \sqrt{A}$. Then, $R(W,V) > 0$ by Theorem 2.1(b). Hence, if $\operatorname{rank} G_\Phi(\gamma) = \operatorname{rank} G_\Psi(\gamma) < \infty$ for *a.e.* $\gamma \in \mathbb{T}^d$, then $R(V,W) > 0$ by

Proposition 3.1. Therefore, (ii) follows by Theorem 2.1 (c). For the proof of (iii) we note that for $n = 1$,

$$G_{\Phi-\Psi}(\gamma) = \sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\gamma + k) - \hat{\psi}(\gamma + k)|^2.$$

Now, the condition that $\mu < \sqrt{A}$ implies that the two sets in (11) coincide, and therefore the rank condition in (ii) is satisfied. □

Explicit calculation of μ in (10) might be complicated, in which case we use the alternative below. The result uses the trace of the Gramian $G_{\Phi-\Psi}(\gamma)$,

$$\mathrm{tr}(G_{\Phi-\Psi}(\gamma)) = \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}^d} |\hat{\phi}_j(\gamma + k) - \hat{\psi}_j(\gamma + k)|^2.$$

Corollary 3.3 *The conclusions of Theorem 3.2 holds if* μ *in* (10) *is replaced by*

$$\mu := \text{ess-sup}_{\gamma\in\mathbb{T}^d} \sqrt{\sum_{j=1}^{n} \sum_{k\in\mathbb{Z}^d} |\hat{\phi}_j(\gamma + k) - \hat{\psi}_j(\gamma + k)|^2}. \tag{12}$$

Proof. Let $\{\lambda_j(\gamma)\}_{j\in J}$ be the eigenvalues of $G_{\Phi-\Psi}(\gamma)$ for *a.e.* $\gamma \in \mathbb{T}^d$. Then,

$$\begin{aligned} ||G_{\Phi-\Psi}(\gamma)|| &= \sup_{j\in J} \lambda_j(\gamma) \\ &\leq \sum_{j=1}^{n} \lambda_j(\gamma) = \mathrm{tr}(G_{\Phi-\Psi}(\gamma)) \\ &= \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}^d} |\hat{\phi}_j(\gamma + k) - \hat{\psi}_j(\gamma + k)|^2. \end{aligned}$$

The corollary now follows from Theorem 3.2. □

We now illustrate the results by some examples.

Example 3.4 Let ϕ_H, ψ_H be the Haar scaling function and wavelet defined by

$$\phi_H(x) := \chi_{[0,1]},\ \psi_H(x) := \chi_{[0,1/2)} - \chi_{(1/2,1]}.$$

It is easy to check that

$$G_{\{\phi_H,\psi_H\}}(\gamma) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{13}$$

We set $\Phi := \{\phi_1, \phi_2\}$ and $\Psi := \{\psi_1, \psi_2\}$, where

$$\begin{aligned}
\hat{\phi}_1(\gamma) &:= \hat{\phi}_H(\gamma)\chi_{[\frac{1}{4},\frac{3}{4}]+\mathbb{Z}}(\gamma) = e^{-i\pi\gamma}\ \frac{\sin(\pi\gamma)}{\pi\gamma}\chi_{[\frac{1}{4},\frac{3}{4}]+\mathbb{Z}}(\gamma);\\
\hat{\phi}_2(\gamma) &:= \hat{\psi}_H(\gamma)\chi_{[\frac{1}{4},\frac{3}{4}]+\mathbb{Z}}(\gamma) = 2ie^{-\pi i\gamma}\ \frac{\sin^2(\pi\gamma/2)}{\pi\gamma}\chi_{[\frac{1}{4},\frac{3}{4}]+\mathbb{Z}}(\gamma);\\
\hat{\psi}_1(\gamma) &:= \hat{\phi}_H(\gamma)\chi_{[-\frac{3}{4},-\frac{1}{4}]\cup[\frac{1}{4},\frac{3}{4}]}(\gamma) = e^{-i\pi\gamma}\ \frac{\sin(\pi\gamma)}{\pi\gamma}\chi_{[-\frac{3}{4},-\frac{1}{4}]\cup[\frac{1}{4},\frac{3}{4}]}(\gamma);\\
\hat{\psi}_2(\gamma) &:= \hat{\psi}_H(\gamma)\chi_{[-\frac{3}{4},-\frac{1}{4}]\cup[\frac{1}{4},\frac{3}{4}]}(\gamma)\\
&= 2ie^{-\pi i\gamma}\ \frac{\sin^2(\pi\gamma/2)}{\pi\gamma}\ \chi_{[-\frac{3}{4},-\frac{1}{4}]\cup[\frac{1}{4},\frac{3}{4}]}(\gamma),
\end{aligned}$$

for *a.e.* $\gamma \in \mathbb{T}$. We then have as in (13)

$$G_\Phi(\gamma) = \begin{bmatrix} \chi_{[\frac{1}{4},\frac{3}{4}]}(\gamma) & 0 \\ 0 & \chi_{[\frac{1}{4},\frac{3}{4}]}(\gamma) \end{bmatrix}, \tag{14}$$

and so $\{T_k\phi_j : k \in \mathbb{Z}, j = 1, 2\}$ is a tight frame with frame bound 1. The Gramian G_Ψ is given by

$$\begin{aligned}
[G_\Psi(\gamma)]_{11} &= \left[\frac{\sin^2(\pi\gamma)}{(\pi\gamma)^2} + \frac{\sin^2(\pi\gamma)}{(\pi(\gamma-1))^2}\right]\chi_{[\frac{1}{4},\frac{3}{4}]}(\gamma);\\
[G_\Psi(\gamma)]_{12} &= 2i\left[\frac{\sin(\pi\gamma)\sin^2(\frac{\pi\gamma}{2})}{(\pi\gamma)^2} - \frac{\sin(\pi\gamma)\cos^2(\frac{\pi\gamma}{2})}{(\pi(\gamma-1))^2}\right]\chi_{[\frac{1}{4},\frac{3}{4}]}(\gamma);\\
[G_\Psi(\gamma)]_{21} &= -2i\left[\frac{\sin(\pi\gamma)\sin^2(\frac{\pi\gamma}{2})}{(\pi\gamma)^2} - \frac{\sin(\pi\gamma)\cos^2(\frac{\pi\gamma}{2})}{(\pi(\gamma-1))^2}\right]\chi_{[\frac{1}{4},\frac{3}{4}]}(\gamma);\\
[G_\Psi(\gamma)]_{22} &= 4\left[\frac{\sin^4(\frac{\pi\gamma}{2})}{(\pi\gamma)^2} + \frac{\cos^4(\frac{\pi\gamma}{2})}{(\pi(\gamma-1))^2}\right]\chi_{[\frac{1}{4},\frac{3}{4}]}(\gamma).
\end{aligned}$$

Direct calculations show that

$$\det G_\Psi(\gamma) = \frac{4\sin^2(\pi\gamma)}{\pi^4\gamma^2(\gamma-1)^2}\ \chi_{[\frac14,\frac34]}(\gamma).$$

Hence, $\operatorname{rank} G_\Phi(\gamma) = \operatorname{rank} G_\Psi(\gamma) = 2\chi_{[\frac14,\frac34]}(\gamma)$ for *a.e.* $\gamma \in [0,1]$. Now, we check the trace condition (12). By (14), we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\hat\phi_1(\gamma+k) - \hat\psi_1(\gamma+k)|^2 &= \sum_{k\in\mathbb{Z}} |\hat\phi_1(\gamma+k)|^2 - |\hat\phi_1(\gamma)|^2 - |\hat\phi_1(\gamma-1)|^2 \\
&= \chi_{[\frac14,\frac34]}(\gamma) - \frac{\sin^2(\pi\gamma)}{(\pi\gamma)^2} - \frac{\sin^2(\pi(\gamma-1))}{(\pi(\gamma-1))^2}, \\
\sum_{k\in\mathbb{Z}} |\hat\phi_2(\gamma+k) - \hat\psi_2(\gamma+k)|^2 &= \sum_{k\in\mathbb{Z}} |\hat\psi_1(\gamma+k)|^2 - |\hat\psi_1(\gamma)|^2 - |\hat\psi_1(\gamma-1)|^2 \\
&= \chi_{[\frac14,\frac34]}(\gamma) - \frac{4\sin^4(\pi\gamma/2)}{(\pi\gamma)^2} - \frac{4\sin^4(\pi(\gamma-1))}{(\pi(\gamma-1))^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{j=1}^{2}\sum_{k\in\mathbb{Z}} |\hat\phi_j(\gamma+k) - \hat\psi_j(\gamma+k)|^2 \\
&= \left(2 - \left(\frac{\sin^2(\pi\gamma)}{(\pi\gamma)^2} + \frac{\sin^2(\pi(\gamma-1))}{(\pi(\gamma-1))^2} + \frac{4\sin^4(\pi\gamma/2)}{(\pi\gamma)^2} + \frac{4\sin^4(\pi(\gamma-1)/2)}{(\pi(\gamma-1))^2}\right)\right)\chi_{[\frac14,\frac34]}(\gamma) \\
&= \left(2 - \left(\frac{\sin(\pi\gamma/2)}{(\pi\gamma/2)}\right)^2 - \left(\frac{\sin(\pi(\gamma-1)/2)}{(\pi(\gamma-1)/2)}\right)^2\right)\chi_{[\frac14,\frac34]}(\gamma) \\
&\le 2 - 2\inf_{\gamma\in[\frac14,\frac34]}\left(\frac{\sin(\pi\gamma/2)}{(\pi\gamma/2)}\right)^2 \\
&= 2 - 2\left(\frac{\sin(3\pi/8)}{(3\pi/8)}\right)^2.
\end{aligned}$$

That is,

$$\mu = \text{ess-sup}_{\gamma\in\mathbb{T}} \sqrt{\sum_{j=1}^{2}\sum_{k\in\mathbb{Z}} |\hat\phi_j(\gamma+k) - \hat\psi_j(\gamma+k)|^2} \le \sqrt{2 - 2\left(\frac{\sin(3\pi/8)}{(3\pi/8)}\right)^2} < 1 = \sqrt{A}.$$

Therefore, $\{T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ is a frame for V with bounds $(1-\mu)^2, (S(V,W)+\mu)^2$ by Corollary 3.3. We can obtain explicit estimates

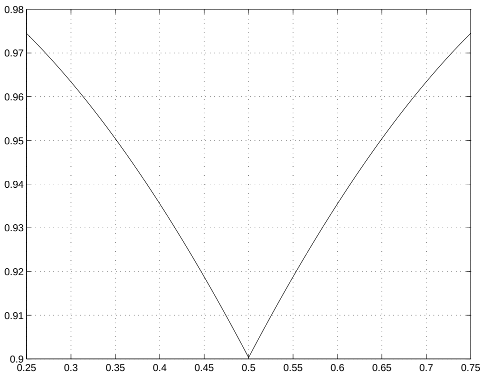

Figure 1: $||(G_\Phi(\gamma)^\dagger)^{1/2}G_{\Phi,\Psi}(\gamma)(G_\Psi(\gamma)^\dagger)^{1/2}||$.

for the frame bounds by using $\sqrt{2-2\left(\frac{\sin(3\pi/8)}{(3\pi/8)}\right)^2} = 0.8775$ as estimate for μ. For $S(V,W)$ we can use a result from [11], showing that if the spectrum of V is defined by $\sigma(V) := \{\gamma \in \mathbb{T}^d : \operatorname{rank} G_\Psi(\gamma) > 0\}$, then

$$\begin{aligned} S(V,W) &= S(W,V) \\ &= \text{ess-sup}_{\gamma\in\sigma(V)\bigcap\sigma(W)} ||(G_\Phi(\gamma)^\dagger)^{1/2}G_{\Phi,\Psi}(\gamma)(G_\Psi(\gamma)^\dagger)^{1/2}||, \quad (15) \end{aligned}$$

where $G_\Phi(\gamma)^\dagger$ and $G_\Psi(\gamma)^\dagger$ denote the pseudo-inverses of $G_\Phi(\gamma)$ and $G_\Psi(\gamma)$, respectively. We plot $||(G_\Phi(\gamma)^\dagger)^{1/2}G_{\Phi,\Psi}(\gamma)(G_\Psi(\gamma)^\dagger)^{1/2}||$ for $\gamma \in \sigma(V) = \sigma(W) = [1/4, 3/4]$ in Figure 1 and estimate $S(V,W) \approx 0.9745$ numerically using (15). As estimates for the frame bounds we finally obtain 0.015 and 3.43, respectively.

Let us finally consider the special case of one generator, i.e., Theorem 3.2(iii). We observe that without the condition (11), the perturbation condition (10) does not imply that $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame:

Example 3.5 The function ϕ given by $\hat{\phi} = \chi_{[1/2,1]}$ generates a frame sequence; but the functions $\psi_\epsilon, \epsilon > 0$, given by

$$\hat{\psi}(\gamma) = \epsilon\gamma\chi_{[0,1/4]}(\gamma) + \chi_{[1/2,1]}(\gamma)$$

do not generate frame sequences regardless how small ϵ is chosen, despite the fact that

$$\text{ess-sup}_{\gamma\in\mathbb{T}^d}||G_{\Phi-\Psi}(\gamma)|| = \text{ess-sup}_\gamma \sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma+k) - \hat{\psi}(\gamma+k)|^2 = \frac{\epsilon^2}{16}$$

can be made arbitrarily small.

Example 3.6 Assume that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame sequence with bounds A, B. If ψ is defined via $\hat{\psi}(\gamma) := \hat{\phi}(\gamma)\chi_{[-\delta,\delta]}(\gamma)$ for some $\delta > 0$, then condition (11) is satisfied. Thus, if μ, defined as in (10), satisfies the condition $\mu < \sqrt{A}$, then $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame sequence. If $\hat{\psi}$ satisfies the decay condition

$$|\hat{\phi}(\gamma)|^2 \leq \frac{C}{1 + |\gamma|^\alpha}$$

for some $\alpha > 1$, then an estimate for δ can be given. We proceed as follows. If $\gamma \in [-1/2, 1/2]$ and $|k| \leq \delta - 1/2$, then we see

$$|\gamma + k| \leq |\gamma| + |k| \leq 1/2 + \delta - 1/2 = \delta.$$

Thus, for *a.e.* $\gamma \in [-1/2, 1/2]$, we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma + k) - \hat{\psi}(\gamma + k)|^2 &= \sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma + k)|^2 \chi_{[-\delta,\delta]^c}(\gamma + k) \\
&\leq \sum_{|k|>\delta-1/2} |\hat{\phi}(\gamma + k)|^2 \\
&\leq C \sum_{|k|>\delta-1/2} \frac{1}{1 + |\gamma + k|^\alpha} \\
&\leq 2C \sum_{k>\delta-1/2} \frac{1}{1 + (k - 1/2)^\alpha} \\
&\leq 2C \sum_{k>\delta-1/2} \frac{1}{(k - 1/2)^\alpha} \\
&\leq 2C \int_{\delta-3/2}^{\infty} \frac{dt}{(t - 1/2)^\alpha} \\
&= \frac{2C}{(\alpha - 1)(\delta - 2)^{\alpha-1}}.
\end{aligned}$$

Thus,

$$\mu \leq \sqrt{\frac{2C}{(\alpha - 1)(\delta - 2)^{\alpha-1}}}.$$

Hence the condition $\mu < \sqrt{A}$, i.e.

$$\delta > \left(\frac{2C}{A(\alpha-1)}\right)^{1/(\alpha-1)} + 2,$$

implies that $\{T_k\psi\}_{k\in\mathbb{Z}}$ is a frame sequence.

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