The infimum cosine angle between two finitely generated shift-invariant spaces and its applications *

Hong Oh Kim[†], Rae Young Kim[‡] and Jae Kun Lim[§]

November 1, 2004

Abstract

An expression of the infimum cosine angle between two finite dimensional subspaces is given in terms of Gramians and is applied to find a useful formula for the infimum cosine angle between two finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ in terms of the

*This work was supported by Korea Research Foundation Grant: KRF-2003-003-C00008.

†Division of Applied Mathematics, KAIST, 373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701, Republic of Korea (hkim@amath.kaist.ac.kr)

‡Division of Applied Mathematics, KAIST, 373-1, Guseong-dong, Yuseong-gu, Daejeon, 305-701, Republic of Korea (rykim@amath.kaist.ac.kr)

§Department of Computational Mathematics and Informatics, Hankyong National University, 67 Sokjong-Dong, Ansung-City, Kyonggi-Do, 456-749, Republic of Korea (jaekun@hknu.ac.kr)

generators. We then present new equivalent conditions for the existence of the oblique projection between two finitely generated shift-invariant subspaces, thereby providing a constructive method to generate oblique dual frames. This result is a generalization of a result of Aldroubi and is closely related with the biorthogonality of two frame multiresolution analyses. Finally, we illustrate our results by examples.

AMS 2000 Subject Classification: 42C15, 42C40, 15A09.
Key Words: Shift-invariant space, Angle between subspaces, Tightization, Oblique projection, Multiresolution analysis, Pseudo-inverse.

1 Introduction

Throughout this article $\mathcal{H}$ denotes a separable Hilbert space over the complex field $\mathbb{C}$. The purpose of this article is to analyze the concept of the *infimum cosine angle* $R(U,V)$ between two closed subspaces U and V of $\mathcal{H}$ which is defined as follows [1, 43]:

$$R(U,V) := \inf_{u\in U\setminus\{0\}} \frac{\|P_V u\|}{\|u\|},$$

where P_V denotes the orthogonal projection onto V. The arc-cosine value of $R(U,V)$ is usually interpreted as the 'largest angle' between U and V [43]. If we take the supremum, instead of the infimum, of the right-hand side of the above equation we have the so-called *supremum cosine angle* $S(U,V)$ of U and V, and the two angles are related by the following relation: $R(U,V) = (1 - S(U,V^{\perp})^2)^{1/2}$ [43]. Similar to $R(U,V)$, the arc-cosine value of $S(U,V)$ is interpreted as the 'smallest angle' between U and V [43]. We use the convention that $R(U,V) = 1$ if U is trivial for the obvious reason. Note also that if U is not trivial and V is trivial, then $R(U,V) = 0$. See [1, 43] for the geometric meaning of this concept and its applications to signal processing, and see [2, 9, 10, 29, 30] for its applications to the theory of wavelets. Even though $R(U,V) = R(V^{\perp}, U^{\perp})$, $R(U,V) \neq R(V,U)$ in general, whereas $S(U,V) = S(V,U)$ [9, 43]. As will be mentioned, $R(U,V)$ is closely related with the biorthogonality of two multiresolution analyses, and the perturbation of frames in shift-invariant subspaces. In this article we concentrate on the infimum cosine angle, and postpone the discussion of the supremum cosine angle to the forthcoming paper [32], in which the connection between $S(U,V)$ and the closedness of the sum $U+V$ is analyzed [32].

We now explain the motivation for investigating the infimum cosine angle. First, the infimum cosine angle between two finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$ is closely related with the biorthogonality of two multiresolution analyses [1, 2, 9, 29, 30, 42]. The infimum cosine angle, however, in the cited papers is considered under various restrictive conditions on the generating sets of the shift-invariant subspaces. See Section 4 for the definition of the shift-invariant subspace of $L^2(\mathbb{R}^d)$. More specifically, the authors in the cited papers consider either the case where the shifts (i.e., (multi-)integer translates) of the multiple generating sets form Riesz bases for the shift-invariant spaces and the cardinalities of the multiple generating sets coincide [1, 9, 42] or the case where the generating sets are singletons [2, 29, 30, 44]. Therefore, the results in the existing literature are insufficient to deal with the case where the shifts of the multiple generating sets form frames for the shift-invariant spaces or the one where the cardinalities of the generating sets are different. Notice that the latter cases occur if we consider frame multiresolution analyses [3, 4, 14, 18, 20, 22, 30, 31, 33, 34, 37, 38]. In this article, we consider the infimum cosine angle between two finitely generated shift-invariant subspaces under no assumption on the generating sets (Theorem 4.7). Therefore, our results can be applied to the more general form of the biorthogonal multiresolution analyses.

Secondly, we mention that the connection between the infimum cosine angle and the perturbation of frames in shift-invariant subspaces will be discussed in a recent paper by Christensen and the authors [16].

Even though many of our results in this article can be generalized to infinitely generated shift-invariant subspaces, we restrict our attention to finitely generated shift-invariant spaces for the following reasons. In the conventional theory of multiresolution analysis, the central space is a finitely generated shift-invariant space. Moreover, if the central space is not regular, (see Section 4 for the definition), then there is no generating set for the space such that the shifts of the generating set form a Riesz basis [39]. On the other hand, for any finitely generated shift-invariant subspaces, there always exists a finite generating set whose shifts form a frame for the space [8, 39]. Now, we have at least two methods to analyze a shift-invariant space: the Gramian approach or the dual Gramian approach [19, 39, 40, 41]. It is generally believed that Gramian analysis is suitable for the analysis of a Riesz basis while the dual Gramian analysis is suitable for the analysis of a frame [39]. The main reason is that there is no good characterization of a shift-invariant frame via Gramians if the shift-invariant space is infinitely generated. On the other hand, there is a nice characterization of a shift-invariant frame via Gramian if the shift-invariant space is finitely generated [39, Theorem 2.3.6] (see also Proposition 4.6). Moreover, if a shift-invariant space is finitely generated, then the Gramian is a finite matrix while the dual Gramian is an infinite matrix. Therefore, if we consider only the finitely generated shift-invariant spaces, then the angle is given by means of finite matrices which are easier to deal with than with the infinite matrices (Theorem 4.7).

The rest of this article is organized in the following manner: The preliminary discussions on the pseudo-inverses and frames are given in Section 2 for the sake of completeness. Then, the infimum cosine angle between two finite dimensional spaces is calculated in Section 3 (Theorem 3.8). The method of the proofs in this section indicates that the language and theory of frames is useful for the finite dimensional spaces even though a finite frame for a finite dimensional space is just a spanning set. Theorem 3.8 is applied to give a concrete closed formula for the angle between two finitely generated shift-invariant spaces (Theorem 4.7) without any assumptions on the generating sets. Finally, we illustrate our results with examples.

2 Preliminary discussions

In this section we review some concepts that will be used later and fix some standing assumptions throughout this article.

First, we fix some notations that will be used throughout this article: Recall that $\mathcal{H}$ always denotes a separable Hilbert space over the complex field $\mathbb{C}$. For a closed subspace S of $\mathcal{H}$, P_S denotes the orthogonal projection onto S, unless stated otherwise explicitly. If H is a closed subspace of a Hilbert space, then I_H is the identity operator on H. Suppose that f is a function from D to C and that E is a subset of D. We write $f|_E$ for the restriction of f on E. For a Lebesgue measurable subset A of $\mathbb{R}^d$, $|A|$ denotes its Lebesgue measure. All set equalities and containments between subsets of $\mathbb{R}^d$ are assumed to be hold almost everywhere with occasional exceptions which are clear from the context. Finally, if M is a matrix, then M^t denotes its transpose, and the adjoint of a matrix or an operator T is denoted by T^*.

We now present well-known basic facts about the pseudo-inverse (or generalized inverse or Moore-Penrose inverse) for the sake of completeness [5, 7, 12, 23]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over $\mathbb{C}$, and $X : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator with closed range. For each $b \in \mathcal{H}_2$, $\{a \in \mathcal{H}_1 : Xa = P_{\operatorname{ran} X} b\}$ is a closed convex subset of $\mathcal{H}_1$. Hence it contains a unique element a of minimal norm. We let $X^\dagger b := a$. It is known that the map: $X^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ is a bounded linear operator, called the *pseudo-inverse* of X [23]. We introduce two results which will be used frequently in this article.

Proposition 2.1 ([23]) *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces over $\mathbb{C}$. Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with closed range. Then the following assertions hold:*

(1) $\operatorname{ran} X^\dagger = \operatorname{ran} X^*$;

(2) $XX^\dagger = P_{\operatorname{ran} X}$;

(3) $X^\dagger X = P_{\operatorname{ran} X^\dagger} = P_{\operatorname{ran} X^*}$.

Proof. (1) is a part of Theorem 2.1.2 of [23]; and (2) and (3) are parts of Theorem 2.2.2 of [23]. □

The following is Theorem 3.1 of [7].

Proposition 2.2 ([7]) *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be separable Hilbert spaces over $\mathbb{C}$ and $X : \mathcal{H}_2 \to \mathcal{H}_3, Y : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operators with closed range. Then $(XY)^\dagger = Y^\dagger X^\dagger$ if and only if*

(i) $\operatorname{ran} XY$ *is closed;*

(ii) $\operatorname{ran} X^*$ *is invariant under* YY^**;*

(iii) $\operatorname{ran} X^* \cap \ker Y^*$ *is invariant under* X^*X.

We now review briefly those parts of the theory of frames which will be used later. Let $\{f_i : i \in I\}$ be a sequence in $\mathcal{H}$, where I is an at most countable, i.e., finite or countably infinite, index set. We say that $\{f_i : i \in I\}$ is a *frame* for $\mathcal{H}$ if there exist positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2$$

for any $f \in \mathcal{H}$. A and B are called *lower and upper frame bounds*, respectively. The infimum of upper frame bounds is said to be the *optimal upper frame bound* and the supremum of lower frame bounds is said to be the *optimal lower frame bound*, and they are upper and lower frame bounds also. If the optimal frame bounds coincide we say that the frame is a *tight frame*. Suppose that $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$ with frame bounds A and B. Define $T : \ell^2(I) \to \mathcal{H}$ via $Tc := \sum_{i \in I} c_i f_i$, where $c := (c_i)_{i \in I}$. It is known that T, usually called the *pre-frame operator*, is an onto bounded linear operator [12, 27]. Moreover, $\|T\| \leq B^{1/2}$. Actually, the converse to this result also holds: $\{f_i : i \in I\}$ is a frame for its closed linear span if and only if the pre-frame operator T is a bounded linear operator onto the closed linear span. ([12, Corollary 4.3], [27, Theorem 2.1]). This proves the following simple fact that will be used repeatedly in this article: A finite sequence is always a frame for its linear span. Now, a direct calculation shows that $T^*f = (\langle f, f_i \rangle)_{i\in I}$. The operator $S := TT^*$, called the *frame operator*, is known to be a strictly positive (and hence self-adjoint) bounded linear operator with a bounded inverse [24]. More precisely, we have $Sf = \sum_{i\in I} \langle f, f_i \rangle f_i$ and $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$. This implies that if the frame is a tight frame with frame bound A, then $S = AI_{\mathcal{H}}$. We need the following fact about the optimal frame bounds which is Proposition 3.4 of [12]. See also Section 1.3 of [39].

Proposition 2.3 ([12, 39]) *Suppose that $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$, and that T and S are the pre-frame operator and the frame operator with respect to the frame, respectively. Then, the optimal lower frame bound is $\|S^{-1}\|^{-1} = \|T^\dagger\|^{-2}$, and the optimal upper frame bound is $\|S\| = \|T\|^2$.*

We say that $\{f_i : i \in I\}$ is a *Riesz basis* for $\mathcal{H}$ with Riesz bounds A and B if it is complete and there exist positive constants A and B such that for any $(c_i)_{i\in I} \in \ell^2(I)$

$$A \sum_{i\in I} |c_i|^2 \leq \| \sum_{i\in I} c_i f_i\|^2 \leq B \sum_{i\in I} |c_i|^2.$$

We refer to [13, 17, 21, 24, 25, 45] for the basic properties of Riesz bases and frames of a separable Hilbert space. In particular, it is shown there that a Riesz basis is a frame. Note also that if I is a finite set, then a Riesz basis is just an ordinary basis treated in Linear Algebra, and a frame is just a spanning set.

Finally, we use the following standard result frequently throughout this article.

Proposition 2.4 ([11]) *Suppose that T is a bounded linear operator from a separable Hilbert space $\mathcal{H}_1$ to another one $\mathcal{H}_2$. T is bounded below, i.e., there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for each $x \in \mathcal{H}_1$, if and only if T^* is onto. In particular, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, then T is one-to-one if and only if T^* is onto.*

3 Infimum cosine angle between two finite dimensional subspaces

In this section we calculate the infimum cosine angle between two finite dimensional spaces via the spanning sets of the spaces. The method of the proofs in this section shows that, even in a finite dimensional space in which a frame is just a spanning set, the theory of frames is useful to unravel some structures of the space. Most of the results in this section hold only for finite dimensional cases. For example, see the comments after the proof of Lemma 3.5.

Throughout the rest of this section we assume the following: Let $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$ be finite sequences of $\mathcal{H}$. Let $U := \text{span}\{u_j\}_{j=1}^m$, and $V := \text{span}\{v_i\}_{i=1}^n$. As noted in Section 2, a finite sequence is a frame for its linear span. Therefore, $\{u_j\}_{j=1}^m$ is a frame for U with frame bounds, say, A_U and B_U, and $\{v_i\}_{i=1}^n$ is a frame V with frame bounds, say, A_V and B_V.

Let $T_U : \mathbb{C}^m \to U$ and $T_V : \mathbb{C}^n \to V$ be the pre-frame operators of $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$, respectively. Also let $S_U : U \to U$ and $S_V : V \to V$ be the frame operators of $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$, respectively, $G := G_{U,V} : \mathbb{C}^m \to \mathbb{C}^n$ be *the mixed Gramian* of the frames $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$ such that $G_{ij} := \langle u_j, v_i \rangle, 1 \leq i \leq n, 1 \leq j \leq m$, and let $G_U : \mathbb{C}^m \to \mathbb{C}^m$ be the *Gramian* of $\{u_j\}_{j=1}^m$ such that $(G_U)_{ij} := \langle u_j, u_i \rangle, 1 \leq i, j \leq m$. The Gramian $G_V : \mathbb{C}^n \to \mathbb{C}^n$ of $\{v_i\}_{i=1}^n$ is defined similarly. Note that our definition of Gramians and mixed Gramians are slightly different from the ones in [39]. We adopt the above definitions since we find them a little more convenient in our situations (for example, as in Lemma 3.1). The negligible digression from the usual definitions, however, does not cause any problems for reading the existing literature. Finally, let $P := P_V|_U : U \to V$ be the restriction of P_V on U throughout this section. We observe that $P^* = (P_V|_U)^* = P_U|_V$.

Lemma 3.1 *Let $G := G_{U,V} : \mathbb{C}^m \to \mathbb{C}^n$ be the mixed Gramian of the frames $\{u_j\}_{j=1}^m$ of U and $\{v_i\}_{i=1}^n$ of V. Then*

$$G = T_V^* P T_U. \tag{3.1}$$

In particular, $G_U = T_U^* T_U$*,* $G_V = T_V^* T_V$*,* $\operatorname{rank} G_U = \dim U$*,* $\operatorname{rank} G_V = \dim V$*, and* $\operatorname{rank} G = \operatorname{rank} P$*.*

Proof. For $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_m)^t \in \mathbb{C}^m$,

$$\begin{aligned}
T_V^* P T_U \alpha &= T_V^* P(\sum_{j=1}^m \alpha_j u_j) = T_V^*(\sum_{j=1}^m \alpha_j P u_j) \\
&= \left(\left\langle \sum_{j=1}^m \alpha_j P u_j, v_i \right\rangle \right)_{i=1}^n = (\sum_{j=1}^m \alpha_j \langle P u_j, v_i \rangle)_{i=1}^n \\
&= (\sum_{j=1}^m \alpha_j \langle P_V u_j, v_i \rangle)_{i=1}^n = (\sum_{j=1}^m \alpha_j \langle u_j, P_V v_i \rangle)_{i=1}^n \\
&= (\sum_{j=1}^m \alpha_j \langle u_j, v_i \rangle)_{i=1}^n = G\alpha.
\end{aligned}$$

This proves (3.1). In particular, $G_U = T_U^*(P_U|_U)T_U = T_U^* T_U$. Similarly, $G_V = T_V^* T_V$. Since T_U and T_V are onto, T_U^* and T_V^* are one-to-one. This shows that the rank conditions hold. □

We need the following fact which is Theorem 1.6 of [5] (see also [12, Lemma 2.4]) combined with Corollary 2.3 of [12].

Proposition 3.2 ([5, 12]) *If T is a bounded linear operator with closed range, then $(T^\dagger)^* = (T^*)^\dagger$, and $(T^*T)^\dagger = T^\dagger (T^*)^\dagger = T^\dagger (T^\dagger)^*$.*

Lemma 3.3 *If $\{u_j\}_{j=1}^n$ is a frame for the closed subspace U of $\mathcal{H}$, then the optimal lower frame bound is $\|G_U^\dagger\|^{-1}$ and the optimal upper frame bound is $\|G_U\|$, where G_U is the Gramian with respect to the frame.*

Proof. We have $G_U = T_U^* T_U$ by Lemma 3.1. Hence, $\|G_U\| = \|T_U\|^2$ is the optimal upper frame bound by Proposition 2.3. Since $G_U^\dagger = T_U^\dagger (T_U^\dagger)^*$ by Proposition 3.2, $\|G_U^\dagger\| = \|T_U^\dagger\|^2$. This completes the proof by Proposition 2.3. □

We now calculate the pseudo-inverse of the mixed Gramian $G := G_{U,V}$ in some special cases.

Lemma 3.4 *If $P := P_V|_U$ is one-to-one and* $\operatorname{ran} P$ *is invariant under S_V, then*

$$G^\dagger = T_U^\dagger P^\dagger (T_V^*)^\dagger = T_U^\dagger P^\dagger (T_V^\dagger)^*. \tag{3.2}$$

In particular, (3.2) holds if either P is invertible (in this case $P^\dagger = P^{-1}$) or P is one-to-one and $\{v_i\}_{i=1}^n$ is a tight frame for V.

Proof. Recall that $G = T_V^* P T_U$ by Lemma 3.1. Let $X := T_V^*$, and $Y := PT_U$. We check Conditions (i), (ii), (iii) of Proposition 2.2. Since $\operatorname{ran} X$ and $\operatorname{ran} Y$ are finite dimensional, they are closed. Moreover, $\operatorname{ran} XY$ is also finite dimensional, hence closed. Since $\operatorname{ran} X^* = \operatorname{ran} T_V = V$, $\operatorname{ran} X^*$ is invariant under YY^*. Since $\ker Y^*$ is a subspace of $V = \operatorname{ran} X^*$ and since $Y^* = T_U^* P^*$, $\operatorname{ran} X^* \cap \ker Y^* = \ker Y^* = \ker(T_U^* P^*)$. Since T_U is onto, T_U^* is one-to-one. Hence, $\ker(T_U^* P^*) = \ker P^* = (\operatorname{ran} P)^\perp$. Now, $\operatorname{ran} X^* \cap \ker Y^* = (\operatorname{ran} P)^\perp$ is invariant under $X^*X = T_V T_V^* = S_V$ since $\operatorname{ran} P$ is assumed to be invariant under the self-adjoint operator S_V ([11, Proposition 3.7]). We have, by Propositions 2.2 and 3.2, $G^\dagger = (PT_U)^\dagger (T_V^*)^\dagger = (PT_U)^\dagger (T_V^\dagger)^*$. We now apply Proposition 2.2 once again with $X := P$ and $Y := T_U$ this time. X, Y and XY have closed range since they are finite dimensional operators. Since P is one-to-one, P^* is onto. Therefore, $\operatorname{ran} X^* = \operatorname{ran} P^* = U$ is invariant under YY^*. Since T_U is onto, $\ker Y^* = \ker T_U^* = \{0\}$. Hence, $\operatorname{ran} X^* \cap \ker Y^* = \{0\}$, which is invariant under X^*X. This shows that $(PT_U)^\dagger = T_U^\dagger P^\dagger$ by Proposition 2.2 again. Therefore, we have shown (3.2): $G^\dagger = T_U^\dagger P^\dagger (T_V^*)^\dagger = T_U^\dagger P^\dagger (T_V^\dagger)^*$. If P is invertible, then clearly P is one-to-one and $\operatorname{ran} P = V$ is invariant under S_V. On the other hand, if $\{v_i\}_{i=1}^n$ is a tight frame for V, then S_V is a constant times I_V. Therefore, $\operatorname{ran} P$, whatever it is, is invariant under S_V. In either case, (3.2) follows. □

We give three examples to show that the injectivity of P and the invariance of $\operatorname{ran} P$ under S_V are independent conditions in Lemma 3.4. In the examples, each space is a subspace of $\mathbb{C}^2$, and all calculations are preformed with respect to the standard orthonormal basis of $\mathbb{C}^2$. Also, the calculations of the pseudo-inverses can be done by resorting to the definition. In the first example, P is one-to-one, $\operatorname{ran} P$ is not invariant under S_V, and (3.2) does not hold. In the second example, P is not one-to-one, $\operatorname{ran} P$ is invariant under S_V, and (3.2) does not hold. In the final example, P is not one-to-one, $\operatorname{ran} P$ is invariant under S_V, but (3.2) does hold. This implies that the conditions in Lemma 3.4 is not necessary for (3.2) to hold.

First, let $\mathcal{H} = \mathbb{C}^2, U := \operatorname{span}\{u_1 := (1,2)^t\}$, and $V := \operatorname{span}\{v_1 := (0,1)^t, v_2 := (1,1)^t\} = \mathbb{C}^2$. Then P_V is the identity map of $\mathbb{C}^2$ and so $P = P_V|_U = I_U$. Hence, P is one-to-one and $\operatorname{ran} P = U$. A direct calculation shows that

$$S_V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Obviously, U is not invariant under S_V. Now, $G : \mathbb{C} \to \mathbb{C}^2$ and $G = \binom{2}{3}$. Therefore, $G^\dagger : \mathbb{C}^2 \to \mathbb{C}$ is $G^\dagger \binom{x}{y} = \frac{2x+3y}{13}$. Since $T_U : \mathbb{C} \to U$ and $T_U z = \binom{z}{2z}$, $T_U^\dagger \binom{z}{2z} = z$ for each complex number z. Likewise

$$T_V^\dagger = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

since $T_V : \mathbb{C}^2 \to \mathbb{C}^2$ and

$$T_V = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence, $(T_V^\dagger)^* = T_V^\dagger$. Note $P\binom{z}{2z} = \binom{z}{2z}$ for each complex number z. Therefore, $P^\dagger\binom{x}{y} = \frac{x+2y}{5}\binom{1}{2}$. Finally,

$$T_U^\dagger P^\dagger (T_V^\dagger)^* \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x+y}{5} \neq G^\dagger \begin{pmatrix} x \\ y \end{pmatrix} = \frac{2x+3y}{13}.$$

On the other hand, let $U := \operatorname{span}\{u_1 := (1,0)^t, u_2 := (1,1)^t\} = \mathbb{C}^2 =: \mathcal{H}$, and $V := \operatorname{span}\{v_1 := (1,2)^t\}$. Then $S_V : V \to V$ is represented by $S_V\binom{z}{2z} = 5\binom{z}{2z}$. Now, $P = P_V|_U = P_V : \mathbb{C}^2 \to V$ is clearly not one-to-one, and $\operatorname{ran} P = V$ is invariant under S_V. Note that $P\binom{x}{y} = \frac{x+2y}{5}\binom{1}{2}$. Since P is onto, $P^\dagger\binom{z}{2z} = \binom{x}{y}$ if

$$\begin{pmatrix} z \\ 2z \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x+2y}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and $x^2 + y^2$ is minimized. A simple calculus shows $P^\dagger\binom{z}{2z} = \binom{z}{2z}$. The mixed Gramian $G : \mathbb{C}^2 \to \mathbb{C}$ is given by $G = \begin{pmatrix} 1 & 3 \end{pmatrix}$. Since G is onto, $G^\dagger z = \binom{x}{y}$ if $G\binom{x}{y} = x + 3y = z$ and $x^2 + y^2$ is minimized. We can easily check that $G^\dagger z = \frac{z}{10}\binom{1}{3}$. We also note that

$$T_U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T_U^\dagger = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Since $T_V : \mathbb{C} \to V$ is given by $T_V z = \binom{z}{2z}$ and is invertible, $T_V^\dagger\binom{z}{2z} = z$. Note that $(T_V^\dagger)^* z = \binom{x}{2x}$ for some $x \in \mathbb{C}$. Hence, $\left\langle (T_V^\dagger)^* z, \binom{w}{2w} \right\rangle_{\mathbb{C}^2} = \left\langle z, T_V^\dagger\binom{w}{2w} \right\rangle_{\mathbb{C}} = \langle z, w\rangle_{\mathbb{C}} = z\overline{w}$ for each $w \in \mathbb{C}$ if and only if $(T_V^\dagger)^* z = (1/5)\binom{z}{2z}$. Now,

$$G^\dagger z = \frac{z}{10} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \neq T_U^\dagger P^\dagger (T_V^\dagger)^* z = \frac{z}{5} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Finally, let $U := \operatorname{span}\{u_1 := (1,0)^t, u_2 := (0,1)^t\} = \mathbb{C}^2 =: \mathcal{H}, V := \operatorname{span}\{v_1 := (1,0)^t\}$. It is easy to see that $G^\dagger z = \binom{z}{0} = T_U^\dagger P^\dagger (T_V^\dagger)^* z$ since

$(T_V^\dagger)^* z = \binom{z}{0}$, $P^\dagger \binom{z}{0} = \binom{z}{0}$ and $T_U^\dagger \binom{z}{0} = \binom{z}{0}$. However, P is not one-to-one, even though $\operatorname{ran} P$ is invariant under S_V.

The following lemma gives a formula for the infimum cosine angle $R(U, V)$ in terms of the pseudo-inverse of $P := P_V|_U$.

Lemma 3.5 *Suppose that U is not trivial. Then*

$$R(U,V) = \begin{cases} 0, & \text{if } P \text{ is not one-to-one,} \\ \|P^\dagger\|^{-1}, & \text{if } P \text{ is one-to-one.} \end{cases} \tag{3.3}$$

In particular, $R(U, V) > 0$ if and only if P is one-to-one.

Proof. If P is not one-to-one, there is $u \in U \setminus \{0\}$ such that $Pu = 0$. Therefore $R(U, V) = 0$. Now suppose that P is one-to-one. Then, P^* is onto. Therefore, $P^\dagger P = P_{\operatorname{ran} P^*} = I_U$ by (3) of Proposition 2.1. It is easy to see that $(\operatorname{ran} P)^\perp \subset \ker P^\dagger$ from the definition of the pseudo-inverse (see [23, p. 52]). For any $v \in V \setminus \ker P^\dagger$, there exist $u \in U \setminus \{0\}$ and $w \in V \ominus \operatorname{ran} P$ such that $v = Pu + w$. Since P is one-to-one, $Pu \neq 0$. Therefore, we have

$$\frac{\|P^\dagger v\|^2}{\|v\|^2} = \frac{\|P^\dagger Pu + P^\dagger w\|^2}{\|Pu\|^2 + \|w\|^2} = \frac{\|P^\dagger Pu\|^2}{\|Pu\|^2 + \|w\|^2} \leq \frac{\|P^\dagger Pu\|^2}{\|Pu\|^2} \leq \|P^\dagger\|^2.$$

This implies that

$$\|P^\dagger\| = \sup_{u \in U \setminus \{0\}} \frac{\|P^\dagger Pu\|}{\|Pu\|}.$$

Recall that $P^\dagger Pu = u$ since P is one-to-one. Now, we have

$$\begin{aligned} R(U,V) &= \inf_{u \in U \setminus \{0\}} \frac{\|Pu\|}{\|u\|} \\ &= \inf_{u \in U \setminus \{0\}} \frac{\|Pu\|}{\|P^\dagger Pu\|} \\ &= \left(\sup_{u \in U \setminus \{0\}} \frac{\|P^\dagger Pu\|}{\|Pu\|} \right)^{-1} \\ &= \|P^\dagger\|^{-1}. \end{aligned}$$

This proves (3.3). If P is one-to-one, then, by the second case in (3.3), $R(U,V) = \|P^\dagger\|^{-1} > 0$ since $\|P^\dagger\| < \infty$. If P is not one-to-one, then $R(U,V) = 0$ by the first case in (3.5) again. $\square$

We mention that the formula (3.3) holds only for the finite dimensional cases. To see this we construct two infinite dimensional subspaces U and V of $\mathcal{H}$ such that $R(U,V) = 0$ and that $P = P_V|_U$ is one-to-one. Once these spaces are constructed, then (3.3) cannot hold since $P^\dagger$ is a bounded operator. Let $\{e_1, e_2, \cdots\}$ be an orthonormal basis of $\mathcal{H}$. Let

$$U := \overline{\text{span}}\left\{e_2 + \frac{1}{3}e_3, e_4 + \frac{1}{5}e_5, e_6 + \frac{1}{7}e_7, \cdots\right\}$$
$$V := \overline{\text{span}}\left\{e_1 + \frac{1}{2}e_2, e_3 + \frac{1}{4}e_4, e_5 + \frac{1}{6}e_6, \cdots\right\}.$$

Define, for $n = 1, 2, \cdots$,

$$u_n := e_{2n} + \tfrac{1}{2n+1}e_{2n+1}, \quad \tilde{u}_n := \frac{u_n}{\|u_n\|} = \frac{e_{2n} + \frac{1}{2n+1}e_{2n+1}}{\sqrt{1 + \left(\frac{1}{2n+1}\right)^2}},$$
$$v_n := e_{2n-1} + \tfrac{1}{2n}e_{2n}, \quad \tilde{v}_n := \frac{v_n}{\|v_n\|} = \frac{e_{2n-1} + \frac{1}{2n}e_{2n}}{\sqrt{1 + \left(\frac{1}{2n}\right)^2}}.$$

Note that $\{\tilde{u}_n\}_{n=1}^\infty$ and $\{\tilde{v}_n\}_{n=1}^\infty$ are orthonormal bases for U and V, respectively. Direct calculations show that:

$$\langle \tilde{u}_n, \tilde{v}_n \rangle = \frac{\frac{1}{2n}}{\sqrt{1 + \left(\frac{1}{2n+1}\right)^2}\sqrt{1 + \left(\frac{1}{2n}\right)^2}};$$
$$\langle \tilde{u}_n, \tilde{v}_{n+1} \rangle = \frac{\frac{1}{2n+1}}{\sqrt{1 + \left(\frac{1}{2n+1}\right)^2}\sqrt{1 + \left(\frac{1}{2n+2}\right)^2}};$$
$$\langle \tilde{u}_n, \tilde{v}_k \rangle = 0 \text{ if } k \neq n \text{ nor } k \neq n+1.$$

Now, $\|P_V\tilde{u}_n\|^2 = |\langle \tilde{u}_n, \tilde{v}_n \rangle|^2 + |\langle \tilde{u}_n, \tilde{v}_{n+1} \rangle|^2 \leq 1/(2n)^2 + 1/(2n+1)^2 \to 0$ as $n \to \infty$. This shows that $R(U,V) = 0$. On the other hand, let $u :=$

$\sum_{n=1}^{\infty} \alpha_n \tilde{u}_n$ with $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. Suppose that $Pu = 0$. Then, $|\langle u, \tilde{v}_k \rangle|^2 = 0$ for each $k = 1, 2, \cdots$. For $k = 1$, $|\langle u, \tilde{v}_1 \rangle|^2 = |\alpha_1 \langle \tilde{u}_1, \tilde{v}_1 \rangle|^2$. Hence $\alpha_1 = 0$. For $k = 2$, $|\langle u, \tilde{v}_2 \rangle|^2 = |\alpha_1 \langle \tilde{u}_1, \tilde{v}_2 \rangle + \alpha_2 \langle \tilde{u}_2, \tilde{v}_2 \rangle|^2 = |\alpha_2 \langle \tilde{u}_2, \tilde{v}_2 \rangle|^2$. Therefore $\alpha_2 = 0$. In this way, we see that $\alpha_k = 0$ for each $k = 1, 2, 3, \cdots$. Hence $u = 0$, which shows that P is one-to-one.

Lemma 3.6 *Suppose that one of the following two conditions holds:*

(1) $0 < \operatorname{rank} G = \dim U = \dim V$*;*

(2) $0 < \operatorname{rank} G = \dim U < \dim V$ *and* $\operatorname{ran} P$ *is invariant under* S_V*.*

Then

$$R(U, V) = \|G_U^{1/2} G^{\dagger} G_V^{1/2}\|^{-1}. \tag{3.4}$$

Proof. In either case, we note that T_U is onto, T_V^* is one-to-one, and $\dim U = \operatorname{rank} G$. By Lemma 3.1, we have

$$\dim \operatorname{dom} P = \dim U = \operatorname{rank} G = \operatorname{rank}(T_V^* P T_U) = \operatorname{rank}(P T_U) = \operatorname{rank} P.$$

This implies that P is one-to-one. If Condition (1) holds, then $\operatorname{ran} P = V$ and it is invariant under S_V. Therefore, in either case, $G^{\dagger} = T_U^{\dagger} P^{\dagger} (T_V^*)^{\dagger}$ by Lemma 3.4. Proposition 2.1 implies that $(T_V^*)^{\dagger} T_V^* = I_V$, and $T_U T_U^{\dagger} = I_U$. Lemma 3.4 then implies that

$$T_U G^{\dagger} T_V^* = (T_U T_U^{\dagger}) P^{\dagger} ((T_V^*)^{\dagger} T_V^*) = I_U P^{\dagger} I_V = P^{\dagger}.$$

Note that $G_U = T_U^* T_U$ and $G_V = T_V^* T_V$ by Lemma 3.1. By Lemma 3.5, we finally compute $R(U,V)$ as follows:

$$\begin{aligned}
R(U,V) &= \|P^\dagger\|^{-1}\\
&= \|P^\dagger(P^\dagger)^*\|^{-1/2}\\
&= \|T_U G^\dagger T_V^* T_V (G^\dagger)^* T_U^*\|^{-1/2}\\
&= \|T_U G^\dagger G_V (G^\dagger)^* T_U^*\|^{-1/2}\\
&= \|T_U G^\dagger G_V^{1/2} G_V^{1/2} (G^\dagger)^* T_U^*\|^{-1/2}\\
&= \|T_U G^\dagger G_V^{1/2}\|^{-1}\\
&= \|G_V^{1/2} (G^\dagger)^* T_U^* T_U G^\dagger G_V^{1/2}\|^{-1/2}\\
&= \|G_V^{1/2} (G^\dagger)^* G_U G^\dagger G_V^{1/2}\|^{-1/2}\\
&= \|G_V^{1/2} (G^\dagger)^* G_U^{1/2} G_U^{1/2} G^\dagger G_V^{1/2}\|^{-1/2}\\
&= \|G_U^{1/2} G^\dagger G_V^{1/2}\|^{-1},
\end{aligned}$$

where we have used $\|X\| = \|XX^*\|^{1/2} = \|X^*X\|^{1/2}$ several times. □

Without the assumptions of the invariance of $\operatorname{ran} P$ under S_V in the second case in Lemma 3.6, $R(U,V)$ is given as in the following lemma.

Lemma 3.7 *If* $0 < \operatorname{rank} G = \dim U < \dim V$, *then*

$$R(U,V) = \|G_U^{1/2}((G_V^\dagger)^{1/2} G)^\dagger|_{(\ker T_V)^\perp}\|^{-1}. \tag{3.5}$$

Proof. We 'tightize' the frame $\{v_j\}_{j=1}^n$ for V: For $i = 1, 2, \cdots, n$, define

$$\tilde{v}_i = \sum_{j=1}^n \overline{((G_V^\dagger)^{1/2})}_{ij} v_j \in V. \tag{3.6}$$

We claim that $\{\tilde{v}_j\}_{j=1}^n$ is a tight frame for V with frame bound 1. Clearly, $\{\tilde{v}_j\}_{j=1}^n$ is a frame for its linear span which is a subspace of V. Let $G_{\tilde{V}}$ denote the Gramian of $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\}$. We show that $G_{\tilde{V}} = (G_V^\dagger)^{1/2} G_V (G_V^\dagger)^{1/2}$. Temporarily, let $M := \overline{(G_V^\dagger)^{1/2}}$. Then,

$$\begin{aligned}
(G_{\tilde{V}})_{ij} &= \langle \tilde{v}_j, \tilde{v}_i \rangle \\
&= \left\langle \sum_{k=1}^{n} M_{jk} v_k, \sum_{l=1}^{n} M_{il} v_l \right\rangle \\
&= \sum_{l,k=1}^{n} \overline{M}_{il} \langle v_k, v_l \rangle M_{jk} \\
&= \sum_{l,k=1}^{n} \overline{M}_{il} (G_V)_{lk} M_{jk} \\
&= (\overline{M} G_V M^t)_{ij} \\
&= ((G_V^\dagger)^{1/2} G_V (G_V^\dagger)^{1/2})_{ij}.
\end{aligned}$$

The last equality follows from the fact that $(G_V^\dagger)^{1/2}$ is positive semi-definite (hence self-adjoint) since G_V is positive semi-definite. The positive semi-definite matrix G_V has the spectral decomposition $G_V = QDQ^*$, where Q is a unitary matrix and $D := \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the diagonal matrix with the (non-negative) eigenvalues $\lambda_i' s$ of G_V as its diagonal entries. For, $i = 1, 2, \cdots, n$, define

$$\mu_i := \begin{cases} 1/\lambda_i, & \text{if } \lambda_i \neq 0, \\ 0, & \text{if } \lambda_i = 0, \end{cases}$$

$D_1 := \text{diag}(\mu_1, \mu_2, \cdots, \mu_n)$, and $D_2 := \text{diag}(\mu_1^{1/2}, \mu_2^{1/2}, \cdots, \mu_n^{1/2})$. Then,

$$G_V^\dagger = QD_1Q^* \text{ and } (G_V^\dagger)^{1/2} = QD_2Q^* \tag{3.7}$$

[23]. Hence,

$$\begin{aligned} G_{\tilde{V}} &= (G_V^{\dagger})^{1/2} G_V (G_V^{\dagger})^{1/2} \\ &= UDD_2^2U^* = UDD_1U^* = UDU^*UD_1U^* = G_V(G_V^{\dagger}) \\ &= P_{\operatorname{ran} G_V} = P_{\operatorname{ran} T_V^* T_V} = P_{\operatorname{ran} T_V^*} = P_{(\ker T_V)^{\perp}}, \end{aligned} \tag{3.8}$$

where we have used Proposition 2.1 (2) in the sixth equality, and the surjectivity of T_V in the penultimate equality. Therefore,

$$\dim \operatorname{span}\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\} = \operatorname{rank} G_{\tilde{V}} = \operatorname{rank} G_V = \dim V,$$

where the first equality holds by Lemma 3.1 and the second one by the sixth equality in (3.8). This shows that $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\}$ is a frame for V. Since $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\}$ is a frame for a non-trivial space V, $G_{\tilde{V}} \neq 0$. Since the eigenvalues of $G_{\tilde{V}}$ are zero or one by the third equality in (3.8), $\|G_{\tilde{V}}\| = \|G_{\tilde{V}}^{\dagger}\| = 1$. Lemma 3.3 implies that $\{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n\}$ is a tight frame with frame bound 1 for V.

Therefore, we have $S_{\tilde{V}} = I_V$; so $\operatorname{ran} P$ is invariant under $S_{\tilde{V}}$. A calculation similar to the one at the beginning of the proof shows that $G_{U,\tilde{V}} = (G_V^{\dagger})^{1/2}G$. We are now able to apply Lemma 3.6 to conclude that

$$R(U, V) = \|G_U^{1/2}((G_V^{\dagger})^{1/2}G)^{\dagger}G_{\tilde{V}}^{1/2}\|^{-1}.$$

Since $G_{\tilde{V}}$ is an orthogonal projection, $G_{\tilde{V}}^{1/2} = G_{\tilde{V}}$. By (3.8) we have

$$R(U, V) = \|G_U^{1/2}((G_V^{\dagger})^{1/2}G)^{\dagger}|_{(\ker T_V)^{\perp}}\|^{-1}.$$

This completes the proof. □

The following, which generalizes Equation (1.3) of [9], is the main result in this section.

Theorem 3.8 *Let U and V be finite dimensional subspaces of a separable complex Hilbert space $\mathcal{H}$. Suppose that $U = \operatorname{span}\{u_1, u_2, \cdots, u_m\}, V = \operatorname{span}\{v_1, v_2, \cdots, v_n\}$. Let G_U, G_V, G, T_V and S_V be as in the paragraph preceding Lemma 3.1. Then,*

$$R(U, V) \tag{3.9}$$
$$= \begin{cases} 1, & \textit{if } U = \{0\}; \\ \|G_U^{1/2} G^\dagger G_V^{1/2}\|^{-1}, & \textit{if } 0 < \operatorname{rank} G = \dim U = \dim V \textit{ or} \\ & \textit{if } 0 < \operatorname{rank} G = \dim U < \dim V \\ & \textit{and } \operatorname{ran} P \textit{ is invariant under } S_V; \\ \|G_U^{1/2}((G_V^\dagger)^{1/2} G)^\dagger|_{(\ker T_V)^\perp}\|^{-1}, & \textit{if } 0 < \operatorname{rank} G = \dim U < \dim V \\ & \textit{and } \operatorname{ran} P \textit{ is not invariant under } S_V; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We only need to consider the following cases: $\operatorname{rank} G \neq \dim U > 0$ or $\dim V < \dim U$. In either case, we show that P is not one-to-one; so $R(U, V) = 0$ by Lemma 3.5. In the first case, note that

$$\operatorname{rank} P = \operatorname{rank}(T_V^* P T_U) = \operatorname{rank} G \neq \dim U = \dim \operatorname{dom} P,$$

where we have used Lemma 3.1 and the fact that T_V^* is one-to-one and that T_U is onto. Hence P is not one-to-one. In the second case, $\dim \operatorname{dom} P = \dim U > \dim V = \dim \operatorname{co-dom} P$. Hence P is not one-to-one. □

We now consider the conditions in order to have $R(U, V) = R(V, U)$.

Lemma 3.9 *Suppose that $R(U, V) > 0$. Then the following assertions are equivalent:*

(1) *$R(U, V) = R(V, U)$;*

(2) $R(V, U) > 0$;

(3) $0 < \dim U = \dim V$;

(4) $0 < \operatorname{rank} G_U = \operatorname{rank} G_V$.

Proof. The implication from (1) to (2) is trivial. Suppose that $R(U, V) > 0$ and that $R(V, U) > 0$. Then neither U nor V is trivial. Now, Lemma 3.5 implies that $P_V|_U$ and $P_U|_V$ are both one-to-one. Therefore $0 < \dim U = \dim V$. This shows that (2) implies (3). Suppose that $R(U, V) > 0$ and that $0 < \dim U = \dim V$. By Lemma 3.5 $P_V|_U$ is one-to-one and onto. Hence $P_U|_V = (P_V|_U)^*$ is also one-to-one and onto. Hence $R(V, U) > 0$ by Lemma 3.5 once again. Finally, (3) and (4) are equivalent by Lemma 3.1. □

Proposition 3.10 *The following assertions hold:*

(1) *If* $\dim U = \dim V$, *then* $R(U, V) = R(V, U)$;

(2) *If* $R(U, V) = R(V, U)$, *then either* $R(U, V) = R(V, U) = 0$ *or* $0 < \dim U = \dim V$;

(3) *If* $R(U, V) > 0$ *and* $R(V, U) > 0$, *then* $R(U, V) = R(V, U)$ *and* $0 < \dim U = \dim V$.

Proof. Suppose that $0 = \dim U = \dim V$. Then $R(U, V) = R(V, U) = 1$ by convention. Now suppose that $0 < \dim U = \dim V$. If $P_V|_U$ is one-to-one, then $R(U, V) > 0$ by Lemma 3.5, and hence $R(U, V) = R(V, U)$ by Lemma 3.9. Now, if $P_V|_U$ is not one-to-one, then it is not onto either since its domain U and co-domain V are of the same dimension. Therefore, $P_U|_V = (P_V|_U)^*$ is nether one-to-one nor onto. Hence $R(U, V) = R(V, U) = 0$ by Lemma

3.5. This proves (1). Now suppose that $0 < R(U,V) = R(V,U)$. Then, $0 < \dim U = \dim V$ by Lemma 3.9. This proves (2). Finally, (3) follows by Lemma 3.9. $\square$

That the converse of the first assertion of Proposition 3.10 does not hold can be seen by noting that $R(U,V) = R(V,U) = 0$ if U and V are mutually orthogonal.

We end this section with a lemma which is needed in Section 5

Lemma 3.11 *If P is invertible, then*

$$\left((G_V^\dagger)^{1/2} G (G_U^\dagger)^{1/2}\right)^\dagger = G_U^{1/2} G^\dagger G_V^{1/2}. \tag{3.10}$$

Proof. Let $X := (G_V^\dagger)^{1/2}$ and $Y := G(G_U^\dagger)^{1/2}$. Observe, by an application of the spectral theorem, that if M is a positive semi-definite matrix, then $\operatorname{ran} M = \operatorname{ran} M^\dagger = \operatorname{ran} M^{1/2} = (\operatorname{ran} M^\dagger)^{1/2}$ and $\ker M = \ker M^\dagger = \ker M^{1/2} = \ker(M^\dagger)^{1/2}$. Since X is positive semi-definite and T_V is onto, $\operatorname{ran} X^* = \operatorname{ran}(G_V^\dagger)^{1/2} = \operatorname{ran} G_V^\dagger = \operatorname{ran} T_V^* T_V = \operatorname{ran} T_V^*$. We note that

$$\begin{aligned} YY^* &= G(G_U^\dagger)^{1/2}(G_U^\dagger)^{1/2}G^* \\ &= GG_U^\dagger G^* \\ &= T_V^* P T_U (T_U^* T_U)^\dagger T_U^* P^* T_V; \end{aligned}$$

And hence $\operatorname{ran} YY^* \subset \operatorname{ran} T_V^*$. Therefore $\operatorname{ran} X^* = \operatorname{ran} T_V^*$ is invariant under

YY^*. On the other hand,

$$\begin{aligned}\ker Y^* &= \ker(G(G_U^\dagger)^{1/2})^* \\ &= \ker(G_U^\dagger)^{1/2}G^* \\ &= \ker G_U^\dagger G^* \quad \text{(above observation)} \\ &= \ker T_U^\dagger (T_U^*)^\dagger T_U^* P^* T_V \quad \text{(Lemma 3.2, Lemma 3.1)} \\ &= \ker T_U^\dagger P_{\operatorname{ran} T_U} P^* T_V \quad \text{(Proposition 2.1)} \\ &= \ker T_U^\dagger P^* T_V \quad (\operatorname{ran} T_U = \operatorname{ran} P^*) \\ &= \ker T_V. \quad (T_U^\dagger \text{ and } P^* \text{ are one-to-one})\end{aligned}$$

Therefore, $\operatorname{ran} X^* \cap \ker YY^* = \operatorname{ran} T_V^* \cap \ker T_V = \{0\}$, which is trivially invariant under X^*X. We have, by Proposition 2.2,

$$((G_V^\dagger)^{1/2} G (G_U^\dagger)^{1/2})^\dagger = (G(G_U^\dagger)^{1/2})^\dagger ((G_V^\dagger)^{1/2})^\dagger = (G(G_U^\dagger)^{1/2})^\dagger G_V^{1/2}. \quad (3.11)$$

Now let $X := G$ and $Y := (G_U^\dagger)^{1/2}$. It is routine to check that $\operatorname{ran} X^* = \operatorname{ran} T_U^*$ and that $YY^* = T_U^\dagger (T_U^*)^\dagger$. Hence $\operatorname{ran} YY^* \subset \operatorname{ran} T_U^\dagger = \operatorname{ran} T_U^* = \operatorname{ran} X^*$. This shows that $\operatorname{ran} X^*$ is invariant under YY^*. On the other hand, by the observation we made and by the fact that T_U^* is one-to-one, $\ker Y^* = \ker G_U = \ker T_U^* T_U = \ker T_U$. This shows that $\operatorname{ran} X^* \cap \ker Y^* = \operatorname{ran} T_U^* \cap \ker T_U = \{0\}$, which is trivially invariant under X^*X. By Proposition 2.2 we have

$$(G(G_U^\dagger)^{1/2})^\dagger = ((G_U^\dagger)^{1/2})^\dagger G^\dagger = G_U^{1/2} G^\dagger. \quad (3.12)$$

Now, the lemma follows from (3.11) and (3.12). □

4 Applications to shift-invariant spaces

In this section we apply Theorem 3.8 to shift-invariant subspaces of $L^2(\mathbb{R}^d)$.

First, we review the basic theory of shift-invariant spaces briefly. All of the results on the theory of shift-invariant spaces we use are contained in [6, 8, 9, 26, 28, 39].

For $y \in \mathbb{R}^d$, define the translation operator $T_y : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ via $(T_y f)(x) := f(x-y)$. A subspace $S \subset L^2(\mathbb{R}^d)$ is said to be a *shift-invariant subspace* of $L^2(\mathbb{R}^d)$ if it is closed and is invariant under each (multi-)integer translation operator $T_k, k \in \mathbb{Z}^d$. For $f \in L^2(\mathbb{R}^d), x \in \mathbb{T}^d$, we let

$$\hat{f}_{||x} := (\hat{f}(x+k))_{k\in\mathbb{Z}^d},$$

which belongs to $\ell^2(\mathbb{Z}^d)$ for almost every $x \in \mathbb{T}^d := [0,1]^d$. Here $\wedge$ denotes the Fourier transform defined by

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(t)^{-2\pi i x\cdot t}\, dt$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and extended to be a unitary operator on $L^2(\mathbb{R}^d)$ by the Plancherel theorem. For a shift-invariant subspace S and $x \in \mathbb{T}^d$ we let

$$\hat{S}_{||x} := \{\hat{f}_{||x} : f \in S\}.$$

It is known that $\hat{S}_{||x}$, called the *fiber* of S at x, is a closed subspace of $\ell^2(\mathbb{Z}^d)$ for almost every $x \in \mathbb{T}^d$. The *spectrum* $\sigma(S)$ of S is defined to be

$$\sigma(S) := \{x \in \mathbb{T}^d : \hat{S}_{||x} \neq \{0\}\}.$$

If there exists $n \in \mathbb{N}$ such that $\dim \hat{S}_{||x} = n$ for almost every $x \in \mathbb{T}$, we say that S is *regular*. If Φ is a subset of $L^2(\mathbb{R}^d)$, then we let

$$\mathcal{S}(\Phi) := \overline{\operatorname{span}}\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\},$$

which is clearly a shift-invariant subspace. We then say that $\mathcal{S}(\Phi)$ is a *shift-invariant space generated by* Φ. In case Φ is finite, we say that it is *finitely generated.* The *length* of a shift-invariant subspace S is defined to be

$$\operatorname{len} S := \min\{\#\Phi : S = \mathcal{S}(\Phi)\}.$$

It is shown in [6, Theorem 3.5] that $\operatorname{len} S = \text{ess-sup}\{\dim \hat{S}_{||x} : x \in \mathbb{T}^d\}$. It is known that a shift-invariant subspace of $L^2(\mathbb{R}^d)$ has a generating set whose cardinality is at most countable. Moreover, it is shown in [6] that S is regular if and only if there exists a finite subset Φ of S such that $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a Riesz basis for S. The following proposition is sometimes called the *fundamental theorem of shift-invariant spaces* [8, Proposition 1.5]. The proof of (the strong form of) the theorem can be found in [6, 8, 26, 28].

Proposition 4.1 ([6, 8, 26, 28]) *For* $\Phi \subset L^2(\mathbb{R}^d)$, $(\mathcal{S}(\Phi))^{\wedge}_{||x} = \overline{\operatorname{span}}\hat{\Phi}_{||x}$, *and* $f \in L^2(\mathbb{R}^d)$ *is an element of* $\mathcal{S}(\Phi)$ *if and only if* $\hat{f}_{||x} \in (\mathcal{S}(\Phi))^{\wedge}_{||x}$ *a.e.* $x \in \mathbb{T}^d$.

The so-called *fiber principle* is roughly stated as follows: A property holds for a shift-invariant space S if and only if it holds for each fiber space of S in a uniform way. It is best understood by looking at examples. Hence we introduce some examples of the fiber principle which will be used later in proving our main results in this section.

The following is Proposition 2.10 of [9], slightly modified for our purposes.

Proposition 4.2 ([9]) *If* U *and* V *are shift-invariant subspaces of* $L^2(\mathbb{R}^d)$, *then*

$$R(U,V) = \begin{cases} \text{ess-inf}_{x\in\sigma(U)} R(\hat{U}_{||x}, \hat{V}_{||x}), & \textit{if } |\sigma(U)| > 0, \\ 1, & \textit{if } |\sigma(U)| = 0. \end{cases}$$

Combining Propositions 3.10 and 4.2 yields the following corollaries. We mention that (1) of Corollary 4.3 is Corollary 2.12 in [9].

Corollary 4.3 *If U and V are finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$, then the following assertions hold:*

(1) *If $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ a.e., then $R(U,V) = R(V,U)$;*

(2) *If $R(U,V) > 0$ and $R(V,U) > 0$, then $R(U,V) = R(V,U)$ and $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ a.e., in particular, $\sigma(U) = \sigma(V)$.*

The following is Theorem 2.3 of [8].

Proposition 4.4 ([8]) *Suppose that $\Phi \subset L^2(\mathbb{R}^d)$ is at most countable. Then $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a frame/Riesz basis for $\mathcal{S}(\Phi)$ with frame/Riesz bounds A and B if and only if, for almost every $x \in \mathbb{T}^d$, $\{\hat{\varphi}_{||x} : \varphi \in \Phi\}$ is a frame/Riesz basis for $(\mathcal{S}(\Phi))^{\wedge}_{||x}$ with frame/Riesz bounds A and B.*

The readers are now convinced that if one is to analyze a shift-invariant subspace, then it probably is best to analyze the fiber spaces separately and then to patch up the fiber-wise analyses together to produce a result on the original shift-invariant space. There is an elegant theory, called the Gramian/dual Gramian analysis, which somehow formalizes this method [6, 8, 40, 41]. The following is an example. First, we need some definitions.

Let $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_m\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$, and let $U := \mathcal{S}(\Phi), V := \mathcal{S}(\Psi)$. The $m \times m$ matrix

$$G_\Phi(x) := \left(\left\langle \hat{\varphi}_{j_{||x}}, \hat{\varphi}_{i_{||x}} \right\rangle\right)_{1\le i,j\le m}$$

is called the *Gramian of* Φ *at* $x \in \mathbb{T}^d$. The $n \times n$ Gramian of Ψ at x, which is denoted by $G_\Psi(x)$, is defined similarly. Finally, let the $n \times m$ matrix

$$G(x) := G_{\Phi,\Psi}(x) := \left(\left\langle \hat{\varphi}_{j\,||x}, \hat{\psi}_{i||x} \right\rangle\right)_{1\leq i\leq n, 1\leq j\leq m}, \; x \in \mathbb{T}^d,$$

is called the *mixed Gramian of* Φ *and* Ψ *at* x. Note that, according to Proposition 4.1, $G_\Phi(x)$ is just the Gramian of the frame (being a finite spanning set) $\hat{\Phi}_{||x}$ for $\hat{U}_{||x}$, and that $G_\Psi(x)$ is the Gramian of the frame $\hat{\Psi}_{||x}$ for $\hat{V}_{||x}$, and that $G(x)$ is the mixed Gramian of $\hat{\Phi}_{||x}$ and $\hat{\Psi}_{||x}$. Let $T_\Phi(x) : \mathbb{C}^m \to \hat{U}_{||x}, S_\Phi(x) : \hat{U}_{||x} \to \hat{U}_{||x}$ be the pre-frame operator and the frame operator of $\hat{\Phi}_{||x}$, respectively. The pre-frame operator $T_\Psi(x)$ and the frame operator $S_\Psi(x)$ are defined similarly. Finally, let $P(x) := P_{\hat{V}_{||x}}|_{\hat{U}_{||x}}$ be the restriction to $\hat{U}_{||x}$ of the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto $\hat{V}_{||x}$ throughout the rest of this article.

Combining Proposition 4.2 and Lemma 3.9 yields the following corollaries.

Corollary 4.5 *Let* $U := \mathcal{S}(\Phi)$ *and* $V := \mathcal{S}(\Psi)$ *be finitely generated shift-invariant subspaces of* $L^2(\mathbb{R}^d)$ *Suppose that* $R(U, V) > 0$. *Then the following assertions are equivalent:*

(1) $R(U, V) = R(V, U)$;

(2) $R(V, U) > 0$;

(3) $0 < \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ *a.e;*

(4) $0 < \operatorname{rank} G_\Phi(x) = \operatorname{rank} G_\Psi(x)$.

The proof of the following proposition is found in [39, Theorem 2.3.6].

Proposition 4.6 ([39]) *Suppose that Φ is finite. Let $\lambda(x), \lambda^+(x)$ and $\Lambda(x)$ denote the smallest eigenvalue, the smallest positive eigenvalue and the largest eigenvalue of $G_\Phi(x)$. $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a Riesz basis for $\mathcal{S}(\Phi)$ with Riesz bounds A and B if and only if*

$$A \le \lambda(x) \le \Lambda(x) \le B$$

for almost every $x \in \mathbb{T}^d$. It is a frame for $\mathcal{S}(\Phi)$ with frame bounds A and B if and only if

$$A \le \lambda^+(x) \le \Lambda(x) \le B$$

for almost every $x \in \sigma(\mathcal{S}(\Phi))$.

Combining Theorem 3.8 and Proposition 4.2 yields the following theorem on the infimum cosine angle between two finitely generated shift-invariant spaces.

Theorem 4.7 *Let $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_m\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$, and let $U := \mathcal{S}(\Phi)$, $V := \mathcal{S}(\Psi)$. Define*

$$\Gamma := \{x \in \sigma(U) : \operatorname{rank} G(x) = \dim \hat{U}_{||x} \le \dim \hat{V}_{||x}\}.$$

Then the following holds;

(1) *If $|\sigma(U)| = 0$, then $R(U, V) = 1$.*

(2) *If $|\sigma(U)| > 0$ and $\Gamma \ne \sigma(U)$, then $R(U, V) = 0$.*

(3) *If $|\sigma(U)| > 0$ and $\Gamma = \sigma(U)$, then*

$$R(U,V) = \min\{\, \text{ess-inf}_{x\in\Gamma_1\cup\Gamma_2} \|G_\Phi(x)^{1/2} G(x)^\dagger G_\Psi(x)^{1/2}\|^{-1}, \\ \text{ess-inf}_{x\in\Gamma_3} \|G_\Phi(x)^{1/2}((G_\Psi(x)^\dagger)^{1/2} G(x))^\dagger|_{(\ker T_\Psi(x))^\perp}\|^{-1}\},$$

where

$$\begin{aligned}
\Gamma_1 &:= \{x \in \sigma(U) : \operatorname{rank} G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}\}, \\
\Gamma_2 &:= \{x \in \sigma(U) : \operatorname{rank} G(x) = \dim \hat{U}_{||x} < \dim \hat{V}_{||x}, \\
&\qquad \operatorname{ran} P(x) \text{ is invariant under } S_\Psi(x)\}, \\
\Gamma_3 &:= \{x \in \sigma(U) : \operatorname{rank} G(x) = \dim \hat{U}_{||x} < \dim \hat{V}_{||x}, \\
&\qquad \operatorname{ran} P(x) \text{ is not invariant under } S_\Psi(x)\},
\end{aligned}$$

Proof. Statement (1) holds since U is trivial if $|\sigma(U)| = 0$. If $|\sigma(U)| > 0$ and $\Gamma \neq \sigma(U)$, then there exists a subset of $\sigma(U)$ having a positive Lebesgue measure such that, for each point of the subset, $R(\hat{U}_{||x}, \hat{V}_{||x}) = 0$ by Theorem 3.8. Now $R(U, V) = 0$ by Proposition 4.2. This proves Statement (2). Statement (3) follows similarly from Theorem 3.8 and Proposition 4.2. □

We now give applications of Theorem 4.7 to the existence problems of the oblique projection if we are given two finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^d)$. First, let us recall the definition of the oblique projection [1, 42]. Let U and V be closed subspaces of $\mathcal{H}$. If $\mathcal{H} = U \dotplus V^\perp$, i.e., $\mathcal{H} = U + V^\perp$ and $U \cap V^\perp = \{0\}$, then we can define the *oblique projection* $P_{U \perp V}$ of $\mathcal{H}$ on U along $V^\perp$ [1]. That is, for any $f \in \mathcal{H}$ there exist unique $u \in U$ and $v^\perp \in V^\perp$ such that $f = u + v^\perp$. We define $P_{U \perp V} f := u$. This concept is closely related with that of the infimum cosine angle between U and V by the following proposition, which is Theorem 2.3 of [42].

Proposition 4.8 ([42]) *Let U and V be closed subspaces of $\mathcal{H}$. The following conditions are equivalent:*

(1) $\mathcal{H} = U \dotplus V^\perp$;

(2) $\mathcal{H} = U^\perp \dot{+} V$;

(3) *There exist Riesz bases* $\{u_i\}_{i\in I}$ *and* $\{v_i\}_{i\in I}$ *for* U *and* V*, respectively, such that* $\{u_i\}_{i\in I}$ *is biorthogonal to* $\{v_i\}_{i\in I}$*;*

(4) $R(U,V) > 0$ *and* $R(V,U) > 0$.

Theorem 4.7 combined with Proposition 4.8 gives us Theorem 4.10 below, which is an extension of $L^2(\mathbb{R})$-version of Theorem 3.1 of [1] (cf. Corollary 4.11). The following lemma is also needed.

Lemma 4.9 *Suppose that* U *and* V *are, not necessarily finitely generated, shift-invariant subspaces of* $L^2(\mathbb{R}^d)$*. If* $L^2(\mathbb{R}^d) = U \dot{+} V^\perp$*, then* $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ *for almost every* $x \in \mathbb{T}^d$*. In particular,* $\sigma(U) = \sigma(V)$*.*

Proof. Since V is shift-invariant, so is $V^\perp$ ([6]). Note that $(L^2(\mathbb{R}^d))^\wedge_{||x} = \ell^2(\mathbb{Z}^d)$ for almost every $x \in \mathbb{T}^d$. Now we have $\ell^2(\mathbb{Z}^d) = \hat{U}_{||x} \dot{+} (V^\perp)^\wedge_{||x}$ for almost every x by an argument similar to the proof of Lemma 3.7 of [30]. This implies that the oblique projection Π_x of $\ell^2(\mathbb{Z}^d)$ on $\hat{U}_{||x}$ along $(V^\perp)^\wedge_{||x}$ is well-defined almost everywhere. Hence, $\ell^2(\mathbb{Z}^d)/\ker \Pi_x = \ell^2(\mathbb{Z}^d)/(V^\perp)^\wedge_{||x}$ is isomorphic to $\operatorname{ran} \Pi_x = \hat{U}_{||x}$. Now $\ell^2(\mathbb{Z}^d)/(V^\perp)^\wedge_{||x}$ is obviously isomorphic to $((V^\perp)^\wedge_{||x})^\perp$. The point-wise projection property of a shift-invariant space ([6, Result 3.7] or [8, Lemma 1.4]) implies that $((V^\perp)^\wedge_{||x})^\perp = \hat{V}_{||x}$. Hence $\hat{U}_{||x}$ is isomorphic to $\hat{V}_{||x}$ for almost every x. In particular, they are of the same dimension for almost every x. □

We postpone the proof of the equivalence of (5) in Theorem 4.10 and in Corollary 4.11 to other conditions in Theorem 4.10 and in Corollary 4.11, respectively, to the next section for the readability of the article since it is slightly long and contains constructive nature which we would like to elucidate further.

Theorem 4.10 *Let* $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_m\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$, *and let* $U := \mathcal{S}(\Phi), V := \mathcal{S}(\Psi)$. *Then the following assertions are equivalent:*

(1) $L^2(\mathbb{R}^d) = U \dot{+} V^\perp$;

(2) $L^2(\mathbb{R}^d) = V \dot{+} U^\perp$;

(3) $R(U, V) > 0$ *and* $R(V, U) > 0$;

(4) $\operatorname{rank} G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ *a.e.* $x \in \mathbb{T}^d$; *and there exists a positive constant* C *such that* $\|G_\Phi(x)^{1/2} G(x)^\dagger G_\Psi(x)^{1/2}\| \leq C$ *a.e.* $x \in \sigma(U)$, *where we recall that* $G(x)$ *denotes the mixed Gramian of* Φ *and* Ψ *at* x;

(5) *There exist* $\tilde{\Phi} := \{\tilde{\varphi}_1, \tilde{\varphi}_2, \cdots, \tilde{\varphi}_r\}, \tilde{\Psi} := \{\tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_r\}$ *such that:*

  (i) $\{T_k \tilde{\varphi}_i : k \in \mathbb{Z}^d, 1 \leq i \leq r\}$ *and* $\{T_k \tilde{\psi}_i : k \in \mathbb{Z}^d, 1 \leq i \leq r\}$ *are frames for* U *and* V, *respectively;*

  (ii) *They are 'oblique'-dual (see [15]) in the sense that for each* $f \in U$ *and* $g \in V$

$$f = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^d} \left\langle f, T_k \tilde{\psi}_i \right\rangle T_k \tilde{\varphi}_i, \text{ and}$$
$$g = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^d} \langle g, T_k \tilde{\varphi}_i \rangle T_k \tilde{\psi}_i.$$

If, in addition, $\{T_k \varphi_j : k \in \mathbb{Z}^d, 1 \leq j \leq m\}$ *and* $\{T_k \psi_i : k \in \mathbb{Z}^d, 1 \leq i \leq n\}$ *are frames for* U *and* V, *respectively, then the above conditions are equivalent to the following condition.*

(6) $\operatorname{rank} G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ *for almost every* $x \in \mathbb{T}^d$*; and there exists a positive constant* C *such that* $\|G(x)^\dagger\| \leq C$ *for almost every* $x \in \sigma(U)$.

Moreover, if any one of the above conditions hold, then $\sigma(U) = \sigma(V)$ *and*

$$
\begin{aligned}
R(U,V) &= R(V,U) \qquad (4.1)\\
&= \begin{cases} 1, & \text{if } U = \{0\};\\ \operatorname{ess-inf}_{x\in\sigma(U)} \|G_\Phi(x)^{1/2} G(x)^\dagger G_\Psi(x)^{1/2}\|^{-1}, & \text{if } U \neq \{0\}. \end{cases}
\end{aligned}
$$

Proof. The equivalences of Conditions (1) (2) and (3) are established in Proposition 4.8.

(1) ⇒ (4): By Lemma 4.9, $\sigma(U) = \sigma(V)$. Hence, if $\sigma(U)$ is of Lebesgue measure zero, then so is $\sigma(V)$. This implies that U and V are trivial. In this case, condition (4) holds trivially. Now, suppose that $\sigma(U)$ and $\sigma(V)$ are of the same positive Lebesgue measure. Then Case (1) of Theorem 4.7 cannot hold. Case (2) of Theorem 4.7 cannot hold either since (1) is equivalent to (3). Therefore, Case (3) of Theorem 4.7 holds. Moreover, by Lemma 4.9, Γ_2 and Γ_3 in Theorem 4.7 are of Lebesgue measure zero. That is, $\sigma(U) = \Gamma = \Gamma_1$. This proves that Condition (4) and the second equality in (4.1) hold by Theorem 4.7 (3).

(4) ⇒ (3): (4) implies that $\sigma(U) = \sigma(V)$. If $\sigma(U)$ is of Lebesgue measure zero, then U and V are trivial. So $R(U,V) = R(V,U) = 1$. Suppose that $\sigma(U) = \sigma(V)$ are of positive Lebesgue measure. In this case, Γ_2 and Γ_3 in Theorem 4.7 are of Lebesgue measure zero and $R(U,V)$ is given by the expression on the last line of (4.1). This implies that $R(U,V) > 0$. Then, $R(V,U) = R(U,V) > 0$ by Corollary 4.3 (1).

The last paragraph also shows that if (4) holds, then $R(U,V) = R(V,U)$ and (4.1) is valid. This proves the equivalence of Conditions (1) to (4) and the validity of (4.1).

Now, suppose that $\{T_k\varphi_j : k \in \mathbb{Z}^d, 1 \le j \le m\}$ and $\{T_k\psi_i : k \in \mathbb{Z}^d, 1 \le i \le n\}$ are frames for U and V, respectively. Then, Propositions 4.4 and 2.3 imply that there exist positive constants A and B such that, for almost every $x \in \mathbb{T}^d$,

$$A \le \min\{\|T_\Phi(x)^\dagger\|^{-1}, \|T_\Psi(x)^\dagger\|^{-1}\}, \tag{4.2}$$

and

$$\max\{\|T_\Phi(x)\|, \|T_\Psi(x)\|\} \le B. \tag{4.3}$$

(6) $\Rightarrow$ (4): By Lemma 3.1, $\|G_\Phi(x)\| = \|T_\Phi(x)^* T_\Phi(x)\| \le B^2$ and $\|G_\Psi(x)\| = \|T_\Psi(x)^* T_\Psi(x)\| \le B^2$ for almost every $x \in \mathbb{T}^d$. This implies (4).
(3) $\Rightarrow$ (6): Corollary 4.3 (2) implies that $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ for almost every $x \in \mathbb{T}^d$, and that $\sigma(U) = \sigma(V)$. If $x \in \mathbb{T}^d \setminus \sigma(U)$, then obviously, $G(x) = 0$, and hence $G(x)^\dagger = 0$. Therefore, $0 = \operatorname{rank} G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$, and $\|G(x)^\dagger\| = 0$. Now, suppose that $\sigma(U)$ is of positive Lebesgue measure. By Proposition 4.2 we see that there exists a positive constant c such that $c \le R(\hat{U}_{||x}, \hat{V}_{||x})$ for almost every $x \in \mathbb{T}^d$. Then, for any $u \in \hat{U}_{||x}$, we have $c\|u\| \le \|P(x)u\|$, where $P(x) := P_{\hat{V}_{||x}}|_{\hat{U}_{||x}}$ as before. This shows that $P(x)$ is one-to-one. It is onto since $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ by Lemma 4.9. It is now easy to see that $\|P(x)^{-1}\| \le c^{-1}$ for almost every $x \in \sigma(U)$. Hence the norm of $G(x)^\dagger$ is bounded uniformly by Lemma 3.4 and (4.2). Recall that

$$G_{\Phi,\Psi}(x) = T_\Psi(x)^* P(x) T_\Phi(x).$$

Since $T_\Phi(x)$ is onto, $\operatorname{ran} T_\Phi(x) = \hat{U}_{||x} = \operatorname{dom} P(x)$. Since $P(x)$ is also onto,

$\operatorname{ran} P(x)T_\Phi(x) = \operatorname{ran} P(x) = \hat{V}_{||x} = \operatorname{dom} T_\Psi(x)^*$. Hence $\operatorname{rank} G_{\Phi,\Psi}(x) = \operatorname{rank} T_\Psi(x)^*$. Now $\operatorname{rank} T_\Psi(x)^* = \operatorname{rank} T_\Psi(x) = \dim \hat{V}_{||x}$ since $T_\Psi(x)$ is onto and since $\operatorname{ran} T_\Psi(x) = \hat{V}_{||x}$. Hence $\operatorname{rank} G(x) = \dim \hat{V}_{||x} = \dim \hat{U}_{||x}$. □

The following corollary is a special case of one of the main results of Aldroubi [1]. He considers the subspace of a general separable Hilbert space $\mathcal{H}$ of the form

$$\left\{ \sum_{i=1}^{r} \sum_{j\in\mathbb{Z}} c_i(j) O^j \varphi_i : c_i \in \ell^2(\mathbb{Z}), 1 \le i \le r \right\},$$

where $\varphi_i \in \mathcal{H}, 1 \le i \le r$ and O is a unitary operator on $\mathcal{H}$. If we let $\mathcal{H} := L^2(\mathbb{R})$ and $O := T_1$, then Theorem 3.1 of [1] reduces to an $L^2(\mathbb{R})$-version of the following corollary.

Corollary 4.11 *Let* $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_n\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$, *and let* $U := \mathcal{S}(\Phi), V := \mathcal{S}(\Psi)$. *Suppose that* $\{T_k\varphi_j : k \in \mathbb{Z}^d, 1 \le j \le n\}$ *and* $\{T_k\psi_i : k \in \mathbb{Z}^d, 1 \le i \le n\}$ *are Riesz bases for* U *and* V*, respectively. Then the following assertions are equivalent:*

(1) $L^2(\mathbb{R}^d) = U \dotplus V^\perp$;

(2) $L^2(\mathbb{R}^d) = V \dotplus U^\perp$;

(3) $R(U,V) > 0$ *and* $R(V,U) > 0$;

(4) $G(x)$ *is invertible for almost every* $x \in \mathbb{T}^d$*; and there exists a positive real number* C *such that* $\|G(x)^{-1}\| \le C$ *for almost every* $x \in \mathbb{T}^d$;

(5) *There exists* $\tilde{\Psi} := \{\tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_n\}$ *such that:*

  (i) $\{T_k\tilde{\psi}_i : k \in \mathbb{Z}^d, 1 \le i \le n\}$ *is a Riesz basis for* V;

(ii) $\left\langle T_k\varphi_i, T_l\tilde{\psi}_j\right\rangle = \delta_{kl}\delta_{ij}$.

If any one of the above conditions is satisfied, then

$$R(U,V) = R(V,U) = \text{ess-inf}_{x\in\mathbb{T}^d} \|G_\Phi(x)^{1/2}G(x)^{-1}G_\Psi(x)^{1/2}\|^{-1}.$$

Proof. We recall that a Riesz basis is a frame. Note that Proposition 4.6 implies that the Gramians $G_\Phi(x)$ and $G_\Psi(x)$ are invertible almost everywhere. Moreover, Proposition 4.4 implies that $\dim \hat{U}_{||x} = \dim \hat{V}_{||x} = n$ almost everywhere, since a finite Riesz basis is a basis in the sense of Linear Algebra.
(1) ⇒ (4): Condition (6) of Theorem 4.10 implies that $\operatorname{rank} G(x) = n$. Therefore $G(x)$ is invertible almost everywhere, and $G(x)^{-1} = G(x)^\dagger$. The proof is complete by Condition (6) of Theorem 4.10.
(4) ⇒ (1): This follows from the equivalence of (1) and (6) of Theorem 4.10.
□

5 Tightization and dualization

In this section we give the remaining implications in Theorem 4.10 and Corollary 4.11, thereby generalizing many of the results on singly generated shift-invariant spaces in [15] to finitely generated shift-invariant spaces.

We first present a lemma which generalizes the well-known orthonormalization technique attributed to Meyer [36]. The proof of the following lemma is already lurking in the proof of Lemma 3.7. Notice the similarity of (3.6) and (5.1). The following lemma provides a process to construct the generators of a tight frame from the generators of a shift invariant subspace.

Lemma 5.1 *Let $F := \{f_1, f_2, \cdots, f_r\} \subset L^2(\mathbb{R}^d)$. Define $\tilde{F} := \{\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_r\}$ via a 'tightization' process:*

$$\hat{\tilde{f}}_i(x) := \sum_{j=1}^{r} \overline{((G_F(x)^\dagger)^{1/2})}_{ij} \hat{f}_j(x), \tag{5.1}$$

where $G_F(x)$ is the Gramian of F at x. Then $\tilde{F} \subset L^2(\mathbb{R}^d)$ and $\{T_k \tilde{f}_i : k \in \mathbb{Z}^d, 1 \leq i \leq r\}$ is a tight frame for $\mathcal{S}(F)$.

Proof. Since $G_F(x)$ is positive semi-definite for a.e. $x \in \mathbb{T}^d$, there exist a unitary matrix $Q(x)$ and a diagonal matrix $D(x)$ such that $G(x) = Q(x)D(x)Q(x)^*$, where the diagonal entries of $D(x)$ is the (non-negative) eigenvalues of $G_F(x)$. Moreover, by [39, Lemma 2.3.5], we may assume that the entries of $U(x)$ and $D(x)$ are measurable 1-(multi)-periodic functions. Then, so are the entries of $G_F(x)^\dagger$ and $(G_F(x)^\dagger)^{1/2}$ (see (3.7)). We now show that $\tilde{f}_i \in L^2(\mathbb{R}^d)$ for each i. First, note that, for a.e. $x \in \mathbb{T}^d$.

$$\hat{\tilde{f}}_{i||x} := \sum_{j=1}^{r} \overline{((G_F(x)^\dagger)^{1/2})}_{ij} \hat{\tilde{f}}_{j||x},$$

which is a well-defined element of $\ell^2(\mathbb{Z}^d)$. A direct calculation shows that

$$\left\| \hat{\tilde{f}}_{i||x} \right\|^2_{\ell^2(\mathbb{Z}^d)} = ((G_F(x)^\dagger)^{1/2} G_F(x) (G_F(x)^\dagger)^{1/2})_{ii} = 0 \text{ or } 1.$$

This implies that $\|\hat{\tilde{f}}_i\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{T}^d} \|\hat{\tilde{f}}_{i||x}\|^2_{\ell^2(\mathbb{Z}^d)}\, dx \leq 1$. Now, to show that $\{T_k \tilde{f}_i : k \in \mathbb{Z}^d, 1 \leq i \leq r\}$ is a frame for $\mathcal{S}(\tilde{F})$, we only need to show that the eigenvalues of $G_{\tilde{F}}(x)$ are 1 or 0 a.e. $x \in \mathbb{T}^d$ by Proposition 4.6. It is straight-forward to see that

$$G_{\tilde{F}}(x) = (G_F(x)^\dagger)^{1/2} G_F(x) (G_F(x)^\dagger)^{1/2} = G_F(x)^\dagger G_F(x) \tag{5.2}$$

Hence the above eigenvalue condition follows. It remains to show that $\mathcal{S}(\tilde{F}) = \mathcal{S}(F)$. Obviously, $(\mathcal{S}(\tilde{F}))^{\wedge}_{||x} \subset (\mathcal{S}(F))^{\wedge}_{||x}$ a.e. Moreover,

$$\dim(\mathcal{S}(\tilde{F}))^{\wedge}_{||x} = \operatorname{rank} G_{\tilde{F}}(x) = \operatorname{rank} G_F(x) = \dim(\mathcal{S}(F))^{\wedge}_{||x}$$

a.e. by (5.2). This shows that $(\mathcal{S}(\tilde{F}))^{\wedge}_{||x} = (\mathcal{S}(F))^{\wedge}_{||x}$ a.e. Hence $\mathcal{S}(\tilde{F}) = \mathcal{S}(F)$ by Proposition 4.1. □

The proof of the following lemma is almost standard. We include it for the sake of completeness.

Lemma 5.2 *Let U and V be $\mathcal{S}(\Phi)$ and $\mathcal{S}(\Psi)$, respectively, where $\Phi := \{\varphi_j\}_{j=1}^r$ and $\Psi := \{\psi_j\}_{j=1}^r$; and $G_\Phi(x), G_\Psi(x), G(x) := G_{\Phi,\Psi}(x)$ be the relevant Gramians or mixed Gramians at $x \in \mathbb{T}^d$. Assume that $\{T_k\varphi_j : k \in \mathbb{Z}^d, 1 \le j \le r\}$ and $\{T_k\psi_j : k \in \mathbb{Z}^d, 1 \le j \le r\}$ are Bessel sequences. Then, the following assertions are equivalent:*

(1) *For each $f \in U$,*

$$f = \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, T_k\psi_j \rangle \, T_k\varphi_j; \tag{5.3}$$

(2) $\hat{\varphi}_{i||x} = \sum_{j=1}^r \left\langle \hat{\varphi}_{i||x}, \hat{\psi}_{j||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_{j||x}$ *for a.e. $x \in \mathbb{T}^d$ and for each $i = 1, 2, \cdots, r$;*

(3) $G_\Phi(x)G(x) = G_\Phi(x)$ *a.e. $x \in \mathbb{T}^d$.*

Proof. The Bessel condition implies that, for each $j = 1, 2, \cdots, r$, the function that maps $x \in \mathbb{T}^d$ to $||\hat{\psi}_{j||x}||_{\ell^2(\mathbb{Z}^d)}$ is in $L^\infty(\mathbb{T}^d)$ by Proposition 4.6. Hence, for each $f \in L^2(\mathbb{R}^d)$ and each $j = 1, 2, \cdots, r$, the function that maps $x \in \mathbb{T}^d$

to $\left\langle \hat{f}_{||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)}$ is in $L^2(\mathbb{R}^d)$. If we take the Fourier transform of the both sides of (5.3), then

$$\begin{aligned}
\hat{f}(x) &= \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}^d} \langle f, T_k\psi_j\rangle\, e^{-2\pi i k\cdot x}\hat{\varphi}_i(x) \\
&= \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}^d} \left(\int_{\mathbb{R}^d} \hat{f}(t)\overline{\hat{\psi}_j}(t) e^{2\pi i k\cdot t}\, dt\right) e^{-2\pi i k\cdot x}\hat{\varphi}_i(x) \\
&= \sum_{j=1}^{r} \sum_{k\in\mathbb{Z}^d} \left(\int_{\mathbb{T}^d} \sum_{l\in\mathbb{Z}^d} \hat{f}(t+l)\overline{\hat{\psi}_j}(t+l) e^{2\pi i k\cdot t}\, dt\right) e^{-2\pi i k\cdot x}\hat{\varphi}_i(x) \\
&= \sum_{j=1}^{r} \left\langle \hat{f}_{||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_j(x),
\end{aligned}$$

where the Parseval's theorem is used in the lase equality. Therefore, (5.3) is equivalent to the following equation:

$$\hat{f}(x) = \sum_{j=1}^{r} \left\langle \hat{f}_{||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_j(x) \text{ for a.e. } x \in \mathbb{R}^d. \tag{5.4}$$

This shows that (1) implies (2) since $\varphi_i \in U$ and the function that maps $x \in \mathbb{R}^d$ to $\left\langle \hat{\varphi}_{i||x}, \hat{\psi}_{j\,||x} \right\rangle$ is 1-periodic.

On the other hand, suppose that (2) holds. Then, for a.e. $x \in \mathbb{R}^d$ and for each $i = 1, 2, \cdots, r$,

$$\hat{\varphi}_i(x) = \sum_{j=1}^{r} \left\langle \hat{\varphi}_{i||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_j(x).$$

Therefore, for a.e. $x \in \mathbb{R}^d$, for each $l = 1, 2, \cdots, r$, and for $k \in \mathbb{Z}^d$

$$e^{-2\pi i k\cdot x}\hat{\varphi}_l(x) = \sum_{j=1}^{r} \left\langle e^{-2\pi i k\cdot x}\hat{\varphi}_{l||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_j(x),$$

which is equivalent to

$$T_k\varphi_l = \sum_{j=1}^{r} \sum_{m\in\mathbb{Z}^d} \langle T_k\varphi_l, T_m\psi_j\rangle\, T_m\varphi_j.$$

This shows that (5.3) holds for each linear combinations of $\{T_k\varphi_i, k \in \mathbb{Z}^d, i = 1, 2, \cdots, r\}$. It is easy to see that the right-hand side of (5.3) defines a bounded linear operator. Hence (1) holds by continuity.

The equivalence of (2) and (3) is seen by direct calculations. $\square$

The following lemma shows how to construct the generators of the oblique dual frame under appropriate conditions.

Lemma 5.3 *Under the hypotheses of Lemma 5.2, suppose also that:*

(i) $\dim \hat{U}_{||x} = \operatorname{rank} G(x)$ *a.e.;*

(ii) *there exists a positive constant* C *such that* $||G(x)^\dagger|| \leq C$ *a.e.* $x \in \sigma(U)$.

Define $\tilde{\Psi} := \{\tilde{\psi}_1, \tilde{\psi}_2, \cdots, \tilde{\psi}_r\}$ *via a 'dualization' process:*

$$\hat{\tilde{\psi}}_{i||x} := \begin{cases} \sum_{j=1}^{r} \overline{G(x)^\dagger}_{ij} \hat{\psi}_{j||x}, & \textit{if } x \in \sigma(U), \\ 0, & \textit{otherwise.} \end{cases} \tag{5.5}$$

Then $\{T_k\tilde{\psi}_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ *is a Bessel sequence and for each* $f \in U$

$$f = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}} \left\langle f, T_k\tilde{\psi}_i \right\rangle T_k\varphi_i.$$

Proof. A direct calculation shows that: $G_{\tilde{\Psi}}(x) = G(x)^\dagger G_\Psi(x)(G(x)^\dagger)^*$ if $x \in \sigma(U)$; $G_{\tilde{\Psi}}(x) = 0$ if $x \in \mathbb{T}^d \setminus \sigma(U)$. Now, $\|G_{\tilde{\Psi}}(x)\|$ is bounded above a.e. by (i) and Proposition 4.6. Hence $\{T_k\tilde{\psi}_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ is a Bessel sequence, again, by Proposition 4.6. If we show that $G_\Phi(x)G_{\Phi,\tilde{\Psi}}(x) = G_\Phi(x)$ a.e., then the lemma follows from Lemma 5.2. It is routine to check that $G_{\Phi,\tilde{\Psi}}(x) = G(x)^\dagger G(x)$ a.e. Recall that $G(x) = T_\Psi(x)^* P(x) T_\Phi(x)$ and $\operatorname{rank} G(x) = \operatorname{rank} P(x)$ by Lemma 3.1, where $T_\Psi(x)$ and $T_\Phi(x)$ are the pre-frame operators and $P(x) = P_{\hat{V}_{||x}}|_{\hat{U}_{||x}}$. In particular, (i) implies that $\dim(\operatorname{dom} P(x)) = \dim \hat{U}_{||x} = \operatorname{rank} G(x) = \operatorname{rank} P(x)$ a.e. Therefore $P(x)$ is one-to-one a.e. Let $X := T_\Psi(x)P(x)$ and $Y := T_\Phi(x)$. Since $P(x)^*$ and $T_\Psi(x)$ are onto, $\operatorname{ran} X^* = \operatorname{ran} P(x)^* T_\Psi(x) = \hat{U}_{||x} = \operatorname{dom} YY^*$. On the other hand, $\ker Y^* = \ker T_\Phi(x)^*$ is trivial since $T_\Phi(x)$ is onto. Therefore, $G(x)^\dagger = T_\Phi(x)^\dagger (T_\Psi(x)^* P(x))^\dagger$ by Proposition 2.2. Now,

$$\begin{aligned}
G_\Phi(x) G_{\Phi,\tilde{\Psi}}(x) &= G_\Phi(x) G(x)^\dagger G(x) \\
&= (T_\Phi(x)^* T_\Phi(x)) T_\Phi(x)^\dagger (T_\Psi(x)^* P(x))^\dagger T_\Psi(x)^* P(x) T_\Phi(x) \\
&= T_\Phi(x)^* (T_\Phi(x) T_\Phi(x)^\dagger)((T_\Psi(x)^* P(x))^\dagger T_\Psi(x)^* P(x)) T_\Phi(x) \\
&= T_\Phi(x)^* P_{\operatorname{ran} T_\Phi(x)} P_{\operatorname{ran}(T_\Psi(x)^* P(x))^*} T_\Phi(x) \\
&= T_\Phi(x)^* P_{\operatorname{ran} T_\Phi(x)} P_{\operatorname{ran} P(x)^* T_\Psi(x)} T_\Phi(x) \\
&= T_\Phi(x)^* P_{\hat{U}_{||x}} P_{\hat{U}_{||x}} T_\Phi(x) \\
&= T_\Phi(x)^* T_\Phi(x) = G_\Phi(x),
\end{aligned}$$

where Lemma 3.1, Proposition 2.1 and the surjectivity of $T_\Phi(x), T_\Psi(x)$ and $P(x)^*$ are used several times. $\square$

Lemma 5.4 *In addition to the hypothesis of Lemma 5.2, suppose also that*

$$\operatorname{rank} G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x} \ \ a.e.$$

Then the following assertions are equivalent:

(1) $f = \sum_{i=1}^r \sum_{k \in \mathbb{Z}^d} \langle f, T_k \psi_i \rangle T_k \varphi$ *for each* $f \in U$*;*

(2) $g = \sum_{i=1}^r \sum_{k \in \mathbb{Z}} \langle g, T_k \varphi_i \rangle T_k \psi_i$ *for each* $g \in V$*.*

If one of the two conditions holds, then $\{T_k \varphi_i : k \in \mathbb{Z}^d, 1 \leq i \leq r\}$ *and* $\{T_k \psi_i : k \in \mathbb{Z}^d, 1 \leq i \leq r\}$ *are frames for* U *and* V*, respectively.*

Proof. The rank condition implies that $P(x)$ is invertible a.e. by Lemma 3.1. Suppose that (1) holds. Then, by Lemma 5.2, $\hat{f}_{||x} = \sum_{j=1}^{r} \left\langle \hat{f}_{||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_{j\,||x}$ for each $f \in U$ and for a.e. $x \in \mathbb{T}^d$. Therefore, for each $f \in U, k = 1, 2, \cdots, r$ and for a.e. $x \in \mathbb{T}^d$,

$$\begin{aligned}
\left\langle P(x)\hat{f}_{||x}, \hat{\psi}_{k||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} &= \left\langle P_{\hat{V}_{||x}} \hat{f}_{||x}, \hat{\psi}_{k||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \\
&= \left\langle \hat{f}_{||x}, P_{\hat{V}_{||x}} \hat{\psi}_{k||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \\
&= \left\langle \hat{f}_{||x}, \hat{\psi}_{k||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \\
&= \left\langle \sum_{j=1}^{r} \left\langle \hat{f}_{||x}, \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\varphi}_{j\,||x}, \hat{\psi}_{k||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \\
&= \left\langle \hat{f}_{||x}, \sum_{j=1}^{r} \left\langle \hat{\psi}_{k||x}, \hat{\varphi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \\
&= \left\langle P(x)\hat{f}_{||x}, \sum_{j=1}^{r} \left\langle \hat{\psi}_{k||x}, \hat{\varphi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\psi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)}.
\end{aligned}$$

Since $P(x)$ is invertible a.e., this shows that

$$\hat{\psi}_{k||x} = \sum_{j=1}^{r} \left\langle \hat{\psi}_{k||x}, \hat{\varphi}_{j\,||x} \right\rangle_{\ell^2(\mathbb{Z}^d)} \hat{\psi}_{j\,||x}$$

for each $k = 1, 2, \cdots, r$ and a.e. $x \in \mathbb{T}^d$. This implies (2) by Lemma 5.2. On the other hand, (2) implies (1) by symmetry.

Finally, the last assertion is a standard fact. ☐

The remaining proofs of Theorem 4.10:

(1) $\Rightarrow$ (5): Assume that (1) holds. Then (4) holds also. In particular, $G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ a.e. Lemma 3.1 implies that $P(x)$ is invertible a.e. Hence, $\operatorname{len} U = \operatorname{len} V$ since we already assumed that U and V are finitely generated. Let r be the common length of U and V. Then, by [39,

Corollary 2.3.8] (see also [8, Theorem 3.3]), there exist $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_r\}$ and $\Psi := \{\psi_1, \psi_2, \cdots, \psi_r\}$ such that $U = \mathcal{S}(\Phi)$ and $V = \mathcal{S}(\Psi)$. Then, again by (4), there exists a positive constant C such that

$$\|G_\Phi(x)^{1/2} G(x)^\dagger G_\Psi(x)^{1/2}\| \leq C \quad \text{a.e. } x \in \sigma(U), \tag{5.6}$$

where the relevant Gramians and the mixed Gramian are $r \times r$ square matrices. Define $\tilde{\Phi}$ and $\tilde{\Psi}$ using the tightization process (5.1) then $\{T_k \tilde{\varphi}_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ and $\{T_k \tilde{\psi}_i : k \in \mathbb{Z}, 1 \leq i \leq r\}$ are tight frames with bound 1 for U and V, respectively. (4) also implies that $\operatorname{rank} G_{\tilde{\Phi},\tilde{\Psi}}(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ a.e. Hence Condition (i) of Lemma 5.3 holds. A direct calculation shows that

$$G_{\tilde{\Phi},\tilde{\Psi}}(x) = (G_\Psi(x)^\dagger)^{1/2} G(x) (G_\Phi(x)^\dagger)^{1/2} \quad \text{a.e.}$$

Since $P(x)$ is invertible a.e.,

$$G_{\tilde{\Phi},\tilde{\Psi}}(x)^\dagger = G_\Phi(x)^{1/2} G(x)^\dagger G_\Psi(x)^{1/2}$$

a.e. by Lemma 3.11. Now (5.6) implies that Condition (ii) of Lemma 5.3 holds. Therefore (5) follows by Lemmas 5.3 and 5.4.

(5) $\Rightarrow$ (1): Assume that (5) holds. Define $\Pi : L^2(\mathbb{R}^d) \to U$ via $\Pi f := \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} \left\langle f, T_k \tilde{\psi}_i \right\rangle T_k \tilde{\varphi}_i$. Then, Π is, not necessarily an orthogonal, projection. Therefore, $L^2(\mathbb{R}^d) = \operatorname{ran}\Pi \dotplus \ker\Pi = U \dotplus \ker\Pi$. We show that $\ker\Pi = V^\perp$. Suppose that $g \in \ker\Pi$. Then there exists $f \in L^2(\mathbb{R}^d)$ such

that $g = f - \Pi f$. Now, for each $h \in V$,

$$\begin{aligned}
\langle g, h\rangle &= \langle f - \Pi f, h\rangle = \langle f, h\rangle - \langle \Pi f, h\rangle \\
&= \langle f, h\rangle - \sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^d} \left\langle f, T_k\tilde{\psi}_i\right\rangle \langle T_k\tilde{\varphi}_i, h\rangle \\
&= \langle f, h\rangle - \left\langle f, \sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^d} \langle h, T_k\tilde{\varphi}_i\rangle T_k\tilde{\psi}_i \right\rangle \\
&= \langle f, h\rangle - \langle f, h\rangle = 0.
\end{aligned}$$

Hence $g \in V^{\perp}$. On the other hand, if $g \in V^{\perp}$, then, trivially, $\Pi g = 0$. □

In the proof of the implication from (1) to (5), we could have assumed, before the tightization process, that $\{T_k\varphi_i : k \in \mathbb{Z}^d, 1 \le i \le r\}$ and $\{T_k\psi_i : k \in \mathbb{Z}^d, 1 \le i \le r\}$ are tight frames with frame bound 1 for U and V, respectively, by resorting either to [6, Theorem 3.5] or to [8, Theorem 3.3]. It, however, is the authors' opinion that the current proof is constructive in the sense that given the generating sets Φ and Ψ of U and V with the same number of generators, respectively, we may first apply the tightization process (5.1) to Φ and Ψ and then we may apply the dualization process (5.5) to one of the resulting tight frame to get the oblique-duality as in Lemma 5.4.

The remaining proofs of Corollary 4.11:

(1) ⇒ (5): Suppose that (1) of Corollary 4.11 holds. Then the conditions of Lemma 5.3 are satisfied since (4) of Corollary 4.11 holds also. Note that $\sigma(U) = \mathbb{T}^d$ by Proposition 4.6. Define $\tilde{\Psi} := \{\tilde{\psi}_i\}_{i=1}^n$ as in (5.5), i.e.,

$$\hat{\tilde{\psi}}_{i||x} := \sum_{j=1}^{n} (\overline{G(x)^{-1}})_{ij}\hat{\psi}_{j||x}, \quad x \in \mathbb{T}^d.$$

Then, by Lemma 5.3,

$$T_k\varphi_i = \sum_{j=1}^{n}\sum_{l\in\mathbb{Z}^d}\left\langle T_k\varphi_i, T_k\tilde{\psi}_j\right\rangle T_l\varphi_j \tag{5.7}$$

for each $k \in \mathbb{Z}^d$ and $i = 1, 2, \cdots, n$. Lemma 5.4 implies that the shifts of $\tilde{\Psi}$ form a frame for V. In particular, the shifts of $\tilde{\Psi}$ are dense in V. A direct calculation shows that $G_{\tilde{\Psi}}(x) = G(x)^{-1}G_\Psi(x)(G(x)^{-1})^*$. Hence $G_{\tilde{\Psi}}(x)$ is invertible a.e. Lemma 3.1 implies that $\|G(x)\|$ is bounded above by a uniform constant a.e. This bound and (4) of Corollary 4.11 show that $\|G_{\tilde{\Psi}}(x)\|$ and $\|G_{\tilde{\Psi}}(x)^{-1}\|$ are bounded above by a uniform constant a.e. Therefore, the shifts of $\tilde{\Psi}$ form a Riesz basis for its closed linear span, which is V, by Proposition 4.6. Moreover, (5.7) implies that $\left\langle T_k\varphi_i, T_l\tilde{\psi}_j\right\rangle = \delta_{kl}\delta_{ij}$ since the shifts of Φ form a Riesz basis.

(5) $\Rightarrow$ (1) follows from the equivalence of (1) and (3) of Proposition 4.8. $\square$

6 Examples

In this section we illustrate our results by concrete examples.

We first give an example in which Theorem 4.7 is used to calculate the angle between two shift-invariant subspaces analytically. Let $\psi_1 := \chi_{[0,1]}$ and $\varphi := \psi_1 * \psi_1 * \psi_1$, where $*$ denotes the convolution. Note that ψ_1 and φ are the B-splines of first and third order, respectively. We recall the following relations:

$$\hat{\psi}_1(x) = e^{-i\pi x}\left(\frac{\sin \pi x}{\pi x}\right) = m_1\left(\frac{x}{2}\right)\hat{\psi}_1\left(\frac{x}{2}\right),$$
$$\hat{\varphi}(x) = e^{-i\pi 3x}\left(\frac{\sin \pi x}{\pi x}\right)^3 = m_3\left(\frac{x}{2}\right)\hat{\varphi}\left(\frac{x}{2}\right),$$

where

$$m_1(x) := e^{-i\pi x}\cos\pi x \quad \text{and} \quad m_3(x) := e^{-i3\pi x}\cos^3\pi x.$$

Let $\psi_2 := \tilde{m}_1(x/2)\hat{\psi}_1(x/2)$, where $\tilde{m}_1(x) := -e^{-2\pi i x}\overline{m_1}(x+1/2)$. We set $U := \mathcal{S}(\Phi)$ and $V := \mathcal{S}(\Psi)$, where $\Phi := \{\varphi\}$ and $\Psi := \{\psi_1, \psi_2\}$. Note that ψ_2 is nothing but the Haar function. It is now easy to see that

$$G_\Psi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, $\dim \hat{V}_{||x} = 2$ a.e. by Lemma 3.1. On the other hand,

$$\begin{aligned} G_\Phi(x) &= \sum_{k\in\mathbb{Z}} |\hat{\varphi}(x+k)|^2 = \left(\frac{\sin\pi x}{\pi}\right)^6 \sum_{k\in\mathbb{Z}} \left(\frac{1}{x+k}\right)^6 \\ &= \cos^2\pi x + \frac{2}{15}\sin^4\pi x \\ &= \frac{16}{30} + \frac{13}{30}\cos 2\pi x + \frac{1}{30}\cos^2 2\pi x, \end{aligned}$$

where we have used the well-known identity ([17, Equation (4.2.9)]):

$$\sum_{k\in\mathbb{Z}} \frac{1}{(x+k)^2} = \frac{\pi^2}{\sin^2\pi x}.$$

It is a standard fact that the shifts of φ form a Riesz basis for its closed linear span [17], which can also be seen by the above calculations and by Proposition 4.6. In particular, $\sigma(U) = \mathbb{T}$ and $\dim \hat{U}_{||x} = 1$ a.e. The mixed Gramian $G(x)$ is also given by

$$G(x) := \begin{pmatrix} \frac{e^{-2\pi i x}}{3}(2+\cos 2\pi x) \\ \frac{-ie^{-2\pi i x}}{4}\sin 2\pi x \end{pmatrix},$$

where we have used the following calculations:

$$\left\langle \hat{\varphi}_{||x}, \hat{\psi}_{1||x} \right\rangle = e^{2\pi i x} \frac{\sin^4(\pi x)}{\pi^4} \sum_{k\in\mathbb{Z}} \frac{1}{(x+k)^4};$$

$$\begin{aligned}\left\langle \hat{\varphi}_{||x}, \hat{\psi}_{2||x} \right\rangle &= m_3\left(\frac{x}{2}\right)\overline{\tilde{m}_1}\left(\frac{x}{2}\right)\left\langle \hat{\varphi}_{||x/2}, \hat{\psi}_{1||x/2} \right\rangle \\ &\quad + m_3\left(\frac{x}{2}+\frac{1}{2}\right)\overline{\tilde{m}_1}\left(\frac{x}{2}+\frac{1}{2}\right)\left\langle \hat{\varphi}_{||x/2+1/2}, \hat{\psi}_{1||x/2+1/2} \right\rangle.\end{aligned}$$

Since

$$\begin{pmatrix} a \\ b \end{pmatrix}^{\dagger} = \frac{1}{|a|^2+|b|^2}\begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix},$$

we have

$$\begin{aligned} G(x)^{\dagger} = & \frac{1}{\frac{4}{9}+\frac{4}{9}\cos(2\pi x)+\frac{1}{9}\cos^2(2\pi x)+\frac{1}{16}\sin^2(2\pi x)} \\ & \times \begin{pmatrix} \frac{e^{2\pi i x}}{3}(2+\cos 2\pi x) & \frac{ie^{2\pi i x}}{4}\sin 2\pi x \end{pmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} & \|G_{\Phi}(x)^{1/2} G(x)^{\dagger} G_{\Psi}(x)^{1/2}\| \\ & = \left\| \frac{\sqrt{\frac{16}{30}+\frac{13}{30}\cos(2\pi x)+\frac{1}{30}\cos^2(2\pi x)}}{\frac{4}{9}+\frac{4}{9}\cos(2\pi x)+\frac{1}{9}\cos^2(2\pi x)+\frac{1}{16}\sin^2(2\pi x)} \right. \\ & \quad \left. \times \begin{pmatrix} \frac{e^{2\pi i x}}{3}(2+\cos 2\pi x) & \frac{ie^{2\pi i x}}{4}\sin 2\pi x \end{pmatrix} \right\| \\ & = \sqrt{\frac{\frac{16}{30}+\frac{13}{30}\cos(2\pi x)+\frac{1}{30}\cos^2(2\pi x)}{\frac{4}{9}+\frac{4}{9}\cos(2\pi x)+\frac{1}{9}\cos^2(2\pi x)+\frac{1}{16}\sin^2(2\pi x)}}, \end{aligned}$$

where we used that $\left\|\begin{pmatrix} a & b \end{pmatrix}\right\| = (|a|^2 + |b|^2)^{1/2}$. Now,

$$
\begin{aligned}
R(U,V) &= \text{ess-inf}_{x\in\mathbb{T}} \, ||G_\Phi(x)^{1/2} G(x)^\dagger G_\Psi(x)^{1/2}||^{-1} \\
&= \text{ess-inf}_{x\in\mathbb{T}} \sqrt{\frac{\frac{4}{9} + \frac{4}{9}\cos(2\pi x) + \frac{1}{9}\cos^2(2\pi x) + \frac{1}{16}\sin^2(2\pi x)}{\frac{16}{30} + \frac{13}{30}\cos(2\pi x) + \frac{1}{30}\cos^2(2\pi x)}} \\
&= \text{ess-inf}_{x\in\mathbb{T}} \sqrt{\frac{\frac{73}{144} + \frac{4}{9}\cos(2\pi x) + \frac{7}{144}\cos^2(2\pi x)}{\frac{16}{30} + \frac{13}{30}\cos(2\pi x) + \frac{1}{30}\cos^2(2\pi x)}} > 0.
\end{aligned}
$$

In order to evaluate $R(U,V)$ analytically, we denote the quantity inside the radical above by $f(x)$. Then we see that

$$
\begin{aligned}
f'(x) &= -2\pi \sin(2\pi x) \frac{5}{8} \frac{25 + 26y + 9y^2}{(16 + 13y + y^2)^2} \\
&= -2\pi \sin(2\pi x) \ \frac{5}{8} \ \frac{9(y + 13/9)^2 + 56/9}{(16 + 13y + y^2)^2},
\end{aligned}
$$

where $y := \cos(2\pi x)$. Then $f'(x) = -f'(-x)$ and $f'(x) > 0$ for $x < 0$. Hence $f(x) \geq f(-1/2) = f(1/2) = 5/6$. Therefore $R(U,V) = \sqrt{30}/6$. $\square$

In the next example we illustrate the use of Theorem 4.10 to construct the generators of the oblique dual frame. Let $1/3 < a < 1/2$ and let $\{I_i\}_{i=1}^3$ be

$$
\begin{aligned}
I_1 &:= [2a - 1, -2a + 1], \\
I_2 &:= \left[-\tfrac{1}{3}, 2a - 1\right] \bigcup \left[-2a + 1, \tfrac{1}{3}\right], \\
I_3 &:= \left[-a, -\tfrac{1}{3}\right] \bigcup \left[\tfrac{1}{3}, a\right].
\end{aligned}
$$

Define $\Phi := \{\varphi_1, \varphi_2\}, \Psi := \{\psi_1, \psi_2\}$ via

$$\hat{\varphi}_1(x) := \chi_{[-\frac{2}{3},a-1]\bigcup[-a,a]\bigcup[-a+1,\frac{2}{3}]}(x);$$

$$\hat{\varphi}_2(x) := \chi_{[-\frac{4}{3},2a-2]\bigcup[-2a,a-1]\bigcup[-a,-\frac{1}{3}]\bigcup[\frac{1}{3},a]\bigcup[-a+1,2a]\bigcup[-2a+2,\frac{4}{3}]}(x);$$

$$\hat{\psi}_1(x) := \chi_{[-a,a]}(x);$$

$$\hat{\psi}_2(x) := \chi_{[-2a,-\frac{2}{3}]\bigcup[-a,-\frac{1}{3}]\bigcup[\frac{1}{3},a]\bigcup[\frac{2}{3},2a]}(x),$$

and let $U := \mathcal{S}(\Phi)$ and $V := \mathcal{S}(\Psi)$. Direct calculations show that

$$G_\Phi(x) = \begin{pmatrix} \chi_{I_1\bigcup I_2}(x) + 2\chi_{I_3}(x) & 2\chi_{I_3}(x) \\ 2\chi_{I_3}(x) & 2\chi_{I_2\bigcup I_3}(x) \end{pmatrix},$$

$$G_\Psi(x) = \begin{pmatrix} \chi_{I_1\bigcup I_2\bigcup I_3}(x) & \chi_{I_3}(x) \\ \chi_{I_3}(x) & \chi_{I_2\bigcup I_3}(x) \end{pmatrix},$$

$$G_{\Phi,\Psi}(x) = G_{\Psi,\Phi}(x) = \begin{pmatrix} \chi_{I_1\bigcup I_2\bigcup I_3}(x) & \chi_{I_3}(x) \\ \chi_{I_3}(x) & \chi_{I_2\bigcup I_3}(x) \end{pmatrix}.$$

We note that

$$\dim \hat{U}_{||x} = \dim \hat{V}_{||x} = \operatorname{rank} G_{\Phi,\Psi}(x) = \begin{cases} 2, & \text{if } x \in I_2; \\ 1, & \text{if } x \in I_1 \bigcup I_3; \\ 0, & \text{otherwise,} \end{cases}$$

and $\sigma(U) = \sigma(V) = I_1 \bigcup I_2 \bigcup I_3$. We can easily check that

$$G_\Phi^{1/2}(x) = \begin{pmatrix} \chi_{I_1\bigcup I_2\bigcup I_3}(x) & \chi_{I_3}(x) \\ \chi_{I_3}(x) & \sqrt{2}\chi_{I_2}(x) + \chi_{I_3}(x) \end{pmatrix}$$

$$G_\Psi^{1/2}(x) = \begin{pmatrix} \chi_{I_1\bigcup I_2}(x) + \frac{1}{\sqrt{2}}\chi_{I_3}(x) & \frac{1}{\sqrt{2}}\chi_{I_3}(x) \\ \frac{1}{\sqrt{2}}\chi_{I_3}(x) & \chi_{I_2}(x) + \frac{1}{\sqrt{2}}\chi_{I_3}(x) \end{pmatrix}$$

$$G_{\Phi,\Psi}^\dagger(x) = \begin{pmatrix} \chi_{I_1\bigcup I_2}(x) + \frac{1}{4}\chi_{I_3}(x) & \frac{1}{4}\chi_{I_3}(x) \\ \frac{1}{4}\chi_{I_3}(x) & \chi_{I_2}(x) + \frac{1}{4}\chi_{I_3}(x) \end{pmatrix}.$$

and

$$G_{\Phi}^{1/2}(x)G_{\Phi,\Psi}^{\dagger}(x)G_{\Psi}^{1/2}(x) = G_{\Psi}^{1/2}(x)G_{\Psi,\Phi}^{\dagger}(x)G_{\Phi}^{1/2}(x)$$
$$= \begin{pmatrix} \chi_{I_1\bigcup I_2}(x) + \frac{1}{\sqrt{2}}\chi_{I_3}(x) & \frac{1}{\sqrt{2}}\chi_{I_3}(x) \\ \frac{1}{\sqrt{2}}\chi_{I_3}(x) & \sqrt{2}\chi_{I_2}(x) + \frac{1}{\sqrt{2}}\chi_{I_3}(x) \end{pmatrix}.$$

Since $\|A\| \leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$ for an $n\times n$ matrix $A := (a_{ij})_{1\leq i,j\leq n}$,

$$\|G_{\Phi}^{1/2}(x)G_{\Phi,\Psi}^{\dagger}(x)G_{\Psi}^{1/2}(x)\| \leq \sqrt{3} \text{ a.e. } x \in \sigma(U).$$

Therefore $R(U,V) = R(V,U) \geq 1/\sqrt{3}$ and so $L^2(\mathbb{R}^d) = U \dotplus V^{\perp}$ by Theorem 4.10. We now construct the generators of the oblique dual frame of $\{T_k\varphi_j : k\in\mathbb{Z}, j=1,2\}$. As in Lemma 5.3, define $\tilde{\psi}_1, \tilde{\psi}_2$ via dualization:

$$\begin{pmatrix} \hat{\tilde{\psi}}_{1||x} \\ \hat{\tilde{\psi}}_{2||x} \end{pmatrix} := \overline{G_{\Phi,\Psi}^{\dagger}(x)} \begin{pmatrix} \hat{\psi}_{1||x} \\ \hat{\psi}_{2||x} \end{pmatrix},$$

that is,

$$\hat{\tilde{\psi}}_1(x) := \chi_{I_1\bigcup I_2}(x) + \tfrac{1}{2}\chi_{I_3}(x),$$
$$\hat{\tilde{\psi}}_2(x) := \tfrac{1}{2}\chi_{I_3}(x) + \chi_{[-2a,-2/3]\bigcup[2/3,2a]}(x).$$

Then $\tilde{\psi}_1, \tilde{\psi}_2$ are the generators of the oblique dual frame of V for $\{T_k\varphi_j : k\in\mathbb{Z}, j=1,2\}$ of U by Lemma 5.3. □

References

[1] A. Aldroubi, Oblique projections in atomic spaces, Proc. Amer. Math. Soc. 124 (1996) 2051–2060.

[2] A. Aldroubi, P. Abry, M. Unser, Construction of biorthogonal wavelets starting from any two multiresolutions, IEEE Trans. Signal Process. 46 (1998) 1130–1133.

[3] J.J. Benedetto, S. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harm. Anal. 5 (1998) 398–427.

[4] J. Benedetto, O.M. Treiber, Wavelet frames: multiresolution analysis and extension principle, in: L.Debnath (Ed.), Wavelet transforms and time-frequency signal analysis, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2001, pp. 3–36.

[5] F.J. Beutler, W.L. Root, The operator pseudo-inverse in control and systems identifications, in: M.Z. Nashed (Ed.), Generalized Inverses and Applications, Academic Press, New York, 1976, pp. 397–494.

[6] C. de Boor, R. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994) 37–78.

[7] R. Bouldin, The pseudo-inverse of a product, SIAM J. Appl. Math. 24 (1973) 489–495.

[8] M. Bownik, The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$, J. Funct. Anal. 177 (2000) 283–309.

[9] M. Bownik, G. Garrigós, Biorthogonal wavelets, MRA's and shift-invariant spaces, Studia Math. 160 (2004) 231–248.

[10] A. Calogero, G. Garrigos, A characterization of wavelet families arising from biorthogonal MRA's of multiplicity d, J. Geom. Anal. 11 (2001) 187–217.

[11] J.B. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1990.

[12] O. Christensen, Frames and pseudo-inverses, J. Math. Anal. Appl. 195 (1995) 401-414.

[13] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[14] O. Christensen, On frame multiresolution analysis, Arab. J. Sci. Eng. Sect. C 28 (2003) 59–72.

[15] O. Christensen, Y.C. Eldar, Oblique dual frames and shift-invariant spaces, Appl. Comput. Harmon. Anal. 17 (2004), 48–68.

[16] O. Christensen, H.O. Kim, R.Y. Kim, J.K. Lim, Perturbation of frame sequences in shift-invariant spaces, Preprint, 2004.

[17] C.K. Chui, An Introduction to Wavelets, Academic Press, San Diego, 1992.

[18] C.K. Chui, W. He, J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, Appl. Comput. Harmon. Anal. 13 (2002) 224–262.

[19] C.K. Chu, X. Shi, J. Stc̆ker, Affine frames, quasi-affine frames, and their duals. Adv. Comput. Math. 8 (1998) 1–17.

[20] X. Dai, Y. Diao, Q. Gu, D. Han, Wavelets with frame multiresolution analysis, J. Fourier Anal. Appl. 9 (2003) 39–48.

[21] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.

[22] I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames. Appl. Comput. Harmon. Anal. 14 (2003) 1–46.

[23] C.W. Groetsch, Generalized Inverses of Linear Operators: Representation and Approximation, Marcel Dekker, New York, 1977.

[24] C. Heil, D. Walnut, Continuous and discrete wavelet transforms, SIAM Rev. 31 (1989) 628–666.

[25] E. Hernández, G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton, 1996.

[26] H. Helson, Lectures on Invariant Subspaces, Academic Press, New York, 1964.

[27] J.R. Holub, Pre-frame operators, Besselian frames, and near-Riesz bases in Hilbert spaces, Proc. Amer. Math. Soc. 122 (1994) 779–785.

[28] R.-Q. Jia, Shift-invariant spaces and linear operator equations, Israel J. Math. 103 (1998) 258–288.

[29] H.O. Kim, R.Y. Kim, J.K. Lim, Characterizations of biorthogonal wavelets which are associated with biorthogonal multiresolution analyses, Appl. Comput. Harmon. Anal. 11 (2001) 263–272.

[30] H.O. Kim, R.Y. Kim, J.K. Lim, Quasi-birothogonal frame multiresolution analyses and wavelets, Adv. Comput. Math. 18 (2003) 269–296.

[31] H.O. Kim, R.Y. Kim, J.K. Lim, Local analysis of frame multiresolution analysis with a general dilation matrix, Bull. Austral. Math. Soc. 67 (2003) 285–295.

[32] H.O. Kim, R.Y. Kim, J.K. Lim, Sum of two finitely generated shift-invariant spaces and its applications, Preprint, 2004.

[33] H.O. Kim, J.K. Lim, On frame wavelets associated with frame multiresolution analysis, Appl. Comput. Harmon. Anal. 10 (2001) 61–70.

[34] J.K. Lim, Gramian analysis of multivariate frame multiresolution analyses, Bull. Austral. Math. Soc. 66 (2002) 291–300.

[35] S. Mallat, Multiresolution approximations and wavelets, Trans. Amer. Math. Soc. 315 (1989) 69–88.

[36] Y. Meyer, Wavelets and Operators, Cambridge University Press, New York, 1992.

[37] M. Papadakis, Generalized frame multiresolution analysis of abstract Hilbert spaces, in: Sampling, wavelets, and tomography, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004, pp. 179–223.

[38] M. Papadakis, G. Gogoshin, I.A. Kakdiaris, D.J. Kouri, D.K. Hoffman, Nonseparable radial frame multiresolution analysis in multidimensions, Numer. Funct. Anal. Optim. 24 (2003) 907–928.

[39] A. Ron, Z. Shen, Frames and stable bases for shift-invarant subspaces of $L_2(\mathbb{R}^d)$, Canad. J. Math. 47 (1995) 1051–1094.

[40] A. Ron, Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$: The analysis of the analysis operator, J. Funct. Anal. 148 (1997) 408–447.

[41] A. Ron, Z. Shen, Affine systems in $L_2(\mathbb{R}^d)$ II: Dual systems, J. Fourier Anal. Appl. 3 (1997) 617–637.

[42] W.-S. Tang, Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces, Proc. Amer. Math. Soc. 128 (1999) 463–473.

[43] M. Unser, A. Aldroubi, A general sampling theory for non-ideal acquisition devices, IEEE Trans. Signal Process. 42 (1994) 2915–2925.

[44] X. Wang, The study of wavelets from the properties of their Fourier transforms, Ph.D Thesis, Washington University in St. Louis, 1995.

[45] R.M. Young, An Introduction to Nonharmonic Fourier Series, Revised First Edition, Academic Press, San Diego, 2001.