# ON TRIANGLE-FREE DISTANCE-REGULAR GRAPHS WITH AN EIGENVALUE MULTIPLICITY EQUAL TO THE VALENCY


Aleksandar Jurišić
IMFM and Nova Gorica Polytechnic
Vipavska 13, p.p. 301
5000 Nova Gorica
Slovenia

Jack Koolen
Divison of Applied Mathematics
KAIST
Daejeon, South Korea

Štefko Miklavič
IMFM and Nova Gorica Polytechnic
Vipavska 13, p.p. 301
5000 Nova Gorica
Slovenia


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**Abstract**

Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and intersection number $a_2 \neq 0$. Assume $\Gamma$ has an eigenvalue with multiplicity $k$. We show that if for $d \geq 4$ we have $a_4 = 0$, then $\Gamma$ is 1-homogeneous in the sense of Nomura. In particular, the following infinite family of feasible intersection arrays

$$\{2\mu^2+\mu, 2\mu^2+\mu-1, \mu^2, \mu, 1; 1, \mu, \mu^2, 2\mu^2+\mu-1, 2\mu^2+\mu\}, \qquad \mu \in \mathbb{N},$$

of double covers with diameter 5 and $4\mu^2(2\mu+3)$ vertices is 1-homogeneous. As a corollary we show that for $\mu = 4$ no such graph exists.

# 1 Introduction

Let $\Gamma$ be a graph with diameter $d$ and $x, y$ its vertices. Let $D_i^j = D_i^j(x, y)$ be the **distance-set** of all the vertices at distance $i$ from $x$ and $j$ from $y$. The set of nonempty distance-sets, i.e.,

$$\{D_i^j \mid 0 \le i, j \le d,\ D_i^j \ne \emptyset\},$$

is called the **distance partition** of the vertices of the graph $\Gamma$ corresponding to vertices $x$ and $y$. Remember, that a partition of the vertices of $\Gamma$ into cells is **equitable** when for any vertex and any cell the number of neighbours the vertex has in the cell is independent of the choice of the vertex in its cell. The graph $\Gamma$ is called $h$**-homogeneous** in the sense of Nomura [10] when for all its vertices $x$ and $y$ at distance $h$ their distance partition is equitable and the corresponding parameters are independent of the choice of vertices $x$ and $y$. Therefore, the graph $\Gamma$ is distance-regular if and only if it is 0-homogeneous. If the graph $\Gamma$ is distance-regular and $h$ is the distance between vertices $x$ and $y$, then the set $D_i^j$ contains $p_{ij}^h$ elements, so $|i-j| > h$ implies $D_i^j = \emptyset$ and $|i-j| = h$ implies $D_i^j \ne \emptyset$.

The well known Terwilliger Tree Bound [12], cf. Brouwer, Cohen and Neumaier [1, p. 163], implies that an eigenvalue multiplicity of a triangle-free distance-regular graph is either 1 or at least its valency, and if equality holds, then the girth is at most 5. There are many interesting distance-regular graphs with an eigenvalue multiplicity equal to its valency. An important class of such examples comes from distance-regular graphs whose association scheme determined by its distance matrices is formally self-dual.

Bipartite distance-regular graphs with an eigenvalue multiplicity equal to its valency have already been classified by Nomura [11] and Yamazaki [13]. Similarly, a triangle- and pentagon-free distance-regular graphs with an eigenvalue multiplicity equal to its valency have been classified in [7]. It turns out that all such graphs are 2-homogeneous. We study triangle-free distance-regular graphs with $a_2 \ne 0$ (which implies an existence of pentagons) and an eigenvalue multiplicity equal to its valency by following the linear algebra approach, cf. Ivanov and Shpectorov [6]. It turns out that very often these graphs are 1-homogeneous. As it is obvious that distance-regular graphs with at most one $i$ such that $a_i \ne 0$ are 1-homogeneous (for example, bipartite graphs, generalized Odd graphs and the Wells graph), we leave them out from the list of all known examples and feasible families that satisfy our assumptions:

(i) $\{3,2,1,1,1;1,1,1,2,3\}$ is intersection array of the dodecahedron, and it is a member of an infinite family of feasible intersection arrays: $\{2\mu^2+\mu, 2\mu^2+\mu-1, \mu^2, \mu, 1; 1, \mu, \mu^2, 2\mu^2+\mu-1, 2\mu^2+\mu\}$, see Section 5 for more details;

(ii) $\{21,20,16;1,2,12\}$ is intersection array of the coset graph of doubly truncated binary Golay code with $v = 512 = 1+21+210+280$ vertices and eigenvalues $21^1, 5^{210}, (-3)^{280}, (-11)^{21}$. It is a primitive graph, see Ivanov et al. [6] and Brouwer et al. [1, Thm. 11.3.6]. This graph belongs to the family of Hermitean forms graphs over $\mathrm{GF}(2^2)$, see Brouwer et al. [1, Subsect. 9.5C], whose members are formally self-dual by Brouwer et al. [1, Thm. 8.4.3];

(iii) $\{21, 20, 16, 6, 2, 1; 1, 2, 6, 16, 20, 21\}$ is intersection array of the coset graph of once shortened and once truncated binary Golay code with $v = 1024 = 1 + 21 + 210 + 560 + 210 + 21 + 1$ vertices and eigenvalues $21^1, 9^{56}, 5^{210}, 1^{336}, (-3)^{280}, (-7)^{120}, (-11)^{21}$. This graph is a unique distance-regular double-cover of the graph in (ii).

(iv) $\{21, 20, 16, 9, 2, 1; 1, 2, 3, 16, 20, 21\}$ is intersection array of the coset graph of a subcode of the doubly truncated binary Golay code with $v = 2048 = 1+21+210+1120+630+63+3$ vertices and eigenvalues $21^1, 9^{168}, 5^{210}, 1^{1008}, (-3)^{280}, (-7)^{360}, (-11)^{21}$. This graph is a unique antipodal cover of the graph in (ii) with the covering index $r = 4$.

This paper is a part of a broader study of $i$-homogeneous graphs and it is organized as follows. In Section 2 we recall some basic facts about distance-regular graphs and cosine sequences. In Section 3 we study properties of eigenspaces of triangle-free distance-regular graphs, whose dimension equals the valency. Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and $a_2 \neq 0$. Suppose that an eigenvalue multiplicity of $\Gamma$ is equal to the valency. We show that the corresponding eigenvalue is nonzero. In Section 4 we show that the graph $\Gamma$ has some additional combinatorial properties. As a corollary we show that under certain conditions the graph $\Gamma$ is 1-homogeneous. In Section 5 we take a closer look at an infinite family of feasible intersection arrays of distance-regular double-covers with diameter 5. We show that a member of this family that has $c_2 = 4$ does not exist.

# 2 Preliminaries

In this section we review some definitions and basic concepts. See Brouwer et al. [1] and Godsil [5] for more background information.

Throughout this paper, $\Gamma$ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, shortest path-length distance function $\partial$, and diameter $d := \max\{\partial(x, y) | x, y \in V\Gamma\}$. For $x \in V\Gamma$ and for an integer $i$ define $\Gamma_i(x)$ to be the set of vertices of $\Gamma$ at distance $i$ from $x$. We abbreviate $\Gamma(x) := \Gamma_1(x)$. The graph $\Gamma$ is said to be **distance-regular** whenever for all integers $h, i, j$ $(0 \leq h, i, j \leq d)$, and all $x, y \in V\Gamma$ with $\partial(x, y) = h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of vertices $x$ and $y$. The constants $p^h_{ij}$ $(0 \leq h, i, j \leq d)$ are known as the **intersection numbers** of $\Gamma$. For notational convenience define $c_i := p^i_{1,i-1}$ $(1 \leq i \leq d)$, $a_i := p^i_{1i}$ $(0 \leq i \leq d)$, $b_i := p^i_{1,i+1}$ $(0 \leq i \leq d-1)$, $k_i := p^0_{ii}$ $(0 \leq i \leq d)$, and set $c_0 = 0 = b_d$. We observe $a_0 = 0$ and $c_1 = 1$. Moreover, $a_i + b_i + c_i = k$ $(0 \leq i \leq d)$, where $k := k_1$.

Let $\Gamma$ be a distance-regular graph with diameter $d$. We recall the distance matrices of $\Gamma$. For each $i$ $(0 \leq i \leq d)$ let $A_i$ be the matrix with rows and columns indexed by $V\Gamma$, and $x, y$ entry of $A_i$ equal to 1 if $\partial(x, y) = i$ and 0 otherwise. We call $A_i$ the $i$**th distance matrix** of $\Gamma$. The distance matrices of $\Gamma$ are symmetric, they sum to the all 1's matrix and $A_0$ is the identity matrix.

The matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra $M$, known as the **Bose-Mesner algebra** of the graph $\Gamma$. The algebra $M$ has a second basis $\{E_0, E_1, \ldots, E_d\}$ such that $E_0 = |V\Gamma|^{-1}J$ where $J$ is the all ones matrix, $E_iE_j = \delta_{ij}E_i$ $(0 \le i, j \le d)$, $E_0 + E_1 + \cdots + E_d = I$ where $I$ is the identity matrix, and $E_i^t = E_i$ $(0 \le i \le d)$, see for example Godsil [5, Thm. 12.2.1]. The elements $E_0, E_1, \ldots, E_d$ are known as the **principal idempotents** of $\Gamma$, and $E_0$ as the **trivial** idempotent.

Set $A := A_1$, and define the real numbers $\theta_i$ $(0 \le i \le d)$ by

$$A = \sum_{i=0}^{d} \theta_i E_i.$$

Then $AE_i = E_iA = \theta_iE_i$ $(0 \le i \le d)$, and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are pairwise distinct, since $A$ generates $M$, see Brouwer et al. [1, pp. 128]. We refer to $\theta_i$ as the **eigenvalue** of $\Gamma$ associated with $E_i$, and call $\theta_0$ the **trivial** eigenvalue. For each integer $i$ $(0 \le i \le d)$, let $m_i$ be the rank of $E_i$. We refer to $m_i$ as the **multiplicity** of $E_i$ (or $\theta_i$). We observe $m_0 = 1$.

For notational convenience, we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space $(V, \langle\, , \rangle)$, where $V = \mathbb{R}^{|V\Gamma|}$ (column vectors), and where $\langle\, , \rangle$ is the dot product

$$\langle u, v\rangle = u^t v \quad (u, v \in V).$$

We now review the cosines. Let $\theta$ be an eigenvalue of the graph $\Gamma$, and let $E$ be the associated principal idempotent. Let $w_0, w_1, \ldots, w_d$ be the real numbers satisfying

$$E = \frac{m_\theta}{|V\Gamma|} \sum_{i=0}^{d} w_i A_i, \tag{1}$$

where $m_\theta$ denotes the multiplicity of $\theta$. We refer to $w_i$ as the $i$**th cosine** of $\Gamma$ with respect to $\theta$ (or $E$), and call $w_0, w_1, \ldots, w_d$ the **cosine sequence** of $\Gamma$ associated with $\theta$ (or $E$). The following basic result can be found for example in Brouwer et al. [1, Prop. 4.1.1] or Jurišić, Koolen and Terwilliger [8, Lem. 2.1].

**Lemma 2.1** *Let $\Gamma$ be a distance-regular graph with diameter $d$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity $m_\theta$, the associated principal idempotent $E$, and the associated cosine sequence $w_0, w_1, \ldots, w_d$. Then the following (i), (ii) hold.*

*(i) For all $x, y \in V\Gamma$ with $\partial(x, y) = i$ we have $\langle Ex, Ey\rangle = w_i\, m_\theta/|V\Gamma|$.*

*(ii) The cosine sequence satisfies $w_0 = 1$ and the three-term recurrence*

$$c_iw_{i-1} + a_iw_i + b_iw_{i+1} = \theta w_i \qquad (0 \le i \le d),$$

*where $w_{-1}$ and $w_{d+1}$ are indeterminates.* ∎

In particular, we have $w_1 = \theta/k$ and for $d \geq 2$ also

$$w_2 = (\theta^2 - a_1\theta - k)/(kb_1) \quad \text{and} \quad kb_1(1 - w_2) = (k - \theta)(\theta + k - a_1).$$

We end this section with the following definition. Let $A \subseteq V\Gamma$ and let $E$ be the principal idempotent of $\Gamma$. Then $\langle A \rangle_E$ is the vector space spanned by $\{Ea \mid a \in A\}$.

# 3 On the eigenvalue multiplicity

Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d$. In this section we discus the case when $\Gamma$ has an eigenvalue $\theta$ with multiplicity equal to its valency. We start with a result, which was proved in [7, Lem. 6] and then prove that $\theta$ is nonzero for $k \geq 3$.

**Lemma 3.1** *Let $\Gamma$ be a triangle-free distance-regular graph with valency $k \geq 3$. Let $\theta \neq \pm k$ be an eigenvalue of $\Gamma$ with multiplicity $m_\theta$ and let $E$ be the associated principal idempotent. Then the following (i) – (iii) hold.*

*(i) $m_\theta \geq k$.*

*(ii) $\theta \neq 0$ if and only if $\langle \Gamma(x) \rangle_E$ has dimension $k$ for all $x \in V\Gamma$.*

*(iii) If $\theta = 0$ then $\langle (\{x\} \cup \Gamma(x)) \setminus \{y\} \rangle_E$ has dimension $k$ for all $x \in V\Gamma$ and for all $y \in \Gamma(x)$.* ∎

**Theorem 3.2** *Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 2$ and valency $k \geq 3$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity equal to $k$. Then $\theta \neq 0$.*

*Proof.* Assume $\theta = 0$. If $d = 2$ then, by Brouwer et al. [1, Thm. 1.3.1(v)], $\Gamma$ is complete multipartite graph. But since $\Gamma$ is triangle-free, it must be complete bipartite. Because the multiplicity of $\theta$ equals valency $k$, we obtain from Brouwer et al. [1, Thm. 1.3.1(vi)] $k = 2$, a contradiction! Suppose now $d \geq 3$. Then $w_1 = 0$, $w_2 = 1/(1-k) \neq 0$ and $w_3 = a_2/((k-1)b_2)$ by Lemma 2.1(ii). Let $x \in V\Gamma$ and let $\{z_1, z_2, \ldots, z_k\}$ be the neighbours of $x$. Let $y$ be the neighbour of $z_1$, which is at distance 2 from $x$. We may assume $\Gamma(x) \cap \Gamma(y) = \{z_1, \ldots, z_{c_2}\}$, $\Gamma(x) \cap \Gamma_2(y) = \{z_{c_2+1}, \ldots, z_{c_2+a_2}\}$ and $\Gamma(x) \cap \Gamma_3(y) = \{z_{c_2+a_2+1}, \ldots, z_k\}$. By Lemma 3.1(iii), the set $\{Ex, Ez_2, \ldots, Ez_k\}$ is the basis of the eigenspace corresponding to $\theta$. Therefore, there exist real numbers $\alpha_2, \ldots, \alpha_k$ and $\delta$, such that

$$Ey = \sum_{i=2}^{k} \alpha_i E z_i + \delta E x. \tag{2}$$

Taking the scalar product of both sides of Equation (2) with $Ex$, $Ez_1$ and $Ez_j$ for $j \in \{2, \dots, k\}$ respectively, we obtain, by Lemma 2.1(i) and multiplication with $|V\Gamma|/k$, the following relations

$$w_2 = \frac{|V\Gamma|}{k}\langle Ey, Ex\rangle = \delta\frac{|V\Gamma|}{k}\langle Ex, Ex\rangle = \delta, \tag{3}$$

$$0 = w_1 = \frac{|V\Gamma|}{k}\langle Ey, Ez_1\rangle = w_2\sum_{i=2}^{k}\alpha_i, \tag{4}$$

$$\alpha_j(w_0 - w_2) = \alpha_j(w_0 - w_2) + w_2\sum_{i=2}^{k}\alpha_i = \begin{cases} w_1 & \text{if } j \in \{2, \dots, c_2\}, \\ w_2 & \text{if } j \in \{c_2+1, \dots, c_2+a_2\}, \\ w_3 & \text{if } j \in \{c_2+a_2+1, \dots, k\}. \end{cases} \tag{5}$$

Since $w_0 - w_2 = 1 - 1/(1-k) \neq 0$, we get

$$\alpha_j = \begin{cases} 0 & \text{if } j \in \{2, \dots, c_2\}, \\ A_2 & \text{if } j \in \{c_2+1, \dots, c_2+a_2\}, \\ B_2 & \text{if } j \in \{c_2+a_2+1, \dots, k\}, \end{cases}$$

where $A_2 = -1/k$ and $B_2 = a_2/(kb_2)$. Since $w_2 \neq 0$, Equation (4) implies $a_2A_2 + b_2B_2 = 0$. Finally, by Lemma 2.1(i), Equation (2) and $w_1 = 0$, we obtain

$$1 = w_0 = (|V\Gamma|/k)\langle Ey, Ey\rangle = Uw_2 + Vw_0, \tag{6}$$

where $V = a_2A_2^2 + b_2B_2^2 + \delta^2$ and $V + U = (a_2A_2 + b_2B_2 + \delta)^2 = \delta^2$. Since $d \geq 3$ we have $b_2 > 0$ and we can multiply the above equation with $b_2k(k-1)$. We get

$$b_2k(k-1) = b_2a_2 + a_2^2 + \frac{b_2k}{k-1}.$$

Hence, by the integrality of the last fraction, $b_2 = k-1$ and thus $a_2 = 0$. But now we obtain from the above equation $k(k-1)^2 = k$, which is clearly impossible. Thus, $\theta \neq 0$. ∎

**Lemma 3.3** *Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 2$ and valency $k \geq 3$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity equal to $k$. Let $E$ be the associated principal idempotent and $w_0, \dots, w_d$ the associated cosine sequence. Let $x \in V\Gamma$ and $y \in \Gamma_i(x)$, where $1 \leq i \leq d$. Then*

$$Ey = C_i\Big(\sum_{z\in\Gamma(x)\cap\Gamma_{i-1}(y)} Ez\Big) + A_i\Big(\sum_{z\in\Gamma(x)\cap\Gamma_i(y)} Ez\Big) + B_i\Big(\sum_{z\in\Gamma(x)\cap\Gamma_{i+1}(y)} Ez\Big),$$

*where*

$$C_i = \frac{w_1w_{i-1} - w_2w_i}{w_1(w_0 - w_2)}, \qquad A_i = \frac{w_1w_i - w_2w_i}{w_1(w_0 - w_2)}, \qquad B_i = \frac{w_1w_{i+1} - w_2w_i}{w_1(w_0 - w_2)}. \tag{7}$$

*The denominators in (7) are nonzero.*

*Proof.* Let us first observe that $m_\theta = k \geq 3$ implies $\theta \neq 0$, i.e., $w_1 \neq 0$, by Theorem 3.2, and that $w_2 \neq 1 = w_0$. So the denominators in (7) are really nonzero. By Lemma 3.1(ii), there exist real numbers $\alpha_z$, $z \in \Gamma(x)$, such that

$$Ey = \sum_{z\in\Gamma(x)} \alpha_z Ez.$$

Taking the scalar product of both sides of the above equation with $Ex$ and $Ev$, $v \in \Gamma(x)$ respectively, we obtain, by Lemma 2.1 and multiplication with $(|V\Gamma|/k)$, the following relations

$$w_1 \sum_{z\in\Gamma(x)} \alpha_z = w_i,$$

$$\alpha_v(w_0 - w_2) + w_i w_2/w_1 = \alpha_v(w_0 - w_2) + w_2 \sum_{z\in\Gamma(x)} \alpha_z = \begin{cases} w_{i-1} & \text{if } v \in \Gamma(x)\cap\Gamma_{i-1}(y), \\ w_i & \text{if } v \in \Gamma(x)\cap\Gamma_i(y), \\ w_{i+1} & \text{if } v \in \Gamma(x)\cap\Gamma_{i+1}(y). \end{cases}$$

Solving the last equation for $\alpha_v$ gives us the desired result. ∎

The main message of the next result is that if we know that a graph has an eigenvalue with multiplicity equal to $k$, then we can recognize from $a_d$ whether it is primitive or not. We need one more notation. Let $\Gamma$ be a distance-regular graph with diameter $d$. For a distinct vertices $x, y \in V\Gamma$ with $\partial(x, y) = d$, we denote the set $\Gamma_{d-1}(x) \cap \Gamma(y)$ by $C(x, y)$.

**Proposition 3.4** *Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 2$ and valency $k \geq 3$, for which $k_d \geq 2$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity equal to $k$ and let $x, y, z$ be three distinct vertices of $\Gamma$ with $\partial(x, y) = d = \partial(x, z)$. Then the following (i)–(iii) hold.*

*(i) If $C(z, x) = C(y, x)$, then $\Gamma$ is bipartite or antipodal.*

*(ii) $\Gamma$ is bipartite or antipodal if and only if $a_d = 0$.*

*(iii) If $a_d \neq 0$ then $k_d \leq \binom{k}{c_d}$.*

*Proof.* (i) Let $E$ be the principal idempotent associated with $\theta$ and let $x_1, \ldots, x_k$ be the neighbours of $x$. Assume $C(y, x) = C(z, x)$. In this case we have $\partial(x_i, y) = \partial(x_i, z)$ for each $i$, $1 \leq i \leq k$. Hence $\langle Ex_i, Ey\rangle = \langle Ex_i, Ez\rangle$ for each $i$, $1 \leq i \leq k$. But, by Theorem 3.2 and Lemma 3.1, the set $\{Ex_i \mid i = 1, \ldots, k\}$ is the basis of the eigenspace associated with $\theta$. Therefore, $Ey = Ez$ and so we obtain

$$1 = w_0 = \frac{|V\Gamma|}{k}\langle Ey, Ey\rangle = \frac{|V\Gamma|}{k}\langle Ey, Ez\rangle = w_{\partial(y,z)}.$$

Observe that $\theta \notin \{k, -k\}$. Hence, by Brouwer et al. [1, Prop 4.4.7], $\Gamma$ is either bipartite or antipodal.

(ii) If $\Gamma$ is bipartite or antipodal, then clearly $a_d = 0$. Suppose now $a_d = 0$. In this case we have $\partial(w, y) = \partial(w, z) = d - 1$ for every $w \in \Gamma(x)$. Hence $C(y, x) = C(z, x) = \Gamma(x)$ and, by (i), the graph $\Gamma$ is bipartite or antipodal.

(iii) Suppose $a_d \neq 0$. Then $\Gamma$ is neither antipodal nor bipartite. By (i), $C(y', x) \neq C(z', x)$ for every $y', z' \in \Gamma_d(x)$. Since $|C(y', x)| = c_d$ for every $y' \in \Gamma_d(x)$ and since there are $\binom{k}{c_d}$ subsets of $\Gamma(x)$ with $c_d$ elements, the statement follows. ∎

# 4 The 1-homogeneous property

Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 2$ and an eigenvalue multiplicity equal to its valency $k \geq 3$. In this section, we will focus our investigation on the 1-homogeneous property. However, in the case $d = 2$ the graph $\Gamma$ obviously has the 1-homogeneous, so we show that it also has the 2-homogeneous property. Then we consider the case $d \neq 2$.

Let $\Gamma$ be a distance-regular graph with diameter $d$ and let $x, y$ be its vertices. Let us repeat that for integers $i$ and $j$ we defined $D_i^j = D_i^j(x,y)$ by $D_i^j = \Gamma_i(x) \cap \Gamma_j(y)$. Suppose $x$ and $y$ are adjacent. Then $D_i^j = \emptyset$ unless $0 \leq i, j \leq d$ and $|i-j| \leq 1$. Moreover, $|D_i^j| = p_{ij}^1$ for $0 \leq i, j \leq d$ implies $D_{i-1}^i \neq \emptyset \neq D_i^{i-1}$ and $D_i^i = \emptyset$ if and only if $a_i = 0$ for $1 \leq i \leq d$.

**Lemma 4.1** (Jurišić et al. [8, Lem. 2.11]) *Let $\Gamma$ be a distance-regular graph with diameter $d$. Fix adjacent vertices $x, y \in V\Gamma$, and pick an integer $i \in \{1, \ldots, d\}$. Then the following (i) and (ii) hold, see Figure 4.1.*

*(i) Each $z \in D_{i-1}^i$ (resp. $D_i^{i-1}$) is adjacent to*
- *(a) precisely* $c_{i-1}$ *vertices in* $D_{i-2}^{i-1}$ *(resp.* $D_{i-1}^{i-2}$*),*
- *(b) precisely* $c_i - c_{i-1} - |\Gamma(z) \cap D_{i-1}^{i-1}|$ *vertices in* $D_i^{i-1}$ *(resp.* $D_{i-1}^i$*),*
- *(c) precisely* $a_{i-1} - |\Gamma(z) \cap D_{i-1}^{i-1}|$ *vertices in* $D_{i-1}^i$ *(resp.* $D_i^{i-1}$*),*
- *(d) precisely* $b_i$ *vertices in* $D_i^{i+1}$ *(resp.* $D_{i+1}^i$*),*
- *(e) precisely* $a_i - a_{i-1} + |\Gamma(z) \cap D_{i-1}^{i-1}|$ *vertices in* $D_i^i$*.*

*(ii) Each $z \in D_i^i$ is adjacent to*
- *(a) precisely* $c_i - |\Gamma(z) \cap D_{i-1}^{i-1}|$ *vertices in* $D_{i-1}^i$*,*
- *(b) precisely* $c_i - |\Gamma(z) \cap D_{i-1}^{i-1}|$ *vertices in* $D_i^{i-1}$*,*
- *(c) precisely* $b_i - |\Gamma(z) \cap D_{i+1}^{i+1}|$ *vertices in* $D_i^{i+1}$*,*
- *(d) precisely* $b_i - |\Gamma(z) \cap D_{i+1}^{i+1}|$ *vertices in* $D_{i+1}^i$*,*
- *(e) precisely* $a_i - b_i - c_i + |\Gamma(z) \cap D_{i-1}^{i-1}| + |\Gamma(z) \cap D_{i+1}^{i+1}|$ *vertices in* $D_i^i$*.* ∎

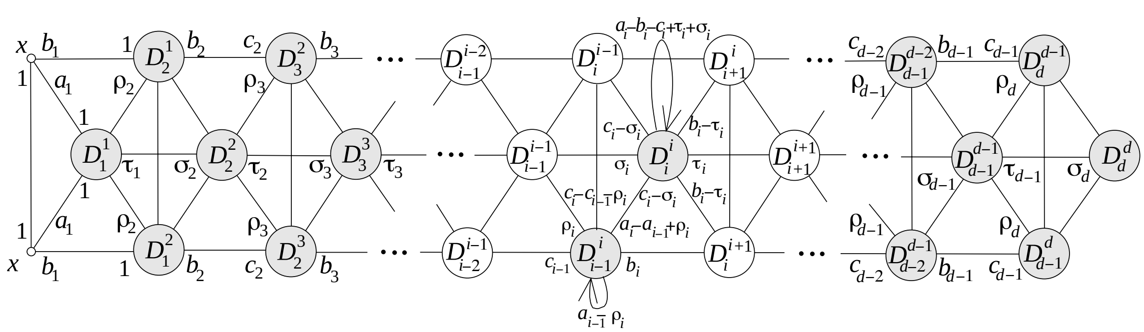



Figure 4.1: The distance partition of the vertex set $V\Gamma$ corresponding to an edge. For $i \in \{1, \ldots, d\}$ let $z \in D_i^i$ and $w \in D_{i-1}^i$. Then $\tau_i(z) = |\Gamma(z) \cap D_{i+1}^{i+1}|$, $\sigma_i(z) = |\Gamma(z) \cap D_{i-1}^{i-1}|$ and $\rho_i(w) = |\Gamma(w) \cap D_{i-1}^{i-1}|$.

## 4.1 The case $d = 2$

Let us suppose $\Gamma$ is a triangle-free distance-regular graph with diameter $d = 2$. i.e., a connected triangle-free strongly regular graph with parameters $(n, k, \lambda, \mu)$ where $\lambda = 0$. Then it is obviously 1-homogeneous. Let us now assume that $\Gamma$ also has an eigenvalue multiplicity equal to its valency $k$. Let $\theta_1$ and $\theta_2$ be the nontrivial eigenvalues of $\Gamma$ with $\theta_1 > \theta_2$ and multiplicities $m_1$ and $m_2$ respectively. We recall, see for example Brouwer et al. [1, Thm. 1.3.1], that $\mu = -(\theta_1 + \theta_2)$, $k = -(\theta_1 + \theta_2 + \theta_1\theta_2)$ and $m_1 = k(k - \theta_2)(\theta_2 + 1)/(\mu(\theta_2 - \theta_1))$, $m_2 = k(k - \theta_1)(\theta_1 + 1)/(\mu(\theta_1 - \theta_2))$. If $\Gamma$ is a conference graph, then $0 = \lambda = \mu - 1$ and $k = 2\mu$, see for example Brouwer et al. [1, Sec. 1.3]. Hence in this case $\mu = 1$, $k = 2$ and $\Gamma$ is isomorphic to the 5-cycle.

Let us now suppose $\Gamma$ is not a conference graph. Then $0 \le \theta_1 \in \mathbb{Z}$ and $-2 \ge \theta_2 \in \mathbb{Z}$, see Brouwer et al. [1, Thm. 1.3.1]. In the case $m_1 = k$, i.e., $\theta_1 = -\theta_2(\theta_2 + 2)$, we have $\theta_2 = -2$ and $\theta_1 = 0$, so $\Gamma$ is the 4-cycle. In the case $m_2 = k$, i.e., $(k - \theta_1)(\theta_1 + 1) = \mu(\theta_1 - \theta_2)$, we have $\theta_2 = -\theta_1(\theta_1 + 2)$ and $k = t(t^2 + 3t + 1)$, $\mu = t(t + 1)$, where $t = \theta_1 \in \mathbb{N}$. For $t = 1$ we get the folded 5-cube, and for $t = 2$ the Higman-Sims graph. If $t > 2$ then the existence of $\Gamma$ is still open, cf. Cameron and Van Lint [3, pp. 29].

Since $\Gamma$ is a generalized Odd graph, there exists its bipartite double $\overline{\Gamma}$. In the case $m_2 = k$ the multiplicity of the second largest eigenvalue of $\overline{\Gamma}$ is equal to its valency, so, by Yamazaki [13], $\overline{\Gamma}$ is 2-homogeneous. Therefore, also $\Gamma$ is 2-homogeneous. In particular, the second subconstituent graph of $\Gamma$ is a triangle-free strongly regular graph with parameters $k' = t^2(t+2)$ and $\mu' = t^2$. For $t = 1$ we get the Petersen graph, and for $t = 2$ we get a unique strongly regular graph with parameters $(77, 16, 0, 4)$, see Brouwer et al. [1, pp. 394].

## 4.2 The case $d \neq 2$

For the rest of this section, we assume that $\Gamma$ is a triangle-free distance-regular graph with diameter $d \ge 3$, valency $k \ge 3$ and $a_2 \neq 0$.

**Theorem 4.2** *Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \ge 3$, valency $k \ge 3$ and $a_2 \neq 0$. Then $a_2 \ge c_2$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity equal to $k$, let $w_0, w_1, \dots, w_d$ be the associated cosine sequence and let $x$ and $y$ be adjacent vertices of $\Gamma$. Then the following (i)–(iii) hold.*

*(i) For all $z \in D^2_2$*

$$|\Gamma(z) \cap D^2_3|\,(c_2 - k + c_2\theta + \theta^2)^2 + b_2^2\big(c_2(\theta+1)^2 - (k-1)(k+\theta)\big) = 0. \qquad (8)$$

*(ii) If $w_2 \neq w_3$ then the number $|\Gamma(z) \cap D^2_3|$ is independent of vertices $x$, $y$, $z$.*

*(iii) $a_2 = c_2$ if and only if $\Gamma$ is the dodecahedron.*

*Proof.* The set $D^2_2$ is nonempty by $a_2 \neq 0$, so let $z \in D^2_2$. Set $S^j_i = S^j_i(x, y, z) = \Gamma(z) \cap D^j_i$ for $i, j \in \{1, 2, 3\}$ and $s = |S^2_3|$. Observe $S^1_1 = \emptyset$, $|S^2_1| = |S^1_2| = c_2$, $|S^3_2| = s$, $|S^3_3| = b_2 - s$ and $|S^2_2| = a_2 - c_2 - s$ by Lemma 4.1. The last equality implies $a_2 \ge c_2$, and $a_2 = c_2$ implies $s = 0$. We postpone the treatment of the equality case until the end of this proof.

(i) Let $E$ be the principal idempotent associated with $\theta$. Then $\theta \neq 0$ by Theorem 3.2. Therefore, we have, by Lemma 3.3,

$$Ex = C\sum_{v\in S_2^1} Ev + A \sum_{v\in S_1^2\cup S_2^2\cup S_3^2} Ev + B\sum_{v\in S_2^3\cup S_3^3} Ev,$$

$$Ey = C\sum_{v\in S_1^2} Ev + A \sum_{v\in S_2^1\cup S_2^2\cup S_2^3} Ev + B\sum_{v\in S_3^2\cup S_3^3} Ev,$$

where $C = C_2$, $A = A_2$ and $B = B_2$ are as defined in Lemma 3.3. Then $\langle Ex, Ey\rangle|V\Gamma|/k = Uw_2+V$, where $V = 2c_2AC+(a_2-c_2-s)A^2+2sAB+(b_2-s)B^2$ and $U+V = (c_2C+a_2A+b_2B)^2$. By Lemma 2.1(i), we obtain $w_1 = Uw_2 + V$, i.e., $Uw_2 + V - w_1 = 0$, which, by Lemma 2.1(ii) and $\theta \neq k$, translates to (8).
(ii) The coefficient beside $s$ is nonzero if and only if $w_2 \neq w_3$, in which case $s$ is independent of choice of vertices $x$, $y$ and $z$.
(iii) It is clear that $a_2 = c_2$ in the dodecahedron. Let us now assume $a_2 = c_2$ and recall that in this case $s = 0$. By Brouwer et al. [1, Lem. 5.5.5], we have $a_2 = 1 = c_2$ and (8) implies $\theta^2 - \theta(k-3) - k^2 + k + 1 = 0$, i.e.,

$$\theta = \frac{k-3\pm(k-1)\sqrt{5}}{2}.$$

Since the above possible values for $\theta$ are conjugate algebraic numbers, they are both the eigenvalues of $\Gamma$. But if $k \geq 5$ then $(k-3+(k-1)\sqrt{5})/2 > k$, which is impossible. Hence $k = 3$ or $k = 4$. By Brouwer et al. [1, Thm. 7.5.1], the only distance-regular graph with valency 3 and with $c_2 = a_2 = 1$ is the dodecahedron. Similarly, by Brouwer and Koolen [2], we find that there is no triangle-free distance-regular graph with valency 4 and $c_2 = a_2 = 1$. ∎

Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and $a_2 \neq 0$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity equal to $k$, The above identity (8) implies that the eigenvalue $\theta$ is a zero of a polynomial of degree at most 4, therefore there are at most 4 eigenvalues with multiplicity $k$ and if $\Gamma(z)\cap D_3^2 = \emptyset$ for some $z \in D_2^2$, then there are at most 2. We believe the following conjecture is true.

**Conjecture 4.3** *Let $\Gamma$ be a primitive distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and $a_1 = 0$. Assume $\Gamma$ has a nonzero eigenvalue $\theta$ with multiplicity equal to $k$. Then $\Gamma$ is 1-homogeneous and $\theta \in \{\theta_1, \theta_d\}$.*

Let $\Gamma$ be an antipodal distance-regular graph with diameter 3 and eigenvalues $\theta_0 > \cdots > \theta_3$. Then $\theta_2 = -1$ has multiplicity $k$. There is an infinite family of such graphs that is triangle-free, see De Caen, Mathon and Moorhouse [4].

Given a nonbipartite distance-regular graph, there exist integers $h$ and $\ell$, such that $a_i \neq 0$ if and only if $i \in \{h, \ldots, \ell\}$, see Brouwer et al. [1, Prop. 5.5.7]. Moreover, $h + \ell \geq d$.

**Theorem 4.4** *Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$ and $a_2 \neq 0$. Let $\theta$ be an eigenvalue of $\Gamma$ with multiplicity equal to $k$, let $w_0, w_1, \dots, w_d$ be the associated cosine sequence, and let $x$ and $y$ be adjacent vertices of $\Gamma$. Let $\ell \geq 2$ be such an integer that $a_i \neq 0$ if and only if $i \in \{2, \dots, \ell\}$. Let $t \in D^\ell_\ell$ and if $\ell \geq 3$ also $u \in D^{i-1}_i \cup D^i_{i-1}$ for $i \in \{3, \dots, \ell\}$. If for $d \geq 4$ and $\ell \geq 3$, we have $w_2 \neq \cdots \neq w_{\min\{\ell, d-1\}}$, then the numbers*

$$|\Gamma(t) \cap D^{\ell-1}_{\ell-1}| \quad \textit{and} \quad \textit{if} \ \ \ell \geq 3 \ \ \textit{also} \quad |\Gamma(u) \cap D^{i-1}_{i-1}|$$

*are independent of vertices $x$, $y$, $t$ and $u$.*

*Proof.* If $w_{d-1} = w_d$, then, by Lemma 2.1 and $m_\theta = k \geq 2$, we have $c_d w_d + (k - c_d) w_d = \theta w_d$, i.e., $w_d = 0$, which implies a contradiction: $0 = w_d = w_{d-1} = \cdots = w_1 \neq 0$. Therefore, $w_{d-1} \neq w_d$.

Since we will now follow closely the approach of the first part of the proof of Theorem 4.2, we record only the essential steps. We set $S^j_i = S^j_i(x, y, t) = \Gamma(t) \cap D^j_i$ and have

$$\begin{aligned} Ex &= C_\ell \sum_{v \in S^\ell_{\ell-1} \cup S^{\ell-1}_{\ell-1}} Ev + A_\ell \sum_{v \in S^{\ell-1}_\ell \cup S^\ell_\ell \cup S^{\ell+1}_\ell} Ev + B_\ell \sum_{v \in S^\ell_{\ell+1}} Ev, \\ Ey &= C_\ell \sum_{v \in S^{\ell-1}_\ell \cup S^{\ell-1}_{\ell-1}} Ev + A_\ell \sum_{v \in S^\ell_{\ell-1} \cup S^\ell_\ell \cup S^\ell_{\ell+1}} Ev + B_\ell \sum_{v \in S^{\ell+1}_\ell} Ev. \end{aligned}$$

If $\ell = d$ then $S^\ell_{\ell+1} = \emptyset = S^{\ell+1}_\ell$ and recall that we assumed $b_d = 0$. We calculate the scalar product $\langle Ex, Ey \rangle$ as $(U w_2 + V w_0) m_\theta / |V\Gamma|$, where

$$V = 2(c_\ell - \sigma_\ell) A_\ell C_\ell + \sigma_\ell C_\ell^2 + (a_\ell - c_\ell + \sigma_\ell) A_\ell^2 + 2 b_\ell A_\ell B_\ell, \qquad U + V = (c_\ell C_\ell + a_\ell A_\ell + b_\ell B_\ell)^2$$

and notice that $U w_2 + V w_0 = w_1$ is a linear equation for $\sigma_\ell$, with a nonzero coefficient beside $\sigma_\ell$ if and only if $A_\ell \neq C_\ell$ if and only if $w_{\ell-1} \neq w_\ell$.

In the last case we set $S^j_{j'} = S^j_{j'}(x, y, u) = \Gamma(u) \cap D^j_{j'}$. We assume $u \in D^i_{i-1}$ (in the case $u \in D^{i-1}_i$ the proof is similar) and have

$$\begin{aligned} Ex &= C_i \sum_{v \in S^{i-1}_{i-2} \cup S^{i-1}_{i-1} \cup S^{i-1}_i} Ev + A_i \sum_{v \in S^i_{i-1} \cup S^i_i} Ev + B_i \sum_{v \in S^{i+1}_i} Ev, \\ Ey &= C_{i-1} \sum_{v \in S^{i-1}_{i-2}} Ev + A_{i-1} \sum_{v \in S^{i-1}_{i-1} \cup S^i_{i-1}} Ev + B_{i-1} \sum_{v \in S^{i-1}_i \cup S^i_i \cup S^{i+1}_i} Ev. \end{aligned}$$

If $i = d$ then $S^{i+1}_i = \emptyset$ (recall again that we assumed $b_d = 0$). Again we calculate the scalar product $\langle Ex, Ey \rangle$ as $(U w_2 + V w_0) m_\theta / |V\Gamma|$, where

$$\begin{aligned} V &= c_{i-1} C_{i-1} C_i + \rho_i A_{i-1} C_i + (c_i - c_{i-1} - \rho_i) B_{i-1} C_i \\ &\quad + (a_{i-1} - \rho_i) A_{i-1} A_i + (a_i - a_{i-1} + \rho_i) B_{i-1} A_i + b_i B_{i-1} B_i, \\ U + V &= (c_i C_i + a_i A_i + b_i B_i)(c_{i-1} C_{i-1} + a_{i-1} A_{i-1} + b_{i-1} B_{i-1}) \end{aligned}$$

and notice that $U w_2 + V w_0 = w_1$ is a linear equation for $\rho_i$, with a nonzero coefficient beside $\rho_i$ if and only if $C_i \neq A_i$ and $A_{i-1} \neq B_{i-1}$ if and only if $w_{i-1} \neq w_i$. ∎

**Corollary 4.5** *Let $\Gamma$ be a triangle-free distance-regular graph with diameter $d \geq 3$, valency $k \geq 3$, $a_2 \neq 0 \neq a_3$ and when $d \geq 4$ also $a_4 = 0$. Assume $\Gamma$ has an eigenvalue $\theta$ with multiplicity equal to $k$. Let $w_0, w_1, \ldots, w_d$ be the associated cosine sequence, and suppose $w_2 \neq w_3$. Then $\Gamma$ is 1-homogeneous.*

*Proof.* Fix adjacent vertices $x, y \in V\Gamma$ and pick $z \in D_2^2$, $t \in D_3^3$ and $u \in D_2^3 \cup D_3^2$ . By Theorems 4.2 and 4.4, the numbers $|\Gamma(z) \cap D_3^3|$, $|\Gamma(t) \cap D_2^2|$ and $|\Gamma(u) \cap D_2^2|$ are independent of the choice of $x, y, z, t, u$. If $d \geq 4$, then $a_4 = 0$ implies $a_i = 0$ for $4 \leq i \leq d$ by Brouwer et al. [1, Prop. 5.5.7]. The assertion now follows from Lemma 4.1. ∎

There is a unique distance-regular graph with intersection array $\{21, 20, 16; 1, 2, 12\}$, see Ivanov et al. [6], Brouwer et al. [1, Thm. 11.3.6], known as the coset graph of doubly truncated binary Golay code. It has spectrum $21^1, (-11)^{21}, 5^{210}, (-3)^{280}$, so it satisfies the assumption of the above statement. Beside the folded 7-cube this is the only known example of a graph with diameter 3 that satisfies the above result. Since these two graphs have classical parameters, they are $Q$-polynomial and they are 1-homogeneous also by [9, Cor. 5.4].

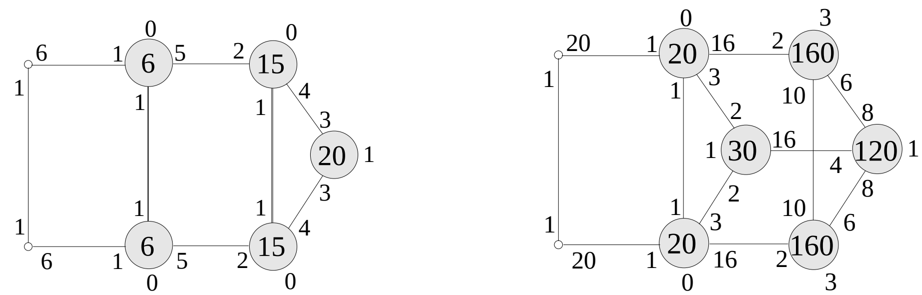


Figure 4.2: The distance partition corresponding to an edge of folded 7-cube and the coset graph of the doubly truncated binary Golay code. The second subconstituent of the second graph consists of 21 disjoint Petersen graphs, see Section 5.

# 5 An infinite family of diameter 5

There is an infinite family of feasible intersection arrays

$$\{2\mu^2 + \mu, 2\mu^2 + \mu - 1, \mu^2, \mu, 1; 1, \mu, \mu^2, 2\mu^2 + \mu - 1, 2\mu^2 + \mu\} \tag{9}$$

of distance-regular double-covers with diameter 5 and $4\mu^2(2\mu + 3)$ vertices. The antipodal quotients of the graphs with these parameters are strongly regular graphs with parameters $(2\mu^2(2\mu + 3), 2\mu^2 + \mu, 0, \mu)$, see Brouwer et al. [1, pp. 417]. Let $\Gamma$ be a distance-regular graph with the above intersection array. Then $\Gamma$ has $a_1 = a_4 = 0$ and $a_2 = a_3 = \mu^2 \neq 0$, so it is not $Q$-polynomial by [9, Thm. 6.3]. Its spectrum is

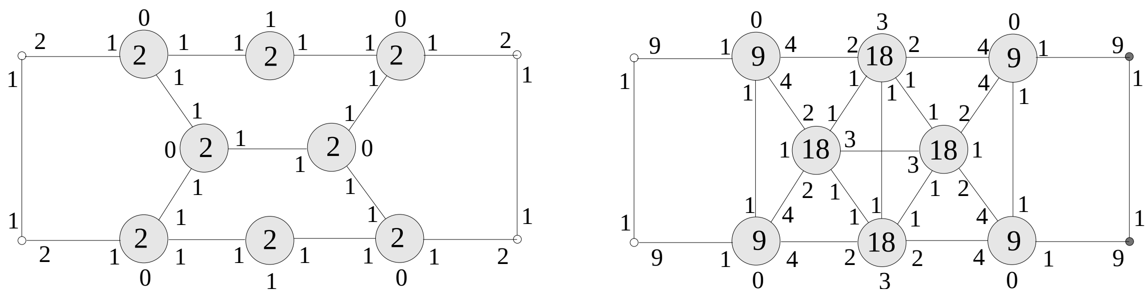
$$k^1, \;\; \theta^k, \;\; \mu^{(2\mu-1)(2\mu+1)(2\mu+3)/3}, \;\; 0^{2\mu(\mu+1)(2\mu-1)}, \;\; (-2\mu)^{2\mu(\mu+1)(2\mu+1)/3}, \;\; (-\theta)^k,$$

where $k = \mu(2\mu + 1)$ and $\theta = \mu\sqrt{2\mu + 3}$. The cosine sequence for the eigenvalue $\theta$ is

$$w_0 = 1, \;\; w_1 = \frac{\sqrt{2\mu + 3}}{2\mu + 1}, \;\; w_2 = \frac{1}{2\mu + 1}, \;\; w_3 = \frac{-1}{2\mu + 1}, \;\; w_4 = -\frac{\sqrt{2\mu + 3}}{2\mu + 1}, \;\; w_5 = -1.$$

(i) In the case $\mu = 1$ the graph $\Gamma$ is realized uniquely by the dodecahedron.

(ii) In the case $\mu = 2$ the existence of the graph $\Gamma$ was first ruled out in Brouwer et al. [1, Prop. 11.4.5], using the structure of the underlying antipodal quotient, namely the Gewirtz graph.

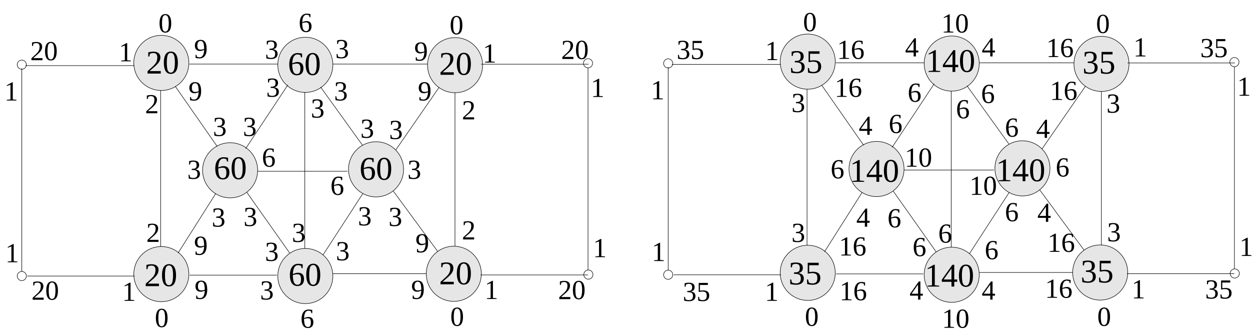


(a) (b)

Figure 5.1: The distance-partition corresponding to an edge is equitable: (a) the dodecahedron and (b) an antipodal 2-cover of the Gewirtz graph, whose nonexistence we show in Theorem 5.3.

(iii) In the case $\mu = 3$ the graph $\Gamma$ has parameters $\{21, 20, 9, 3, 1; 1, 3, 9, 20, 21\}$, $v = 1 + 21 + 140 + 140 + 21 + 1$ vertices, eigenvalues $21^1$, $9^{21}, 3^{105}, 0^{120}, -6^{56}, -9^{21}$ and the antipodal quotient is a strongly regular graph with parameters $(162, 21, 0, 3)$. In this case the existence of $\Gamma$ is still open.

(iv) In the case $\mu = 4$ the graph $\Gamma$ has parameters $\{36, 35, 16, 4, 1; 1, 4, 16, 35, 36\}$, $v = 1 + 36 + 315 + 315 + 36 + 1$ vertices, eigenvalues $36^1$, $(4\sqrt{7})^{36}$, $4^{231}, 0^{280}$, $-8^{120}$, $-(4\sqrt{7})^{36}$ and the antipodal quotient is a strongly regular graph with parameters $(325, 36, 0, 4)$. In this case the nonexistence of $\Gamma$ is shown in Theorem 5.3, despite the fact that we do not know the existence of its quotient.



(c) (d)

Figure 5.3: The distance partition corresponding to an edge of (c) the graph $\Gamma$ for $\mu = 3$, (d) the graph $\Gamma$ for $\mu = 4$.

The following result is an immediate consequence of Corollary 4.5, Theorems 4.2, 4.4 and Lemma 4.1.

**Corollary 5.1** *Let $\Gamma$ be a distance-regular graph with intersection array (9). Then $\Gamma$ is 1-homogeneous with parameters as on Figure 5.2(b), where $s = \mu(\mu - 1)/2$.* ∎

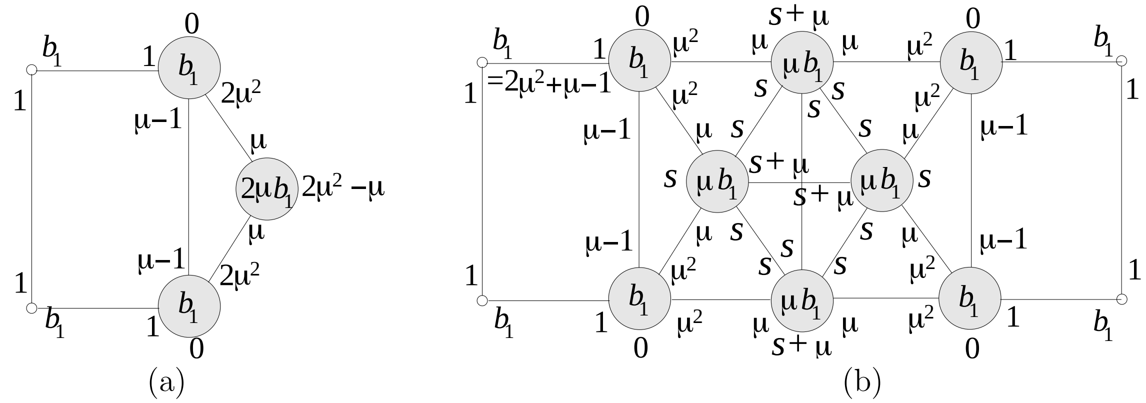



Figure 5.2: The distance partition corresponding to an edge of (a) the antipodal quotient of Γ and (b) the graph Γ that is a double-cover.

**Lemma 5.2** *Let Γ be a distance-regular graph with intersection array (9) and fix vertices* $x \in V\Gamma$ *and* $y \in \Gamma_2(x)$. *Then there exist a set of vertices* $\{v_1, \ldots, v_\mu\} \subset \Gamma_2(x) \cap \Gamma(y)$, *such that all the vertices of this set have a common neighbour in* $\Gamma(x)$.

*Proof.* Let us choose a vertex $w \in \Gamma(x) \cap \Gamma_2(y) \neq \emptyset$. Since $\partial(w, y) = 2$, the vertices $y$ and $w$ must have $\mu$ common neighbours. But Γ is triangle-free, so all of this common neighbours are in $\Gamma_2(x)$. Their common neighbour is the vertex $w$. ∎

**Theorem 5.3** *There are no distance-regular graphs with intersection arrays*

$$\{10, 9, 4, 2, 1; 1, 2, 4, 9, 10\} \quad \textit{and} \quad \{36, 35, 16, 4, 1; 1, 4, 16, 35, 36\}.$$

*Proof.* Let Γ be a distance-regular graph with intersection array (9) and assume $\mu > 1$. Let $x \in V\Gamma$, $y \in \Gamma_2(x)$ and let us choose vertices $v_1, v_2 \in \Gamma_2(x) \cap \Gamma(y)$, $v_1 \neq v_2$. Then $\partial(v_1, v_2) = 2$. Let $\gamma_j$ (resp. $\alpha_j$, $\beta_j$) be the number of vertices in $D^2_j(v_1, y) \cap \Gamma(x)$ at distance 1 (resp. 2, 3) from the vertex $v_2$. Then we have

$$\begin{aligned} \gamma_1 + \gamma_2 + \gamma_3 &= |S^2_1(v_2, y, x)| = \mu, & \alpha_1 + \alpha_2 + \alpha_3 &= |S^2_2(v_2, y, x)| = m, \\ \beta_1 + \beta_2 + \beta_3 &= |S^2_3(v_2, y, x)| = m, & \alpha_1 + \beta_1 + \gamma_1 &= |S^2_1(v_1, y, x)| = \mu, \\ \alpha_2 + \beta_2 + \gamma_2 &= |S^2_2(v_1, y, x)| = m, & \alpha_3 + \beta_3 + \gamma_3 &= |S^2_3(v_1, y, x)| = m. \end{aligned}$$

Finally, let $\omega$ be the number of vertices in $D^3_2(v_1, y)$, which are at distance 2 from the vertex $v_2$. The scalar product $\langle \overline{v}_1, \overline{v}_2 \rangle |V\Gamma|/k$ can be written as $Uw_2 + Vw_0$, where

$$V = A^2(\mu + \alpha_2 + \omega) + B^2(\beta_3 + \mu + \omega) + C^2\gamma_1 + AB(2m - 2\omega + \beta_2 + \alpha_3) + BC(\beta_1 + \gamma_3) + AC(\gamma_2 + \alpha_1),$$

$$U + V = (a_2 A + b_2 B + c_2 C)^2 = \big((2m + \mu)(A + B) + \mu C\big)^2,$$

$A = A_2$, $B = B_2$ and $C = C_2$. Then the relation $\langle \overline{v}_1, \overline{v}_2 \rangle = (k/|V\Gamma|)w_2$ is equivalent to

$$(\gamma_3 - \alpha_1)(\sqrt{2\mu + 3} + 1) = \mu(1 - \mu) + \gamma_1(\mu + 1) + 2(\alpha_2 + \omega). \tag{10}$$

Suppose $\sqrt{2\mu+3}$ is irrational. Then we have $\gamma_3 = \alpha_1$ and $\mu(\mu-1) = \gamma_1(\mu+1) + 2(\alpha_2 + \omega)$.

Suppose first $\mu = 2$, i.e., the graph $\Gamma$ has intersection array $\{10, 9, 4, 2, 1; 1, 2, 4, 9, 10\}$. Since $\sqrt{7}$ is irrational, we obtain from Equation (10)

$$2 = 3\gamma_1 + 2(\alpha_2 + \omega),$$

implying $\gamma_1 = 0$ for all pairs of distinct elements in $\Gamma_2(x) \cap \Gamma(y)$. But this is of course not possible by Lemma 5.2.

Assume now $\mu = 4$, i.e., $\Gamma$ has intersection array $\{36, 35, 16, 4, 1; 1, 4, 16, 35, 36\}$. Since $\sqrt{11}$ is irrational, we obtain from Equation (10)

$$12 = 5\gamma_1 + 2(\alpha_2 + \omega),$$

implying $\gamma_1$ is even. By Lemma 5.2, there exist vertices $v_1, v_2, v_3, v_4$ in $\Gamma_2(x) \cap \Gamma(y)$, which have a common neighbour $w \in \Gamma(x) \cap \Gamma_2(y)$. So $\gamma_1 \geq 1$ for all pairs of distinct elements in $\{v_1, v_2, v_3, v_4\}$. But since $\gamma_1$ is even, this implies $\omega \in \{0, 1\}$ for all pairs of distinct elements in $\{v_1, v_2, v_3, v_4\}$. Let $P := \Gamma(x) \cap \Gamma_3(y)$ and $P_i := P \cap \Gamma_2(v_i)$, $i \in \{1, 2, 3, 4\}$. Observe $|P| = 16$ and $|P_i| = 6$ for $i \in \{1, 2, 3, 4\}$. Furthermore, since $\omega \in \{0, 1\}$, we have $|P_i \cap P_j| \leq 1$ for $1 \leq i < j \leq 4$. This is a contradiction, because $4 \times 6 - \binom{4}{2} = 18 > 16$. ∎

**Remark 5.4** In the case $\mu = 3$ the condition (10) is equivalent to $9 = 4\gamma_1 + 2\gamma_2 + 2\alpha_1 + \alpha_2 + \omega$, so we have only one condition instead of two (in the irrational case).

We end the paper with the following conjecture.

**Conjecture 5.5** *A distance-regular graph with intersection array (9) exists if and only if* $\mu = 1$.

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