TRIANGLE- AND PENTAGON-FREE DISTANCE-REGULAR GRAPHS WITH AN EIGENVALUE MULTIPLICITY EQUAL TO THE VALENCY

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Abstract

We classify triangle- and pentagon-free distance-regular graphs with diameter $d \geq 2$, valency k, and an eigenvalue multiplicity k. In particular, we prove that such a graph is isomorphic to a cycle, a k-cube, a complete bipartite graph minus a matching, a Hadamard graph, a distance-regular graph with intersection array $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$, $\gamma \in \mathbb{N}$, or a folded k-cube, k odd and $k \geq 7$. This is a generalization of the results of Nomura [10] and Yamazaki [13], where they classified bipartite distance-regular graphs with an eigenvalue multiplicity k and showed that all such graphs are 2-homogeneous.
We also classify bipartite almost 2-homogeneous distance-regular graphs with diameter $d \geq 4$. In particular, we prove that such a graph is either 2-homogeneous (and thus classified by Nomura and Yamazaki), or a folded k-cube for k even, or a generalized $2d$-gon with order $(1, k-1)$.

CONTENTS

1 Introduction

Let Γ be a distance-regular graph with diameter d and let $r \in \{1,\dots,d-1\}$. We say that the parameter γ_r *exists*, when for all vertices z of Γ the following holds:

$$|\Gamma(x) \cap \Gamma(y) \cap \Gamma_{r-1}(z)| = \gamma_r, \qquad \text{for all } x, y \in \Gamma_r(z), \text{ such that } \partial(x,y) = 2,$$

i.e., two vertices at distance two and at distance r from z have precisely γ_r common neighbours that are at distance $r-1$ from z. If Γ is bipartite, then it is called 2-*homogeneous* in the sense of Nomura [9], when the parameters γ_r exist for all $r \in \{1,\dots,d-1\}$. Nomura [10] (see also Curtin [3]) showed that if the graph Γ is a 2-homogeneous bipartite distance-regular graph, then Γ is (i) a cycle, (ii) a hypercube, (iii) a complete bipartite graph, (iv) a complete bipartite graph minus a matching, (v) a Hadamard graph, i.e., a distance-regular graph with intersection array $\{4\gamma, 4\gamma-1, 2\gamma, 1; 1, 2\gamma, 4\gamma-1, 4\gamma\}$ where $\gamma \in \mathbb{N}$, or (vi) a graph with intersection array $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, where $k = \gamma(\gamma^2+3\gamma+1)$, $c = \gamma(\gamma+1)$ and $\gamma \in \mathbb{N}$.

The well known Terwilliger tree bound [12] (see Godsil [5, Lemma 13.4.4]) implies that an eigenvalue multiplicity of a triangle-free distance-regular graph is either 1 or at least its valency. If a distance-regular graph has an eigenvalue with multiplicity equal to its valency, then this often yields some additional combinatorial properties. For example, Yamazaki [13] showed that if a bipartite distance-regular graph Γ of valency k has an eigenvalue of multiplicity k, then the graph Γ is 2-homogeneous. Combining Nomura's and Yamazaki's results gives the following classification of bipartite distance-regular graphs with an eigenvalue multiplicity equal to its valency:

Theorem 1 (Nomura & Yamazaki) *Let Γ be a bipartite distance-regular graph with valency k. Then Γ has an eigenvalue with multiplicity k if and only if Γ is one of the following graphs:*

(i) the $(2n)$-gon for $n \geq 2$;

(ii) the k-cube for $k \geq 1$;

(iii) the complete bipartite graph $K_{k+1,k+1}$ minus a matching for $k \geq 2$;

(iv) a Hadamard graph, i.e., a distance-regular graph with intersection array

$$\{4\gamma, 4\gamma-1, 2\gamma, 1; 1, 2\gamma, 4\gamma-1, 4\gamma\}, \quad \text{where } k = 4\gamma,\ \gamma \in \mathbb{N};$$

(v) a distance-regular graph with intersection array

$$\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\},$$

where $k = \gamma(\gamma^2+3\gamma+1)$, $c = \gamma(\gamma+1)$, $\gamma \in \mathbb{N}$.

Let Γ be a triangle-free distance-regular graph with valency $k \geq 3$ and with eigenvalue multiplicity k. It turns out that the girth of Γ is at most five, see Lemma 7(iv). Therefore, we can consider separately the following three cases: (i) $c_2 = 1$, $a_2 \geq 1$; (ii) $c_2 \geq 2$, $a_2 = 0$; and (iii) $c_2 \geq 2$, $a_2 \geq 1$. A complete classification of cases (i) and (iii) still seems to be beyond reach. In this paper we will give the complete classification of the case (ii), i.e., triangle- and pentagon-free distance-regular graphs with diameter $d \geq 2$, valency k, and an eigenvalue multiplicity k. The following is our first result.

Theorem 2 *Let Γ be a triangle- and pentagon-free distance-regular graph with diameter $d \geq 2$ and valency k. Then Γ has an eigenvalue with multiplicity k if and only if Γ is one of the following graphs:*

(i) the n-gon for $n \geq 6$;

(ii) the k-cube for $k \geq 2$;

(iii) the complete bipartite graph $K_{k+1,k+1}$ minus a matching for $k \geq 2$;

(iv) a Hadamard graph, i.e., a distance-regular graph with intersection array

$$\{4\gamma, 4\gamma - 1, 2\gamma, 1; 1, 2\gamma, 4\gamma - 1, 4\gamma\}, \quad \text{where } k = 4\gamma,\ \gamma \in \mathbb{N};$$

(v) a distance-regular graph with intersection array

$$\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\},$$

where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$, $\gamma \in \mathbb{N}$;

(vi) the folded k-cube for k odd and $k \geq 7$.

Remarks: (a) It is not known whether a Hadamard graph exists for every positive integer γ, however, its existence is equivalent to the existence of a Hadamard matrix of order 4γ. For the survey of Hadamard matrices see for example Hedayat and Wallis [6] or Seberry and Yamada [11].
(b) In the case (v) there are only two examples known, namely for $\gamma = 1$ the 5-cube and for $\gamma = 2$ the bipartite double of the Higman-Sims graph. For integers $\gamma \geq 3$ the existence is still undecided.

A bipartite distance-regular graph Γ with diameter $d \geq 3$ is called *almost* 2*-homogeneous*, if the parameter γ_r exists for all $r \in \{1, \ldots, d-2\}$. Almost 2-homogeneous graphs were introduced and studied by Curtin [4]. Observe that $\gamma_1 = 1$, so γ_1 exists for every distance-regular graph. Therefore, all bipartite distance-regular graphs with diameter 3 are almost 2-homogeneous. So we will investigate only almost 2-homogeneous distance-regular graphs with diameter $d \geq 4$. Our second result is the classification of these graphs. We will prove the following result.

Theorem 3 *A bipartite distance-regular graph with diameter $d \geq 4$ and valency k is almost 2-homogeneous if and only if Γ is one of the following graphs:*

(i) *the $(2d)$-gon;*

(ii) *the d-cube;*

(iii) *a Hadamard graph, i.e., a graph with intersection array*

$$\{4\gamma, 4\gamma - 1, 2\gamma, 1; 1, 2\gamma, 4\gamma - 1, 4\gamma\}, \quad \text{where } k = 4\gamma,\ \gamma \in \mathbb{N};$$

(iv) *a graph with intersection array*

$$\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\},$$

where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $c = \gamma(\gamma + 1)$, $\gamma \in \mathbb{N}$;

(v) *the folded $2d$-cube;*

(vi) *a generalized 8-gon with order $(1, k-1)$, i.e., a distance-regular graph with intersection array $\{k, k-1, k-1, k-1; 1, 1, 1, k\}$;*

(vii) *a generalized 12-gon with order $(1, k-1)$, i.e., a distance-regular graph with intersection array $\{k, k-1, k-1, k-1, k-1, k-1; 1, 1, 1, 1, 1, k\}$.*

Our paper is organized as follows. After preliminaries in Section 2, we will derive some results concerning triangle- and pentagon free distance-regular graphs with eigenvalue multiplicity equals to the valency in Section 3. In particular, we will show that such graphs have some additional combinatorial properties. In Section 4 we prove Theorems 2 and 3.

2 Partitions

In this section we recall some definitions and basic concepts about distance-regular graphs and their partitions. See Brouwer, Cohen and Neumaier [1] and Godsil [5] for more background information.

Let Γ be a finite, undirected, connected graph, without loops or multiple edges. We denote the vertex set of Γ by $V\Gamma$. For arbitrary $x, y \in V\Gamma$, let $\partial_\Gamma(x, y) = \partial(x, y)$ denote the distance between x and y, i.e., the length of a shortest path connecting x and y. Let $d = d(\Gamma) := \max\{\partial(x, y) | x, y \in V\Gamma\}$ denote the diameter of Γ. For $x \in V\Gamma$ and for an integer i we denote by $\Gamma_i(x)$ the set of vertices of Γ at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$ and $k_i(x) = |\Gamma_i(x)|$.

If Π is an arbitrary partition of the vertex set of a graph Γ, then there is an obvious concept of a quotient graph Γ/Π. Given a partition Π on the vertex set of a graph Γ

(into nonempty classes), we define the quotient graph Γ/Π on the classes of Π by defining two distinct classes $C, C' \in \Pi$ to be adjacent if Γ contains an edge joining a vertex of C to a vertex of C'. The partition Π is called *regular* (also equitable) if for any two classes $C, C' \in \Pi$, the number of vertices in C' adjacent to $x \in C$ is a constant $e(C, C')$ independent of $x \in C$.

A notion of a *covering graph* is an opposite concept of the quotient graph corresponding to the regular partition Π with $e(C, C') \in \{0, 1\}$ and $e(C, C) = 0$ for every $C, C' \in \Pi$. In the next lemma it is demonstrated that some special graphs with valency k have k-cubes for covering graphs. For, we need two more definitions.

A connected graph Γ is called a *rectagraph* if it is triangle free and if any two vertices of Γ at distance 2 have exactly 2 common neighbours. By Brouwer et al. [1, Prop. 1.1.2], every rectagraph is regular. An *s-claw* in a graph Γ is a subgraph of Γ which is isomorphic to the complete bipartite graph $K_{1,s}$.

Lemma 4 (Brouwer et al. [1, Prop. 4.3.6, Cor. 4.3.7]) *Let Γ be a rectagraph with v vertices and valency k such that any 3-claw determines a unique 3-cube. Then the following (i), (ii) hold.*

(i) There exists a map π from a k-cube to Γ preserving distances ≤ 2. If Γ does not contain pentagons, then π also preserves distance 3.

(ii) If x and y are two adjacent vertices of Γ, then each vertex in $\pi^{-1}(x)$ is adjacent to a unique vertex in $\pi^{-1}(y)$. In particular, $|\pi^{-1}(x)| = |\pi^{-1}(y)|$. It follows that $v \mid 2^k$.

∎

We now define distance-regular graphs. A connected graph Γ with diameter d is said to be *distance-regular*, whenever for all integers h, i, j $(0 \leq h, i, j \leq d)$, and all $x, y \in V\Gamma$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\{z \mid z \in V\Gamma,\ \partial(x, z) = i,\ \partial(y, z) = j\}| \tag{1}$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1i-1}$ for $1 \leq i \leq d$, $a_i := p^i_{1i}$ for $0 \leq i \leq d$, $b_i := p^i_{1i+1}$ for $0 \leq i \leq d-1$, and $c_0 = b_d = 0$. Observe that $k_i = k_i(x) = p^0_{ii}$ for $0 \leq i \leq d$ and for all $x \in V\Gamma$. It is well known that $k_i = (b_0 \cdots b_{i-1})/(c_1 \cdots c_d)$ $(0 \leq i \leq d)$. Moreover, for $0 \leq i \leq d$ we have

$$c_i + a_i + b_i = k,$$

where $k := k_1$.

Let Γ be a distance-regular graph with diameter d. For each integer i $(0 \leq i \leq d)$, let A_i be the matrix with rows and columns indexed by $V\Gamma$, and x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i, \\ 0 & \text{if } \partial(x,y) \neq i. \end{cases}$$

We call A_i the *ith distance matrix* of Γ. Observe (i) $A_0 = I$, where I denotes the identity matrix; (ii) $A_i^T = A_i$ $(0 \leq i \leq d)$; (iii) $\sum_{h=0}^{d} A_h = J$, where J denotes the all 1's matrix, and (iv) $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$ $(0 \leq i, j \leq d)$. These properties imply that the matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra M, known as the *Bose-Mesner algebra*. By Godsil [5, Thm. 12.2.1], the algebra M has a second basis $E_0, E_1, \ldots, E_d$ such that (i) $E_0 = |V\Gamma|^{-1} J$; (ii) $E_i^T = E_i$ $(0 \leq i \leq d)$; (iii) $\sum_{h=0}^{d} E_h = I$, and (iv) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq d)$. We refer to $E_0, E_1, \ldots, E_d$ as the *principal idempotents* of Γ, and E_0 as the *trivial* idempotent.

Set $A := A_1$ and let $\theta_0, \theta_1, \ldots, \theta_d$ denote the real numbers which satisfy

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

Then $AE_i = E_i A = \theta_i E_i$ $(0 \leq i \leq d)$, and $\theta_0 = k$. We refer to θ_i as the *eigenvalue* of Γ associated with E_i, and call θ_0 the *trivial* eigenvalue of Γ. For each integer i $(0 \leq i \leq d)$, let m_i be the rank of E_i. We refer to m_i as the *multiplicity* of E_i (or θ_i).

For notational convenience, we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space $(V, \langle\, , \rangle)$, where $V = \mathbb{R}^{|V\Gamma|}$ (column vectors), and where $\langle\, , \rangle$ is the inner product

$$\langle u, v \rangle = u^t v \quad (u, v \in V).$$

We now review the cosine sequence of Γ. Let θ be an eigenvalue of Γ, and let E be the associated principal idempotent. Let $w_0, w_1, \ldots, w_d$ be the real numbers satisfying

$$E := \frac{m_\theta}{|V\Gamma|} \sum_{i=0}^{d} w_i A_i, \tag{2}$$

where m_θ denotes the multiplicity of θ. We refer to w_i as the *ith cosine* of Γ with respect to θ (or E), and call $w_0, w_1, \ldots, w_d$ the *cosine sequence* of Γ associated with θ (or E). The following basic result can be found for example in Brouwer et al. [1, Prop. 4.1.1].

Lemma 5 *Let Γ be a distance-regular graph with diameter d. Let θ be an eigenvalue of Γ with multiplicity m_θ, the associated principal idempotent E, and the associated cosine sequence $w_0, w_1, \ldots, w_d$. Then the following (i), (ii) hold.*

(i) For all $x, y \in V\Gamma$ with $\partial(x,y) = i$ we have $\langle Ex, Ey \rangle = w_i \cdot m_\theta / |V\Gamma|$.

(ii) The cosine sequence satisfies $w_0 = 1$ *and the three-term recurrence*

$$c_i w_{i-1} + a_i w_i + b_i w_{i+1} = \theta w_i \qquad (0 \le i \le d). \quad \blacksquare \tag{3}$$

In particular, we have $w_1 = \theta/k$ and for $d \ge 2$ also

$$w_2 = (\theta^2 - a_1\theta - k)/(kb_1) \quad \text{and} \quad kb_1(1 - w_2) = (k - \theta)(\theta + k - a_1). \tag{4}$$

We end this section with the following definitions. Let $A \subseteq V\Gamma$ and let E be a principal idempotent of Γ. Then $\langle A \rangle_E$ will denote the vector space spanned by $\{Ea \mid a \in A\}$. Let ℓ and n be positive integers and let $v_1, v_2, \ldots, v_\ell$ be vectors in the Euclidean space $\mathbb{R}^n$. Then the *Gram matrix* of vectors $v_1, v_2, \ldots, v_\ell$ is the matrix G of dimension $\ell \times \ell$ defined by $G_{ij} = \langle v_i, v_j \rangle$, $1 \le i, j \le \ell$. Observe that the determinant of G is zero if and only if the vectors $v_1, v_2, \ldots, v_\ell$ are linearly dependent.

3 On the eigenvalue multiplicity

In this section we will show that a triangle- and pentagon-free distance-regular graph Γ with an eigenvalue multiplicity equal to the valency has some additional combinatorial properties. We begin with a simple observation.

Lemma 6 *Let* n *be a positive integer,* I *be the* $n \times n$ *identity matrix and* J *all ones matrix of the same dimension. Then* $\det(aI + bJ) = a^{n-1}(a + nb)$ *for any scalars* a *and* b.

Proof. For the all 1's vector $\boldsymbol{j}$ we have $(aI + bJ)\boldsymbol{x} = \lambda\boldsymbol{x}$ if and only if $(\lambda - a)\boldsymbol{x} = b\langle \boldsymbol{x}, \boldsymbol{j} \rangle \boldsymbol{j}$. $\blacksquare$

Lemma 7 *Let* Γ *be a triangle-free distance-regular graph with valency* $k \ge 3$. *Let* $\theta \ne \pm k$ *be an eigenvalue of* Γ *with multiplicity* m_θ *and let* E *be the associated principal idempotent. Then the following (i)–(iv) hold.*

(i) $m_\theta \ge k$.

(ii) $\theta \ne 0$ *if and only if* $\langle \Gamma(x) \rangle_E$ *has dimension* k *for all vertices* $x \in V\Gamma$.

(iii) If $\theta = 0$ *then* $\langle (\{x\} \cup \Gamma(x)) \setminus \{y\} \rangle_E$ *has dimension* k *for all* $x \in V\Gamma$ *and for all* $y \in \Gamma(x)$.

(iv) If $m_\theta = k$, *then the girth of* Γ *is at most five.*

Proof. (i) This is an immediate consequence of Terwilliger Tree Bound, see [12].

(ii) Let $\Gamma(x) = \{y_1, \ldots, y_k\}$ and let G be the Gram matrix of vectors Ey_i, $i = 1, 2, \ldots, k$. By Lemma 5(i), we have

$$G = \frac{m_\theta}{v}\left(I_k + w_2(J_k - I_k)\right) = \frac{m_\theta}{v}\left((1 - w_2)I_k + w_2 J_k\right),$$

where I_k and J_k are the identity and the all 1's matrix with dimension $k \times k$, and $v = |V\Gamma|$. By Lemma 6, we observe that $(v/m_\theta)^k \det(G) = (1 - w_2)^{k-1}(1 + (k-1)w_2)$. Since $\theta \neq \pm k$ we have $1 = w_0 \neq w_2$. Thus $\det(G) \neq 0$ if and only if $w_2 \neq -1/(k-1)$, implying that the set of vectors $\{Ey_i \mid i = 1, 2, \ldots, k\}$ is linearly independent if and only if $w_2 \neq -1/(k-1)$, i.e., $\theta \neq 0$ by $a_1 = 0$ and (4).

(iii) Suppose $\theta = 0$, i.e., $w_1 = 0$ and $w_2 = -1/(k-1)$. Let $\Gamma(x) = \{y_1, \ldots, y_k\}$. Without loss of generality we may assume $y = y_k$. Let G be the Gram matrix of vectors $Ex, Ey_1, Ey_2, \ldots, Ey_{k-1}$. Using Lemma 5(i), Gauss elimination and Lemma 6, we calculate

$$(v/m_\theta)^k \det(G) = (1 - w_2)^{k-2}(1 + (k-2)w_2).$$

But since $w_2 = -1/(k-1) \neq 1$, we find $\det(G) \neq 0$, showing that the vectors Ex, Ey_1, Ey_2, $\ldots$, Ey_{k-1} are linearly independent.

(iv) Assume the girth of Γ is greater then 5. Choose $x \in V\Gamma$ and $y \in \Gamma(x)$ and let T be the graph induced on $\Gamma(x) \cup \Gamma(y)$. Since the girth of Γ is greater then 5, T is a tree and $\partial_T(v_1, v_2) = \partial_\Gamma(v_1, v_2)$ for every $v_1, v_2 \in VT$. So, by Terwilliger tree bound, $m_\theta \geq 2(k-1) = k + k - 2 > k$, a contradiction. ∎

Let Γ be a triangle- and pentagon-free distance-regular graph with an eigenvalue multiplicity equal to the valency. Since the parameter γ_1 always exists, the parameter γ_2 is the next to consider. In the next lemma we will show, that the parameter γ_i exists when $a_i = 0$, so in particular γ_2 exists.

Lemma 8 *Let Γ be a triangle- and pentagon-free distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$, and an eigenvalue θ with multiplicity k. Then the following (i)–(iii) hold.*

(i) $c_2 \geq 2$.

(ii) $\theta \neq 0$.

(iii) For all $1 \leq i \leq d-1$ with $a_i = 0$ the parameters γ_i exist. In particular, $\gamma_1 = 1$ and

$$\gamma_2 = \frac{(k - \theta^2)(2c_2 - k) + c_2^2(\theta^2 - 1)}{\theta^2(k-1)}.$$

Proof. Let E be the principal idempotent associated with the eigenvalue θ. For a vertex u of Γ we denote by $\hat{u}$ the vector Eu. Let $x \in V\Gamma$ and $\Gamma(x) = \{y_1, \dots, y_k\}$.

(i) An immediate consequence of Lemma 7(iv).

(ii) Assume $\theta = 0$. Then $a_1 = a_2 = 0$, (4) and (3) implies $w_1 = 0 = w_3$ and $w_2 = -1/(k-1)$. By Lemma 7(iii), $\langle (\{x\} \cup \Gamma(x)) \setminus \{y_k\} \rangle_E$ has dimension k. Hence, since the multiplicity of θ is k, the set $\{\hat{x}, \hat{y}_1, \dots, \hat{y}_{k-1}\}$ is a basis for the vector subspace $\langle V\Gamma \rangle_E$. Let z be a neighbour of y_k different from x. So $\partial(x, z) = 2$, because Γ is triangle-free, and there exists real numbers $\alpha_1, \dots, \alpha_{k-1}, \delta$ such that

$$\hat{z} = \sum_{i=1}^{k-1} \alpha_i \, \hat{y}_i + \delta \, \hat{x}. \tag{5}$$

Without loss of generality we may assume that $y_1, y_2, \dots, y_{c_2-1}$ are adjacent to z. Because $a_1 = a_2 = 0$, we have $\partial(z, y_i) = 3$ for $c_2 \le i \le k-1$. By taking the inner product of both sides of Equation (5) with $\hat{y}_j$, for $j = k$ and $j = 1, \dots, k-1$, and using Lemma 5(i) and $w_1 = 0 = w_3$, we find

$$\sum_{\ell=1}^{k-1} \alpha_\ell w_2 = 0 \quad \text{and} \quad \sum_{\ell=1}^{k-1} \alpha_\ell w_2 + (1 - w_2)\alpha_j = 0 \qquad \text{for } j \in \{1, \dots, k-1\}. \tag{6}$$

Therefore, since $w_2 = -1/(k-1) \neq 1$, we obtain $\alpha_1 = \cdots = \alpha_{k-1} = 0$. Hence $\hat{z} = \delta \hat{x}$. As $\hat{x}$ and $\hat{z}$ have both the same length, it follows that $\delta = \pm 1$ and so also $w_2 = \langle \hat{x}, \hat{z} \rangle \, v/k = \delta \langle \hat{x}, \hat{x} \rangle \, v/k = \pm 1$, where v is the number of vertices of Γ. Since $w_2 = -1/(k-1) \neq 1$ we have $-1/(k-1) = -1$, i.e., $k = 2$. A contradiction! Therefore, $\theta \neq 0$.

(iii) Suppose $a_i = 0$ for some $i \in \{1, \dots, d-1\}$. By (ii) above, we have $\theta \neq 0$ and so also $w_1 = \theta/k \neq 0$. Let $y \in \Gamma_2(x)$ and $z \in \Gamma_i(x) \cap \Gamma_i(y)$. Obviously $\gamma_1 = 1$, so we assume $i \ge 2$. Furthermore, without loss of generality we assume $\partial(y, y_j) = 1$ for $1 \le j \le c_2$. Since $a_1 = a_2 = 0$, we have $\partial(y, y_j) = 3$ for $c_2 + 1 \le j \le k$. By Lemma 7(ii), there exists real numbers α_j such that

$$\hat{y} = \sum_{\ell=1}^{k} \alpha_\ell \, \hat{y}_\ell. \tag{7}$$

Similarly as in (ii) above, by taking the inner product of both sides of Equation (7) with $\hat{y}_j$ for $j = 1, \dots, k$, we find

$$\sum_{\ell=1}^{k} \alpha_\ell w_2 + (1 - w_2)\alpha_j = \begin{cases} w_1 & \text{if } j \in \{1, 2, \dots, c_2\} \\ w_3 & \text{if } j \in \{c_2 + 1, \dots, k\}, \end{cases} \tag{8}$$

and hence $\alpha_1 = \cdots = \alpha_{c_2} =: \alpha$, $\alpha_{c_2+1} = \cdots = \alpha_k =: \beta$. From this, (8) and, by taking the inner product of both sides of (7) also with $\hat{x}$, we obtain,

$$w_1 = \alpha + ((c_2 - 1)\alpha + \beta(k - c_2)) w_2 \quad \text{and} \quad w_2 = (c_2 \alpha + \beta(k - c_2)) w_1. \tag{9}$$

As $w_2 \neq 1$, $w_1 \neq 0$ and $k \neq c_2$, we can solve the above two equalities for α and β and obtain

$$\alpha = \frac{(w_1 - w_2)(w_1 + w_2)}{w_1(1 - w_2)} \quad \text{and} \quad \beta = \frac{w_2(w_2 - 1) + c_2(w_1 - w_2)(w_1 + w_2)}{(k - c_2)w_1(w_2 - 1)}.$$

Let $\gamma = \gamma_i(x, y, z)$ be the cardinality of the set $\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z)$. We may assume $\Gamma(x) \cap \Gamma(y) \cap \Gamma_{i-1}(z) = \{y_1, \ldots, y_\gamma\}$. Observe that $a_i = 0$ implies $\partial(z, y_j) = i + 1$ for $\gamma + 1 \leq j \leq c_2$. Since $|\Gamma(x) \cap \Gamma_{i-1}(z)| = c_i$, we may assume $\partial(z, y_j) = i - 1$ for $c_2+1 \leq j \leq c_2+c_i-\gamma$. Again, since $a_i = 0$, we have $\partial(z, y_j) = i+1$ for $c_2+c_i-\gamma+1 \leq j \leq k$. By calculating the inner product of $\hat{y}$ and $\hat{z}$, we get

$$w_i = \gamma\alpha w_{i-1} + (c_2 - \gamma)\alpha w_{i+1} + (c_i - \gamma)\beta w_{i-1} + (k - c_2 - c_i + \gamma)\beta w_{i+1}. \qquad (10)$$

Observe that (10) is a linear equation for γ with the coefficient beside γ equal to $(\alpha - \beta)(w_{i-1} - w_{i+1})$. Let us show that $(\alpha - \beta)(w_{i-1} - w_{i+1}) \neq 0$. Observe $\alpha - \beta = \theta/(k - c_2) \neq 0$ since $\theta \neq 0$. Let us suppose $w_{i-1} = w_{i+1}$. Then, by $a_i = 0$ and (3), we obtain $\theta w_i = c_i w_{i-1} + b_i w_{i+1} = k w_{i+1}$ and thus $w_{i+1} = w_1 w_i$. By Equation (10), we find $w_i = w_{i+1}(\alpha c_2 + \beta(k - c2))$, and hence, by the second equation of (9), we have $w_2 w_{i+1} = w_1 w_i$. Therefore, $w_2 w_1 w_i = w_1 w_i$. If $w_i = 0$, then also $w_{i-1} = w_{i+1} = 0$. But then, by the recursion relation of the cosine sequence $\{w_i\}$, we have also $w_1 = 0$, a contradiction. So $w_i \neq 0$ and hence $w_2 = 1$, which is equivalent to $\theta = \pm k$ and this is clearly impossible. Hence we can calculate γ from Equation (10) and is therefore independent of the choice of x, y, z.

We obtain the formula for γ_2 from (10) for $i = 2$. ∎

The following result and its proof are essentially the same as in Nomura [10, Lemma 5.1].

Lemma 9 *Let Γ be a triangle- and pentagon-free distance-regular graph with diameter $d \geq 3$, valency k. Pick an integer i, $2 \leq i \leq d - 1$, such that $a_i = 0$ and that the parameter γ_i exists. Then*

(i) If γ_2 exists, then $(k - 2)(\gamma_2 - 1) = (c_2 - 1)(c_2 - 2)$;

(ii) $\gamma_i(c_{i+1} - 1) = c_i(c_2 - 1)$.

Proof. (i) Let $u \in V\Gamma$ and $v \in \Gamma(u)$, $w \in \Gamma_2(u) \cap \Gamma(v)$. Count the number of edges between $\Gamma(u) \cap \Gamma(w) \cap \Gamma_2(v)$ and $\Gamma(v) \cap \Gamma_2(u) \cap \Gamma_2(w)$ in two different ways.

(ii) Let $u \in V\Gamma$ and $v \in \Gamma_i(u)$, $w \in \Gamma_{i+1}(u) \cap \Gamma(v)$. Count the number of edges between $\Gamma_{i-1}(u) \cap \Gamma(v)$ and $\Gamma_i(u) \cap \Gamma(w) \cap \Gamma_2(v)$ in two different ways. ∎

4 Proofs of Theorems 2 and 3

In this section we prove our main results. In order to prove Theorem 2 and Theorem 3 we will first prove Theorem 10 and Theorem 11, which are also of independent interest. Recall that we have identified the vertex set of an arbitrary graph Δ with the standard orthonormal basis in $\mathbb{R}^n$, where $n = |V\Delta|$. For an arbitrary vector $\boldsymbol{v} \in \mathbb{R}^n$ and for every $x \in V\Delta$ we denote by $\boldsymbol{v}_x = \langle \boldsymbol{v}, x \rangle$ the component of $\boldsymbol{v}$ corresponding to the vertex x.

Theorem 10 *Let Γ be a distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$, $a_1 = a_2 = 0$ and $c_i = i$ for $i = 1, 2, 3$. Then Γ has an eigenvalue $\theta \in \{k-2, 2-k\}$ if and only if Γ is the k-cube or k odd, $k \geq 7$ and Γ is the folded k-cube.*

Proof. If Γ is the k-cube or k odd, $k \geq 7$ and Γ is the folded k-cube, then Γ has an eigenvalue $\theta = k-2$ or $\theta = 2-k$ respectively, see Brouwer et al. [1, p. 261, p. 264].

Let us now assume the graph Γ has an eigenvalue $\theta \in \{k-2, 2-k\}$. Since Γ is a rectagraph (i.e., $a_1 = 0$ and $c_2 = 2$), $c_3 = 3$ and $a_2 = 0 \leq 3$, every 3-claw in Γ determines a unique 3-cube by Brouwer et al. [1, Lemma 4.3.5 (ii)]. Therefore, by Lemma 4(i), there exists a map π from the k-cube Δ to Γ, which preserves distances 1, 2 and 3. Let $\Pi = \{\pi^{-1}(x) \mid x \in V\Gamma\}$. Then two vertices in the same class are at distance at least 7. By Lemma 4(ii), between two classes of Π there is either a perfect matching or nothing (so the partition Π of the vertex set of Δ is uniformly regular), and the quotient graph Δ/Π is the graph Γ. Hence the eigenvalue θ is also an eigenvalue of Δ, see Godsil [5, Lemma 5.2.2(a)].

We construct an eigenvector of θ in Δ from the cosine sequence $w_0, w_1, \ldots, w_d$ of Γ corresponding to θ. Choose $x_0 \in V\Gamma$. So $(w_0, w_1, \ldots, w_d)^T$ is a (right) eigenvector of the tridiagonal matrix corresponding to the distance partition of the graph Γ, see Brouwer et al. [1, Sect. 4.1B], and a vector $\boldsymbol{v} \in \mathbb{R}^n$, where $n = |V\Gamma|$, defined by $\boldsymbol{v}_x = w_i$ if and only if $\partial(x_0, x) = i$, is an eigenvector of Γ corresponding to θ by Godsil [5, Lemma 5.2.2(a)]. For $i \in \{0, 1, \ldots, d\}$ define

$$D_i = \bigcup \{\pi^{-1}(x) \,|\, x \in V\Gamma,\ \partial(x_0, x) = i\}.$$

Observe that, by Brouwer et al. [1, Lemma 11.1.4], $D_i = \{\overline{x} \in V\Delta \mid \partial(\overline{x}, D_0) = i\}$, where $\partial(\overline{x}, D_0) = \min\{\partial(\overline{x}, \overline{y}) \mid \overline{y} \in D_0\}$. Therefore, the vector $\overline{\boldsymbol{v}} \in \mathbb{R}^{2^k}$, defined by $\overline{\boldsymbol{v}}_{\overline{x}} = (k/2^k)w_i$ if and only if $\overline{x} \in D_i$, is an eigenvector of Δ corresponding to θ, such that $\overline{\boldsymbol{v}}_{\overline{x}} = (k/2^k)w_0 = k/2^k$ if $\overline{x} \in D_0$.

Choose $\overline{x}_0 \in D_0$ and let $\overline{x}_i$, $1 \leq i \leq k$, be the neighbours of $\overline{x}_0$ in Δ. They are members of D_1, since the map π preserves adjacency. Let E be the principal idempotent of Δ corresponding to θ. We will now show that $\overline{\boldsymbol{v}} = E\overline{x}_0$. Note that we have $E\overline{\boldsymbol{v}} = \overline{\boldsymbol{v}}$, since $\overline{\boldsymbol{v}}$ is an eigenvector for Δ corresponding to θ, and that the vectors $E\overline{x}_i$, $1 \le i \le k$, are a basis of the eigenspace corresponding to θ, by Lemma 7, as the multiplicity of θ in Δ is k, see Brouwer et al. [1, p. 261]. We have, by Lemma 5 and the definition of the vector $\overline{\boldsymbol{v}}$,

$$\langle E\overline{x}_0, E\overline{x}_i\rangle = \frac{k}{2^k}\,\frac{\theta}{k} = \frac{k}{2^k}\,w_1 = \overline{\boldsymbol{v}}_{\overline{x}_i} = \langle \overline{\boldsymbol{v}}, \overline{x}_i\rangle = \langle E\overline{\boldsymbol{v}}, \overline{x}_i\rangle = \langle \overline{\boldsymbol{v}}, E\overline{x}_i\rangle \quad \text{for} \quad 1 \le i \le k.$$

Therefore, $\overline{\boldsymbol{v}} = E\overline{x}_0$. Let $w_0(\Delta), w_1(\Delta), \ldots, w_k(\Delta)$ be the cosine sequence corresponding to θ in Δ. Observe that $(E\overline{x}_0)_{\overline{x}} = (k/2^k)w_i(\Delta)$ if and only if $\partial(\overline{x}_0, \overline{x}) = i$, since $E = (k/2^k)\sum_{j=0}^{d} w_j(\Delta)A_j$, where A_j is the j-th distance matrix of Δ. So, by Brouwer et al. [1, Prop. 4.4.7], the set $\{\overline{x} \in V\Delta \mid (E\overline{x}_0)_{\overline{x}} = k/2^k\}$, which contains D_0, is either $\{\overline{x}_0\}$ or $\{\overline{x}_0, \overline{y}_0\}$, where $\overline{y}_0$ is the unique vertex in Δ, which is at distance k from $\overline{x}_0$. It follows Γ is either the k-cube, or the folded k-cube. Assume Γ is the folded k-cube. If k is even, then neither $k-2$ nor $2-k$ is eigenvalue of Γ, see Brouwer et al. [1, p. 264]. Hence k must be odd. If k is 5, then $d = 2$, thus $k \ge 7$. ∎

Theorem 11 *Let Γ be a distance-regular graph with diameter $d \ge 3$, valency $k \ge 3$, $a_1 = a_2 = 0$, for which the parameter γ_2 exist. Then (i) $\gamma_2 = c_2 = 1$, or (ii) $\gamma_2 = 1$, $c_2 = 2$ and $c_3 = 3$, or (iii) Γ is bipartite 2-homogeneous graph.*

Proof. Set $\gamma := \gamma_2$. By $k \ge 3$, it follows that $\gamma \ne 0$. We first consider the case $\gamma = 1$. If $c_2 = 1$ then we obtain the case (i), so we may assume $c_2 \ge 2$. Then it follows from Lemma 9(i) that $c_2 = 2$ and hence, by Lemma 9(ii), $c_3 = 3$. So we obtain the case (ii). It remains to consider the case $\gamma \ge 2$. Then we have, by Lemma 9(i), $c_2 > 2$. If $\gamma \ge c_2$, then we obtain from Lemma 9(i) $c_2 \ge k$, which is not possible as $d \ge 3$. Thus $c_2 > \gamma$. Let us show $2c_3 \ge k+3$. As $c_2 > \gamma$, it follows that $c_2\gamma - 2c_2 + 2\gamma = c_2(\gamma-2) + 2\gamma > \gamma^2 > \gamma(\gamma-1)$, so

$$\frac{c_2\gamma - 2c_2 + 2\gamma}{\gamma(\gamma-1)} > 1.$$

On the other hand, from Lemma 9 we calculate

$$k = \frac{(c_2-1)(c_2-2)}{\gamma-1} + 2 \quad \text{and} \quad c_3 = \frac{c_2(c_2-1)}{\gamma} + 1, \tag{11}$$

so we have

$$2c_3 - k = (c_2-1)\frac{c_2\gamma - 2c_2 + 2\gamma}{\gamma(\gamma-1)}.$$

It follows that $2c_3 - k > c_2 - 1 \ge 2$, i.e., $2c_3 \ge k+3$.

We will now derive two consequences, namely that Γ is bipartite and has diameter at most 5. Suppose Γ is not bipartite and let i be the minimal integer such that $a_i > 0$. By

Brouwer et al. [1, Prop 5.5.4 (ii)], $i \geq 3$ and $2c_3 \geq k+3$, we find $a_i \geq c_i \geq c_3 > k/2$, which is clearly impossible. So Γ is bipartite. The bound $d \leq 5$ follows immediately from $b_3 < c_3$, see Brouwer et al. [1, Prop. 4.1.6(ii)].

Let us now show that Γ is 2-homogeneous. We will first show that $c_2 \,|\, 2\gamma(\gamma+1)$. Since $k_2 = k(k-1)/c_2$ is integral, c_2 divides $k(k-1)$. From Lemma 9(i) we calculate $k(\gamma-1)$ and $(k-1)(\gamma-1)$ in order to find $(\gamma-1)^2 k(k-1) = (c_2^2 - 3c_2 + 2\gamma)(c_2^2 - 3c_2 + \gamma + 1)$. Thus c_2 must divide $2\gamma(\gamma+1)$.
Since Γ is bipartite, we obtain from (11)

$$b_3 = k - c_3 = \frac{(c_2-\gamma)(c_2-\gamma-1)}{\gamma(\gamma-1)}. \tag{12}$$

Let us now consider separately the cases $d=3$, $d=4$ and $d=5$. If $d=3$, then $b_3=0$ and hence, by (12), $(c_2-\gamma)(c_2-\gamma-1)=0$. Since $c_2 > \gamma$, we obtain $c_2 = \gamma+1$. But then, by (11), $k = \gamma+2$ and Γ is the complete bipartite graph $K_{k+1,k+1}$ minus a matching, which is 2-homogeneous by Theorem 1.

Assume $d=4$. By Brouwer et al. [1, Lemma 4.1.7], we find $p^4_{42} = k(k-1-c_3)/c_2$. But since p^4_{42} is integral, c_2 divides $k(k-1-c_3)$. By direct computation, we find from (11) $\gamma(\gamma-1)^2 k(k-1-c_3) = (c_2-1)(c_2-2\gamma)(c_2^2-3c_2+2\gamma)$, hence c_2 divides $4\gamma^2$. But c_2 divides also $2\gamma(\gamma+1)$, so $c_2 \,|\, 4\gamma$. Since $c_2 > \gamma$, we obtain $c_2 \in \{4\gamma/3, 2\gamma, 4\gamma\}$. We will consider each of this three cases separately. If $c_2 = 4\gamma$ then, by the integrality of k, $\gamma \in \{2,3,4,7\}$. Only for $\gamma = 3$ the number k_2 is integral, but in this case the number p^4_{42} is not integral. If $c_2 = 2\gamma$ then, by (11), $k = 4\gamma$ and $c_3 = 4\gamma - 1$. Hence Γ is a distance-regular graph with intersection array $\{4\gamma, 4\gamma-1, 2\gamma, 1; 1, 2\gamma, 4\gamma-1, 4\gamma\}$, i.e., a Hadamard graph, which is 2-homogeneous by Theorem 1. Finally, if $c_2 = 4\gamma/3$, then, by the integrality of k, $\gamma = 3$. But in this case we have $c_2 = 4$, $k = 5$ and $c_3 = 5$, which is in contradiction with $d = 4$. This completes the proof of the case $d = 4$.

Assume $d=5$. Then $b_3 \geq c_2$ by Brouwer et al. [1, Prop. 4.1.6(ii)], and from (12) we obtain $(c_2-\gamma)(c_2-\gamma-1) \geq c_2\gamma(\gamma-1)$, i.e., $c_2 \geq \gamma(\gamma+1)$. Since $c_2 \,|\, 2\gamma(\gamma+1)$, we have $c_2 \in \{\gamma(\gamma+1), 2\gamma(\gamma+1)\}$. Suppose first $c_2 = 2\gamma(\gamma+1)$. By the integrality of k, we obtain $(\gamma-1) \,|\, 6$, so $\gamma \in \{2,3,4,7\}$. But for none of this possibilities the number k_3 is integral. So we can assume $c_2 = \gamma(\gamma+1)$. Then we have $c_2 = b_3$ and $c_3 = b_2$. Therefore, $c_4 \,|\, k(k-1)$ by the integrality of k_4. If $c_4 = k-1$, then Γ is a distance-regular graph with intersection array $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, where $k = \gamma(\gamma^2+3\gamma+1)$, $c = c_2 = \gamma(\gamma+1)$. In this case Γ is 2-homogeneous by Theorem 1. Assume now $c_4 < k-1$. Since $c_4 \,|\, k(k-1)$, we have $c_4 = k(k-1)/(k+a)$ for some positive integer a and, by $c_4 \geq c_3$, we obtain $a \leq \gamma^2 + 2\gamma - 1 - 1/(\gamma(\gamma+2))$. Hence, by the integrality of a, $a \leq \gamma^2 + 2\gamma - 2$. Observe that Γ is not antipodal because $b_4 = k - c_4 > c_1 = 1$. Hence, by Brouwer et al. [1, Prop. 5.6.1] and $a_5 = 0$, we must have $k_2 \leq k_5(k_5-1)$. By direct computation, we obtain

$k_2 = (\gamma^2 + 3\gamma + 1)(\gamma^2 + 2\gamma - 1)$ and $k_5 = a + 1$. Thus

$$(\gamma^2 + 3\gamma + 1)(\gamma^2 + 2\gamma - 1) \leq (a+1)a \leq (\gamma^2 + 2\gamma - 1)(\gamma^2 + 2\gamma - 2),$$

which gives us $\gamma \leq -3$, a contradiction! ∎

We are now ready to give the proofs of Theorem 2 and Theorem 3.

Proof of Theorem 2. Since distance-regular graphs with $k = 2$ are cycles, we can assume $k \geq 3$. If $d = 2$, then Γ is a bipartite strongly regular graph, i.e., a complete bipartite graph $K_{k,k}$. But the only complete bipartite graph with an eigenvalue multiplicity equal to its valency is the 4-gon, i.e., the 2-cube. Thus we can assume $d \geq 3$. If Γ has an eigenvalue with multiplicity k, then, by Lemma 8, $c_2 \geq 2$, parameter γ_2 exists and it is equal to

$$\gamma_2 = \frac{(k - \theta^2)(2c_2 - k) + c_2^2(\theta^2 - 1)}{\theta^2(k-1)}. \tag{13}$$

By Theorem 11, either Γ is bipartite and 2-homogeneous, or $\gamma_2 = 1$, $c_2 = 2$ and $c_3 = 3$. In the first case we are done by Theorem 1, so assume $\gamma_2 = 1$, $c_2 = 2$ and $c_3 = 3$. From (13) we find $\theta \in \{k-2, 2-k\}$, so, by Theorem 10, Γ is the k-cube or $k \geq 7$, k odd and Γ is the folded k-cube.

On the other hand, graphs (ii)-(v) from Theorem 2 are bipartite (and hence triangle- and pentagon-free) and they all have an eigenvalue with multiplicity k by Theorem 1. Any n-gon, $n \geq 6$, is also triangle- and pentagon-free and it has an eigenvalue $2\cos(2\pi/n)$ with multiplicity 2. Finally, folded k-cube, k odd and $k \geq 7$, is triangle- and pentagon-free and it has eigenvalue $2 - k$ with multiplicity k. ∎

Proof of Theorem 3. Since distance-regular graphs with $k = 2$ are cycles, we can assume $k \geq 3$. If Γ is almost 2-homogeneous bipartite distance-regular graph with diameter $d \geq 4$, then parameter γ_2 exists. So, by Theorem 11, either Γ is bipartite and 2-homogeneous, or $\gamma_2 = 1$. If the first case the result follows from Nomura [10, Thm. 1.2]. Therefore, assume $\gamma_2 = 1$.

If $c_2 = 1$, then, by Curtin [4, Thm. 4.4], Γ is a regular generalized $2d$-gon of order $(1, k-1)$. But, by $d \geq 4$ and Brouwer et al. [1, Thm. 6.5.1], $2d \in \{8, 12\}$. If $c_2 \geq 2$, then, by Curtin [4, Thm. 4.7], Γ is the d-cube or the folded $2d$-cube.

On the other hand, graphs (i)-(iv) from Theorem 3 are almost 2-homogeneous since they are 2-homogeneous by Theorem 1, while the folded $2d$-cube and a regular generalized $2d$-gon of order $(1, k-1)$ are almost 2-homogeneous by Curtin [4, Thm. 4.4, Thm. 4.7]. ∎

5 Conclusions and comments

The conditions $c_3 = 3$ and $a_2 = 0$ in Theorem 10 are probably not necessary. If $\theta = k-2$, these conditions are not necessary, see Brouwer et al. [1, Thm. 4.4.11], but our proof does not seem to generalize in order to show this.

The argument of the proof of Theorem 10 follows an approach due to Meyerowitz [7], who classified the completely regular codes of strength 0, that is a completely regular code, whose quotient matrix has $k-2$ as an eigenvalue, in the Hamming schemes, see also [2].

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