Wavelet frames from Butterworth filters *

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Abstract

The unitary extension principle(UEP) of Ron and Shen is applied to two parameter family of the Butterworth-type filters

$$\left(\frac{\cos^{2n}(w/2)}{\cos^{2n}(w/2)+\sin^{2n}(w/2)}\right)^K$$

to obtain multiwavelet frames. The parameter K determines the number of generators of the multiwavelet frames and n determines the smoothness of the generators of the multiwavelet frames.

The generators of the wavelet frames does not have compact support unlike the multiwavelet frames with B-spline generators, but they have an exponential decay. For fixed K we also show that the generators tends to the generator of the Shannon wavelet as n tends to infinity.

1 Introduction

The Unitary Extension Principle(UEP) of Ron and Shen [2] gives a very efficient way to construct the multiwavelet frames with finite generators. This is illustrated by the standard application of UEP for the construction of $B-$spline multiwavelet frames. See Chapter 14 of [1] for very well organized developments and other variations of the theory and examples. These multiwavelet frames are compactly supported but the number of generators must be increased to enhance the smoothness of the multiwavelet frames.

The fundamental idea of Ron and Shen is followed by further developments of the theory (Oblique Extension Principle, for example) for more applicable

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constructions of multiwavelet frames. See [7, 8], for example. For the history of the developments after UEP we refer to the introduction sections of [7, 8]. For an easy introduction of the multiwavelet frames we refer to Chapter 14 of [1]. We will mostly follow the expressions and notations therein.

In this paper, we are concerned only with another interesting examples of multiwavelet frames with rational filters. We apply UEP to the spectral decompositions of the Butterworth-type filters of the form

$$\left(\frac{\cos^{2n}(w/2)}{\cos^{2n}(w/2)+\sin^{2n}(w/2)}\right)^K$$

to construct the multiwavelet frames of K generators with smoothness increasing with Kn. For the complete study and applications of the Butterworth filters, we refer to [4, 6, 9]. The generators of the multiwavelet frames are not compactly supported but have the exponential decay to compensate for the lack of compact support. The major advantage of the so-called Butterworth multiwavelet frames is that we can enhance the smoothness of generators without increasing the number of generators. For the design purposes we have a useful flexibility we may have seen in Section 2. Finally, we also show that for K fixed, the multiwavelet frames for a proper sequence Butterworth filter system reduce to the Shannon wavelet as n tends to the infinity.

We recall the Unitary Extension Principle of Ron and Shen and state it in the following form for our convenience. We will mostly follow the notations in [1] except for $\mathbb{T}=[-\pi,\pi]$ here.

Theorem 1.1 (UEP) *Given $H_0 \in L^\infty(\mathbb{T})$ with $H_0(0)=1$ such that $\widehat{\psi}_0(w)=\prod_{j=1}^{\infty} H_0(w/2^j)$ converges to $L^2(\mathbb{R})$ function (For example, $H_0 \in C^1(\mathbb{T})$ will suffice), find $H_1, H_2, \cdots, H_n \in L^\infty(\mathbb{T})$ such that*

$$\sum_{l=0}^{n} |H_l(w)|^2 = 1, \tag{1.1}$$

$$\sum_{l=0}^{n} H_l(w)\overline{H_l(w+\pi)} = 0, \tag{1.2}$$

for a.e. $w \in \mathbb{T}$. Define

$$\widehat{\psi_l}(w) = H_l(w/2)\widehat{\psi_0}(w/2), \quad l = 1, \cdots, n.$$

Then, the family $\{D^j T_k \psi_l\}_{j,k\in Z, l=1,\cdots,n}$ constitutes a tight frame for $L^2(\mathbb{R})$ with frame bound equal to 1.

We will call such a system of filters as a *UEP filter system* and such a frame as a *multiwavelet frame*. A canonical example of UEP filter system is given by the filters

$$H_l(w) = \sqrt{\tbinom{2m}{l}} \cos^{2m-l}(w/2)\sin^l(w/2), \quad l = 0, 1, \cdots, 2m,$$

with corresponding B-spline multiwavelet frame. For the explicit expressions and the graphs for the case of $B-$spline multiwavelet frames, see Example 14.3.2 and Figures 14.1 and 14.2 of [1].

2 Wavelet frames from Butterworth filters

For the parameters $n \in \mathbb{N}$ and $K \in \mathbb{N}$, we consider the following identity:

$$\begin{aligned} 1 &= \left(\frac{\cos^{2n}(w/2) + \sin^{2n}(w/2)}{\cos^{2n}(w/2) + \sin^{2n}(w/2)}\right)^K \\ &= \sum_{l=0}^{K} \binom{K}{l} \left(\cos^{2n}(w/2)\right)^{K-l} \left(\sin^{2n}(w/2)\right)^l / \left(\cos^{2n}(w/2) + \sin^{2n}(w/2)\right)^K \end{aligned}$$

If we have a family of spectral decompositions $\{H_l^{n,K}(w)\}_{l=0,\cdots,K}$ of the summands above as in

$$|H_l^{n,K}(w)|^2 = \binom{K}{l} \left(\cos^{2n}(w/2)\right)^{K-l} \left(\sin^{2n}(w/2)\right)^l / \left(\cos^{2n}(w/2) + \sin^{2n}(w/2)\right)^K \tag{2.1}$$

then the filters obviously satisfy Equation (1.1). Furthermore, the decompositions can be chosen to satisfy Equation (1.2) in Theorem 1.1 as well. Therefore, the corresponding UEP filter system $\{H_l^{n,K}\}_{l=0,1,\cdots,K}$ will generate a multiwavelet system $\{D^j T_k \psi_l\}_{j,k\in Z, l=1,\cdots,n}$, where $\{\psi_l\}_{l=1,\cdots,K}$ is induced by the family $\{H_l^{n,K}(w)\}_{l=1,\cdots,K}$, as in Theorem 1.1. We will call this as Butterworth multiwavelet frame.

2.1 Spectral Decomposition

A special class of spectral decompositions can be obtained directly from (2.1). That is, writing $H_l^{n,K}(z) = H_l^{n,K}(w)$ interchangably with $z = e^{iw}$, we have

$$\begin{aligned} H_l^{n,K}(z) = H_l^{n,K}(w) &= \frac{\sqrt{\tbinom{K}{l}} \left(\cos^n(w/2)\right)^{K-l} \left(\sin^n(w/2)\right)^l}{\left(\cos^n(w/2) \pm i \sin^n(w/2)\right)^K} \\ &= i^{nl} \frac{\sqrt{\tbinom{K}{l}}(1+z)^{n(K-l)}(1-z)^{nl}}{\{(1+z)^n \pm i^{n+1}(1-z)^n\}^K}. \end{aligned}$$

From the choices of $+$ or $-$ for each factor of the denominator, we have $K+1$ different choices of spectral decompositions $H_l^{n,K}$ in this special class. For example, if we choose j times $+$ and $K-j$ times $-$ then we get a spectral decomposition of the form

$$H_l^{n,K,j}(z) := \frac{\sqrt{\binom{K}{l}}\left(\cos^n(w/2)\right)^{K-l}\left(\sin^n(w/2)\right)^l}{\left(\cos^n(w/2)+i\sin^n(w/2)\right)^j\left(\cos^n(w/2)-i\sin^n(w/2)\right)^{K-j}}.$$

The balanced choices of $+$ and $-$ for the factors of denominator will be given by taking $j=\left[\frac{K}{2}\right]$. That is,

$$H_l^{n,K,j}(z) = \begin{cases} \dfrac{\sqrt{\binom{K}{l}}\left(\cos^n(w/2)\right)^{K-l}\left(\sin^n(w/2)\right)^l}{\left(\cos^{2n}(w/2)+\sin^{2n}(w/2)\right)^k}, & K=2k, \\ \dfrac{\sqrt{\binom{K}{l}}\left(\cos^n(w/2)\right)^{K-l}\left(\sin^n(w/2)\right)^l}{\left(\cos^{2n}(w/2)+\sin^{2n}(w/2)\right)^k\left(\cos^n(w/2)\pm i\sin^n(w/2)\right)}, & K=2k+1. \end{cases}$$

An extreme choice of signs can be obtained by taking all $+$ signs or all $-$ signs. For example, by taking all $+$ signs, we get

$$\begin{aligned} H_l^{n,K,j}(z) &= \frac{\sqrt{\binom{K}{l}}\left(\cos^n(w/2)\right)^{K-l}\left(\sin^n(w/2)\right)^l}{\left(\cos^n(w/2)+i\sin^n(w/2)\right)^K} \\ &= \frac{\sqrt{\binom{K}{l}}(1+z)^{n(K-l)}(1-z^{-1})^{nl}}{\left((1+z)^n+i^{n+1}(1-z)^n\right)^{K/2}}\left(\frac{e^{iw}}{i}\right)^{nl}. \end{aligned}$$

Now, we consider the general form of the spectral decompositions. We write

$$\begin{aligned} H_l^{n,K}(z)\overline{H_l^{n,K}}(z^{-1}) &= |H_l^{n,K}(w)|^2 \\ &= \binom{K}{l}\left(\cos^{2n}(w/2)\right)^{K-l}\left(\sin^{2n}(w/2)\right)^l/\left(\cos^{2n}(w/2)+\sin^{2n}(w/2)\right)^K \\ &= \frac{\binom{K}{l}(1+z)^{n(K-l)}(1+z^{-1})^{n(K-l)}(1-z)^{nl}(1-z^{-1})^{nl}}{\{(1+z)^n(1+z^{-1})^n+(1-z)^n(1-z^{-1})^n\}^K} \\ &=: \frac{F(z)}{G(z)}. \end{aligned} \tag{2.2}$$

In order to get a spectral decomposition $H_l^{n,K}$ we consider the spectral decompositions of the numerator and denominator seperately as

$$F(z) = f(z)\overline{f}(z^{-1}), \tag{2.3}$$

$$G(z) = g(z)\overline{g}(z^{-1}). \tag{2.4}$$

For the spectral decomposition of the numerator $F(z) = f(z)\overline{f}(z^{-1})$, we have essentially only one choice as

$$f(z) = \sqrt{\binom{K}{l}}(1+z)^{n(K-l)}(1-z^{-1})^{nl}. \tag{2.5}$$

Now, we recall a well-known lemma by Riesz. A positive trigonometric polynomial $T(z) = T(w) = \sum_{k=0}^{N} \alpha_k \cos kw$ with $z = e^{iw}$ has a factorization of the form

$$T(z) = a_N \prod_{m=1}^{M} (z - r_m)(z^{-1} - r_m) \prod_{j=1}^{J} (z - z_j)(z^{-1} - z_j)(z - \overline{z}_j)(z^{-1} - \overline{z}_j),$$

where $N = M + 2J$, $a_N = \frac{\alpha_N}{2} / \prod_{m=1}^{M}(-r_m) \prod_{j=1}^{J} |z_j|^2$, and $\{r_m\}$ are real zeros and $\{z_j\}$ are complex zeros inside unit disc. According to the proof of Riesz lemma, a spectral decomposition can be obtained by choosing one factor out of $(z - r_m)$ and $z^{-1} - r_m$ for each m, by choosing one factor out of $z - z_j$ and $z^{-1} - \overline{z}_j$, and finally by choosing one factor out of $z^{-1} - z_j$ and $z - \overline{z}_j$ for each j. Therefore, we have 2^{M+2J} choices of factors for the spectral decompositions of $T(z)$. See [5, 10] .

For the spectral decompositions of the denominator $G(z) = g(z)\overline{g}(z^{-1})$, we consider the zeros of the positive trigonometric polynomial, with $z = e^{iw}$,

$$\begin{aligned} B(z) &= (1+z)^n(1+z^{-1})^n + (1-z)^n(1-z^{-1})^n \\ &= 2^{2n}\left(\cos^{2n}\frac{w}{2} + \sin^{2n}\frac{w}{2}\right). \end{aligned}$$

If n is even, the zeros of $B(z)$ are $z = z_j, \overline{z}_j, z_j^{-1}, \overline{z}_j^{-1}$, $j = 0, 1, \cdots, \frac{n}{2} - 1$, where

$$z_j = i \tan \frac{(2j+1)\pi}{4n}, \quad j = 0, \cdots, \frac{n}{2} - 1. \tag{2.6}$$

If n is odd, the zeros of $B(z)$ are $z = 0$ and $z = z_j, \overline{z}_j, z_j^{-1}, \overline{z}_j^{-1}$, $j = 1, \cdots, \frac{n-1}{2} - 1$, where

$$z_j = i \tan \frac{j\pi}{2n}, \quad j = 1, \cdots, \frac{n-1}{2}. \tag{2.7}$$

According to the choices of zeros of $B(z)$, we have 2^n(for n =even) or 2^{n-1}(for n =odd) choices of factors for the spectral decompositions of $B(z)$.

For a fixed K, we select one choice $b_k(z)$ of spectral decompositions of $B(z)$ up to a unimodular monomial multiple for each $k = 1, \cdots, K$. Then, we get a spectral decomposition of the denominator $G(z)$ of the form

$$g(z) = \prod_{k=1}^{K} b_k(z). \tag{2.8}$$

Therefore, we have $\binom{2^n+K-1}{K}$ or $\binom{2^{n-1}+K-1}{K}$ choices of the spectral decompositions of the denominator $G(z)$ depending on whether n is even or odd. Thus, a general form of the spectral decompositions of (2.1) or (2.2) can be expressed as

$$H_l^{n,K}(z) = \sqrt{\tbinom{K}{l}}(1+z)^{n(K-l)}(1-z^{-1})^{nl}/g(z), \tag{2.9}$$

for a choice g of the spectral decompositions of the denominator $G(z)$.

If we require a certain property (for example, symmetry or real-valued) of the generators ψ_l's we have to design the corresponding filters $H_l^{n,K}$'s with the proper property (for example, linear phase or real coefficients). This can be done by properly distributing the zeros of $G(z)$ for the appropriate decomposition $g(z)$. We refer to [4, 10] for the design process of the filters. For example, the maximum (or minimum) phase among the all possible spectral decompositions of the denominator $G(z)$, which will produce the most asymmetric generators ψ_l's, can be obtained by selecting always the zeros outside (or inside) the unit disk from each reciprocal pair of zeros of $G(z)$ for its spectral decomposition g(z). In order to generate the symmetric generators ψ_l's we have to take the spectral decomposition $g(z)$ of $G(z)$ by selecting each zero with its reciprocal zero to stay together as zeros of $g(z)$ for the filters $H_l^{n,K}$'s. Fortunately, This can always be possible in our spectral decompositions of Butterworth-type filters as an advantage for applications. Note that for compactly supported wavelets the symmetry and orthogonality conflict each other except for the simplest Haar wavelet. These designs will be illustrated in Section 3 with examples.

2.2 UEP filter systems and multiwavelet frames

In order to choose the spectral decompositions $H_l^{n,K}(z)$ so that they satisfy (1.2) as well, we consider the modified filters

$$\widetilde{H}_l^{n,K}(z) := \sqrt{\binom{K}{l}}\frac{(1+z)^{n(K-l)}(1-z^{-1})^{nl}}{g(z)}B_l(z) \tag{2.10}$$

where $B_l(z)$ is a unimodular monomial of z and $g(z)$ is a fixed choice of spectral decomposition of the denominator $G(z)$ as in Section 2.1. Obviously, with $z = e^{iw}$,

$$\sum_{l=0}^{K} |\widetilde{H}_l^{n,K}(w)|^2 = 1. \tag{2.11}$$

We will choose $B_l(z)$ so that the filters $\widetilde{H}_l^{n,K}$ also satisfy

$$\sum_{l=0}^{K} \widetilde{H}_l^{n,K}(w)\overline{\widetilde{H}_l^{n,K}(w+\pi)} = 0. \tag{2.12}$$

Substituting the equation (2.10) into the above equation (2.12), we have

$$\begin{aligned} 0 &= \sum_{l=0}^{K} \binom{K}{l} \frac{(1+z)^{n(K-l)}(1-z^{-1})^{nl}(1-z^{-1})^{n(K-l)}(1+z)^{nl}}{g(z)\overline{g(-z)}} B_l(z)\overline{B_l(-z)} \\ &= \sum_{l=0}^{K} \binom{K}{l} \frac{(1+z)^{nK}(1-z^{-1})^{nK}}{g(z)\overline{g(-z)}} B_l(z)\overline{B_l(-z)} \\ &= \frac{(1+z)^{nK}(1-z^{-1})^{nK}}{g(z)\overline{g(-z)}} \sum_{l=0}^{K} \binom{K}{l} B_l(z)\overline{B_l(-z)}. \end{aligned}$$

Thus, we have to choose the unimodular monomials $B_l(z)$'s so that

$$\sum_{l=0}^{K} \binom{K}{l} B_l(z)\overline{B_l(-z)} = 0. \tag{2.13}$$

This is satisfied by choosing $B_l(z) = \lambda_l z^l$ for arbitrary unimodular constant λ_l for $1 \leq \lambda \leq K$. We always choose λ_0 so that $\widetilde{H}_0^{n,K}(0) = 1$. That is,

$$\widetilde{H}_l^{n,K}(z) = \sqrt{\binom{K}{l}} \frac{(1+z)^{n(K-l)}(1-z^{-1})^{nl}}{g(z)} \lambda_l z^l, \quad l = 0, 1, \cdots, K. \tag{2.14}$$

The choice λ_l may be utilized to make the filters have real coefficients. By applying the UEP to $H_l^{n,K}$ we have the following theorem.

Theorem 2.1 *The filter system $\{\widetilde{H}_l^{n,K}\}_{l=0,\cdots,K}$ of (2.14) is a UEP filter system. That is, they satisfy the equations (1.1) and (1.2). Therefore, the K functions $\psi_1, \cdots, \psi_K$ defined by*

$$\widehat{\psi}_l(w) = \widetilde{H}_l^{n,K}\left(\frac{w}{2}\right)\widehat{\psi}_0\left(\frac{w}{2}\right), \tag{2.15}$$

where ψ_0 is defined as

$$\widehat{\psi}_0(w) = \prod_{j=0}^{\infty} \widetilde{H}_0^{n,K}(w/2^j),$$

generate a tight multiwavelet frame $\{D^i T_j \psi_l\}_{i,j \in Z, l=1,\cdots,k}$ for $L^2(\mathbb{R})$

We will call the wavelet frames as the *Butterworth wavelet frames* .

We now consider the special class of filters $H_l^{n,K,j}(z)$ in Section 2.1, and modify them to form a UEP system. Fix n, K and j. We set

$$\widetilde{H}_l^{n,K,j}(z) = H_l^{n,K,j}(z)B_l(z),$$

where $B_l(z)$ is a unimodular monomial of z. Again, $\widetilde{H}_l^{n,K,j}$'s satisfy (1.1). To choose $B_l(z)$ so that they also satisfy (1.2), we must have

$$0 = \sum_{l=0}^{K} \binom{K}{l} (-1)^{n(K-l)} B_l(z)\overline{B_l(-z)}.$$

Thus, we may choose, for $l = 0, 1, \cdots, K$,

$$\begin{cases} B_l(z) = \lambda_l, & n = \text{odd} \\ B_l(z) = \lambda_l z^l, & n = \text{even}, \end{cases} \tag{2.16}$$

for arbitrary unimodular constants λ_l. Again, we set $\lambda_0 = 1$ to have $\widetilde{H}_0^{n,K,j}(w) = 1$ at $w = 0$. Other λ_l's may be adjusted for the filters $\widetilde{H}_l^{n,K,j}$ have real coefficients.

Theorem 2.2 *The filters* $\{\widetilde{H}_l^{n,K,j}\}_{l=0,1,\cdots,K}$ *with the choice of* $B_l(z)$ *as in (2.16) form a UEP filter system. Therefore, the* K *functions* $\psi_1, \cdots, \psi_K$ *defined by*

$$\widehat{\psi}_l(w) = \widetilde{H}_l^{n,K,j}(w/2)\widehat{\psi}_0(w/2),$$

where ψ_0 *is defined as*

$$\widehat{\psi}_0(w) = \prod_{m=1}^{\infty} \widetilde{H}_0^{n,K,j}(w/2^m),$$

generate a tight multiwavelet frame $\{D^i T_j \psi_l\}_{i,j\in\mathbb{Z},\ l=1,\cdots,K}$ *for* $L^2(\mathbb{R})$.

We discuss more on the filters $\widetilde{H}_l^{n,K,j}$. For the purpose of applications, it is desirable to have real-valued symmetric multiwavelet frames. We will show that among the Butterworth multiwavelets which comes from the UEP filter system $\{\widetilde{H}_l^{n,K,j}\}$, the real-valued symmetric multiwavelets are only possible when K is even and $j = K/2$. First, we deal with the real-valuedness and symmetry separately. The real-valued multiwavelet frames come from the filters with real-coefficients.

Theorem 2.3 *The filters* $\widetilde{H}_0^{n,K,j}(z)$ *have real-coefficients iff either* $n =$ *odd, or* $n =$ *even,* $K =$*even and* $j = K/2$.

In this case, $\widetilde{H}_l^{n,K,j}(z)$*'s,* $l = 1, 2, \cdots, K$*, can be chosen to have real-coefficients by the appropriate selections of unimodular constants* λ_l*'s.*

Therefore, this appropriate choices of $B_l(z)$*'s in (2.16) will give rise to real-coefficient UEP filter system* $\widetilde{H}_l^{n,K,j}(z)$ *with the corresponding real-valued generators* ψ_l*'s.*

Proof. Note that the filter

$$\widetilde{H}_0^{n,K,j}(z) \quad = \quad \frac{(1+z)^{nK}}{\{(1+z)^n + i^{n+1}(1-z)^n\}^j \{(1+z)^n - i^{n+1}(1-z)^n\}^{K-j}}$$

has real-coefficients iff its denominator $D(z)$ has real-coefficients.

If n is odd, it clearly has real-coefficients. Suppose n is even and note $i^{n+1} = \pm i$. The denominator $D(z)$ has real coefficients iff $D(z) = \overline{D}(z)$; that is

$$\begin{aligned}&\{(1+z)^n + i^{n+1}(1-z)^n\}^j \{(1+z)^n - i^{n+1}(1-z)^n\}^{K-j} \\ &\quad = \{(1+z)^n - i^{n+1}(1-z)^n\}^j \{(1+z)^n + i^{n+1}(1-z)^n\}^{K-j}.\end{aligned}$$

Therefore,

$$\{(1+z)^n + i^{n+1}(1-z)^n\}^{K-2j} = \{(1+z)^n - i^{n+1}(1-z)^n\}^{K-2j}$$

which is possible only when $K - 2j = 0$. *i.e.*, only when K is even and $j = K/2$. □

The symmetry of the Butterworth multiwavelet frame comes from the symmetric UEP filters. A filter $H(z) = N(z)/D(z)$ with $\deg N = n$ and $\deg Q = d$ is symmetric if $z^n N(1/z) = N(z)$ and $z^d D(1/z) = D(z)$. To get a symmetric spectral decompositions, the zeros and poles z_j, $\frac{1}{z_j}$ must stay in the same decomposition. For the case of the filters $\widetilde{H}_l^{n,K,j}$ in the special class, we can prove

Theorem 2.4 *The filters* $\widetilde{H}_l^{n,K,j}(z)$ *are symmetric iff either n=even, or K=even and j=K/2.*

In this case, the UEP filter system $\widetilde{H}_l^{n,K,j}(z)$ *will give rise to symmetric generators from* ψ_l*'s.*

Proof. As in the proof of Theorem 2.3, $\widetilde{H}_l^{n,K,j}$ is real symmetric iff its denominator $D(z)$ is symmetric; *i.e.*, $z^{nK} D(z^{-1}) = D(z)$. Therefore,

$$\begin{aligned}&\{(1+z)^n + (-1)^n i^{n+1}(1-z)^n\}^j \{(1+z)^n - (-1)^n i^{n+1}(1-z)^n\}^{K-j} \\ &\quad = \{(1+z)^n + i^{n+1}(1-z)^n\}^j \{(1+z)^n - i^{n+1}(1-z)^n\}^{K-j}. \quad (2.17)\end{aligned}$$

If n is even, then (2.17) is obviously true. Assume n is odd and note $i^{n+1} = \pm 1$. As in the proof of Theorem 2.3 again, (2.17) is true iff $K = 2j$; *i.e.*, K is even and $j = K/2$. □

Finally,we classify the symmetric UEP filters $\widetilde{H}_l^{n,K,j}$ with real coefficients, which generates real-valued symmetric multiwavelet frames.

Theorem 2.5 *The filters $\widetilde{H}_0^{n,K,j}(z)$ with appropriate selection of λ_l's are symmetric with real coefficients iff K is even and $j = K/2$.*

Proof follows from Theorem 2.3 and 2.4.

The next Theorem shows that Butterworth multiwavelet frame does not have the classical MRA-wavelet structures. In fact, $V_1 = W_0$ where $W_0 = \overline{\text{span}}\{\psi_l(t-k) : k \in \mathbb{Z}, l = 1, \cdots, N\}$ and $V_1 = \overline{\text{span}}\{\psi_0(2t-k) : k \in \mathbb{Z}\}$. Thus we do not have a direct sum decomposition $V_1 = V_0 \dot{+} W_0$.

Theorem 2.6 *Let ψ_l's be as in Theorem 2.1. Then $V_1 = W_0$.*

Proof. According to [11], it is enough to check that the matrix

$$\begin{pmatrix} H_1^{n,K}(z) & \overline{H_1^{n,K}}(-z) \\ H_2^{n,K}(z) & \overline{H_2^{n,K}}(-z) \\ \vdots & \vdots \\ H_K^{n,K}(z) & \overline{H_K^{n,K}}(-z) \end{pmatrix}$$

has rank 2 for *a.e.* on $|z| = 1$. We can compute the first minor

$$\begin{aligned}
&\begin{vmatrix} H_1^{n,K}(z) & \overline{H_1^{n,K}}(-z) \\ H_2^{n,K}(z) & \overline{H_2^{n,K}}(-z) \end{vmatrix} \\
= &\begin{vmatrix} \sqrt{\binom{K}{1}}\dfrac{(1+z)^{n(K-1)}(1-z^{-1})^n}{g(z)}z & \sqrt{\binom{K}{1}}\dfrac{(1-z)^{n(K-1)}(1+z^{-1})^n}{\overline{g(-z)}}(-z) \\ \sqrt{\binom{K}{2}}\dfrac{(1+z)^{n(K-2)}(1-z^{-1})^{2n}}{g(z)}z^2 & \sqrt{\binom{K}{2}}\dfrac{(1-z)^{n(K-2)}(1+z^{-1})^{2n}}{\overline{g(-z)}}(-z)^2 \end{vmatrix} \\
= &\frac{\sqrt{\binom{K}{1}}\sqrt{\binom{K}{2}}}{g(z)\overline{g}(-z)}\begin{vmatrix} (1+z)^{n(K-1)}(1-z^{-1})^n z & (1-z)^{n(K-1)}(1+z^{-1})^n(-z) \\ (1+z)^{n(K-2)}(1-z^{-1})^{2n}z^2 & (1-z)^{n(K-2)}(1+z^{-1})^{2n}(-z)^2 \end{vmatrix} \\
= &\frac{\sqrt{\binom{K}{1}}\sqrt{\binom{K}{2}}(1-z^2)^{n(K-2)}(1-z^{-2})^n z^3((1+z)^n(1+z^{-1})^n + (1-z)^n(1-z^{-1})^n)}{g(z)\overline{g(-z)}} \\
= &\frac{\sqrt{\binom{K}{1}}\sqrt{\binom{K}{2}}(1-z^2)^{n(K-2)}(1-z^{-2})^n z^3}{\{(1+z)^n(1+z^{-1})^n + (1-z)^n(1-z^{-1})^n\}^{K-1}} \\
\neq &\ 0, \quad a.e. \text{ on } |z| = 1.
\end{aligned}$$

2.3 Smoothness of ψ_l

We estimate the smoothness of the refinable function ψ_0 defined by

$$\begin{aligned}\widehat{\psi_0}(w) &= \widetilde{H}_0^{n,K}(w/2)\widehat{\psi_0}(w/2) \\ &= \prod_{j=1}^{\infty} \widetilde{H}_0^{n,K}(w/2^j)\end{aligned}$$

with the normalization $\psi_0(0) = 1$. One way to estimate the smoothness is given in [3] as Theorem 5.5. Note that

$$\widetilde{H}_0^{n,K}(z) = \left(\frac{1+z}{2}\right)^{nK} \Big/ \frac{g(z)}{2^{nK}} =: \left(\frac{1+z}{2}\right)^{nK} S(z)$$

where g is a spectral decomposition of

$$\begin{aligned}|g(z)|^2 &= \left\{(1+z)^n(1+z^{-1})^n + (1-z)^n(1-z^{-1})^n\right\}^K \\ &= 2^{2nK}\left\{\cos^{2n}(w/2) + \sin^{2n}(w/2)\right\}^K .\end{aligned}$$

We estimate

$$\begin{aligned}B_1 &:= \sup_{|z|=1} |S(z)| = \sup_{0\le w\le 2\pi} \frac{2^{nK}}{|g(w)|} \\ &= \frac{1}{\inf_{0\le w\le 2\pi}\left(\cos^{2n}(w/2) + \sin^{2n}(w/2)\right)^{K/2}} \\ &= \frac{1}{\inf_{0\le y\le 1}(y^n + (1-y)^n)^{K/2}}, \quad \left(y = \cos^2(w/2)\right) \\ &= \frac{1}{\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^n\right]^{K/2}}, \quad (\text{at } y = 1/2) \\ &= 2^{(n-1)K/2}.\end{aligned}$$

By Theorem 5.5 in [3], we have

$$|\widehat{\psi_0}(w)| \le C(1+|w|)^{-\frac{K(n+1)}{2}}.$$

Thus, $\psi_0 \in C^{\frac{K(n+1)}{2}-1-\epsilon}$ for any $\epsilon > 0$. Since the coefficients of Laurent expansion of the rational filters $\widetilde{H}_l^{n,K}(z)$'s decay exponentially, all other ψ_l's have the same smoothness and decay as ψ_0.

We note that K determines the number of generators $\psi_1, \psi_2, \cdots, \psi_K$ and Kn determines their smoothness. For the case of $B-$spline multiwavelet frames in order to enhance the smoothness, the number of generators for the wavelet frames must be increased. As a merit of this Butterworth multiwavelet frames we can enhance the smoothness of the generators by increasing n with fixed number K of the generators. The disadvantage of Butterworth wavelet frames is the lack of compact support but the exponential decay of the generators may compensate for it.

3 Illustrations and Examples

We illustrate the spectral decompositions of the denominator for the simple case $K = 2$ and $n = 2$. We write $G(z) = B(z)^2$ where

$$\begin{aligned} B(z) &= (1+z)^2(1+z^{-1})^2 + (1-z)^2(1-z^{-1})^2 \\ &= 16\left(\cos^4\frac{w}{2} + \sin^4\frac{w}{2}\right). \end{aligned}$$

$B(z)$ has four zeros at $z_0 = i\tan\frac{\pi}{8}, \overline{z_0}, z_0^{-1}, \overline{z}_0^{-1}$. According to the choice of the zeros as explained in Section 2.1, $B(z)$ has four spectral decompositions as follows:

$$\begin{aligned} b_1(z) &:= (2+\sqrt{2})(z-z_0)(z-\overline{z}_0) = (2+\sqrt{2})(z^2+\tan^2\frac{\pi}{8}) \\ &= \sqrt{2}((\sqrt{2}+1)z^2+\sqrt{2}-1), \\ b_2(z) &:= (2+\sqrt{2})(z-z_0)(z^{-1}-z_0) = (2+\sqrt{2})(1-\tan^2\frac{\pi}{8} - i\tan\frac{\pi}{8}(z+z^{-1})) \\ &= \sqrt{2}(2-i(z+z^{-1})), \\ b_3(z) &:= (2+\sqrt{2})(z^{-1}-\overline{z}_0)(z-\overline{z}_0) = (2+\sqrt{2})(1-\tan^2\frac{\pi}{8} + i\tan\frac{\pi}{8}(z+z^{-1})) \\ &= \sqrt{2}(2+i(z+z^{-1})), \\ b_4(z) &:= (2+\sqrt{2})(z^{-1}-\overline{z}_0)(z^{-1}-z_0) = (2+\sqrt{2})(z^{-2}+\tan^2\frac{\pi}{8}) \\ &= \sqrt{2}((\sqrt{2}+1)z^{-2}+\sqrt{2}-1). \end{aligned}$$

Thus, we have the $4 = 2^2$ spectral decompositions according to the choice of the factors of $B(z)$. Then the spectral decompositions $g(z)$ of the $G(z)$ has the following forms:

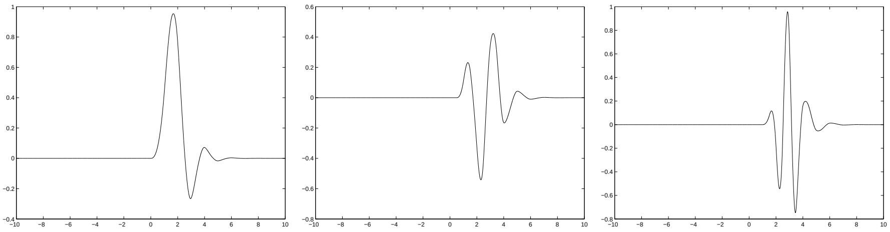
$$
\begin{aligned}
g_1(z) &:= b_1(z)b_1(z) = (6+4\sqrt{2})z^4 + 4z^2 + 6 - 4\sqrt{2}, \\
g_2(z) &:= b_4(z)b_4(z) = (6+4\sqrt{2})z^{-4} + 4z^{-2} + 6 - 4\sqrt{2}, \\
g_3(z) &:= b_2(z)b_2(z) = 8 - 8i(z+z^{-1}) - 2(z+z^{-1})^2, \\
g_4(z) &:= b_3(z)b_3(z) = 8 + 8i(z+z^{-1}) - 2(z+z^{-1})^2, \\
g_5(z) &:= b_1(z)b_2(z), \\
&= -i(2\sqrt{2}+2)z^3 + (4\sqrt{2}+4)z^2 - i4\sqrt{2}z + 4\sqrt{2} - 4 - i(2\sqrt{2}-2)z^{-1}, \\
g_6(z) &:= b_3(z)b_4(z), \\
&= i(2\sqrt{2}+2)z^{-3} + (4\sqrt{2}+4)z^{-2} + i4\sqrt{2}z^{-1} + 4\sqrt{2} - 4 + i(2\sqrt{2}-2)z, \\
g_7(z) &:= b_1(z)b_3(z), \\
&= i(2\sqrt{2}+2)z^3 + (4\sqrt{2}+4)z^2 + i4\sqrt{2}z + 4\sqrt{2} - 4 + i(2\sqrt{2}-2)z^{-1}, \\
g_8(z) &:= b_2(z)b_4(z), \\
&= -i(2\sqrt{2}+2)z^{-3} + (4\sqrt{2}+4)z^{-2} - i4\sqrt{2}z^{-1} + 4\sqrt{2} - 4 - i(2\sqrt{2}-2)z, \\
g_9(z) &:= b_1(z)b_4(z) = 2(6+z^2+z^{-2}) = (1+z)^2(1+z^{-1})^2 + (1-z)^2(1-z^{-1})^2, \\
g_{10}(z) &:= b_2(z)b_3(z) = 2(6+z^2+z^{-2}) = 16\left(\cos^4\frac{w}{2} + \sin^4\frac{w}{2}\right).
\end{aligned}
$$

Thus, we have the $\binom{2^2+2-1}{2} = 10$ spectral decompositions $g(z)$ of the $G(z)$. Note that $g_{2l}(z) = \overline{g}_{2l-1}(z^{-1})$ and $g_9 = g_{10}$ is positive; so that $g_{2l}(z)g_{2l-1}(z) = G(z)$. Therefore, we have essentially $5 = 10/2$ spectral decompositions.

The filters with complex coefficient generate complex valued functions. Among the spectral factorizations above, only g_1, g_2 and $g_9 = g_{10}$ will give rise to the filter $H_l^{2,2}(z)$ with real coefficients which will give rise to real-valued wavelet frames. From the choice of zeros, the filters $H_l^{2,2}(z)$ which come from the choice $g = g_3, g_4$ or $g_9 = g_{10}$ will generate the symmetric wavelet frames and the filters $H_l^{2,2}(z)$ from the choice $g = g_1$ or g_2 will give rise to the most asymmetric wavelet frames as explained at the end of Section 2.1. See [5]. The factorization $g_9 = g_{10}$ gives rise to the filter

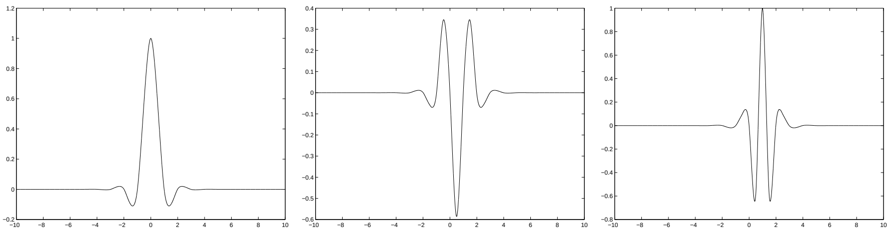
$$
\begin{aligned}
H_l^{2,2}(z) &= \frac{\sqrt{\binom{2}{l}}(1+z)^{4-2l}(1-z^{-1})^{2l}}{(1+z)^2(1+z^{-1})^2 + (1-z)^2(1-z^{-1})^2} z^l \\
&= \frac{\sqrt{\binom{2}{l}}(\cos^{2-l}\frac{w}{2}\sin^l\frac{w}{2})}{\cos^4\frac{w}{2} + \sin^4\frac{w}{2}} e^{i2w}
\end{aligned}
$$

which generates the symmetric real-valued wavelet frames. The generators of



(a) (b) (c)

Figure 1: The generators by taking $g = g_1$ are real-valued but nonsymmetric (a) ψ_0 (b) ψ_1 (c) ψ_2



(a) (b) (c)

Figure 2: The generators by taking $g = g_9$ are real-valued and symmetric (a) ψ_0 (b) ψ_1 (c) ψ_2

multiwavelet frame coming from the choice of the spectral decompositions g of the denominator are illustrated by the Figures 1, 2 and 3.

We now give more examples to illustrate Theorem 2.3, 2.4 and 2.5. Figure 4 shows the real-valued generators for the case $n = 3$(odd). The case $K = 2$(even) and $j = K/2$ gives symmetric generators. Figure 5 shows the generators which are symmetric but complex-valued. Finally, Figure 6 shows the real-valued symmetric generators for the case $n = 4$(even).

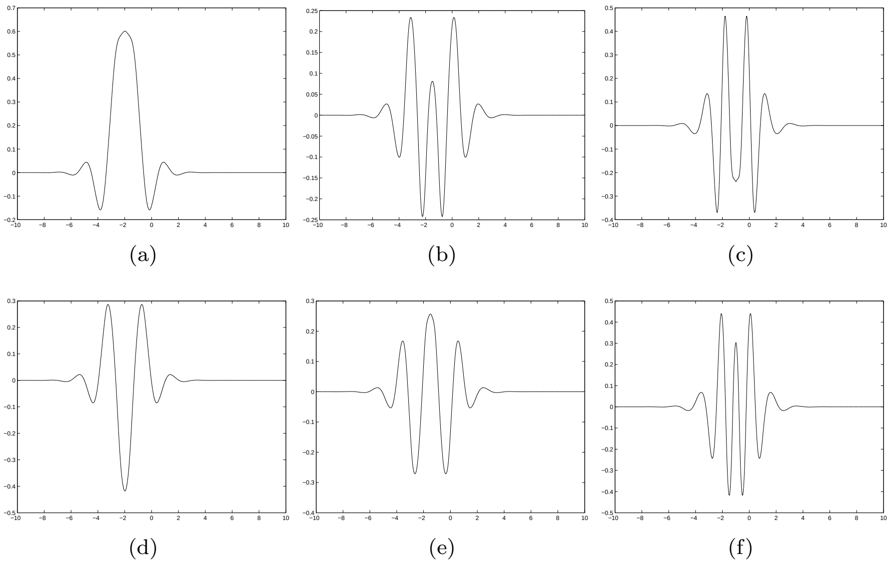


Figure 3: The generators by taking $g = g_3$ are symmetric but complex-valued. (a) Re ψ_0 (b) Re ψ_1 (c) Re ψ_2. (d) Im ψ_0 (e) Im ψ_1 (f) Im ψ_2.

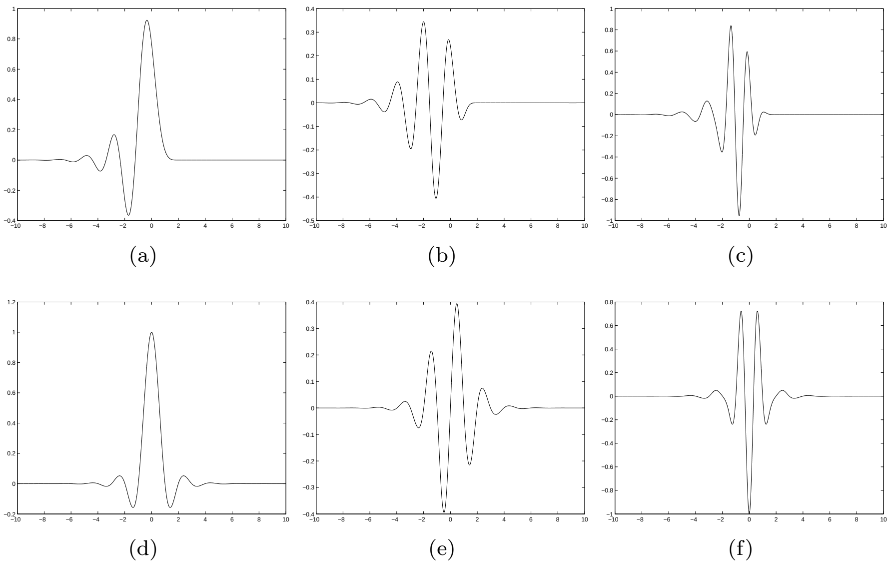


Figure 4: Top : The real-valued generators obtained by the filter $\widetilde{H}_l^{3,2,0}$ (a)ψ_0 (b) ψ_1 (c) ψ_2; Bottom : The real-valued symmetric generators obtained by the filter $\widetilde{H}_l^{3,2,1}$ (d) ψ_0 (e) ψ_1 (f) ψ_2.

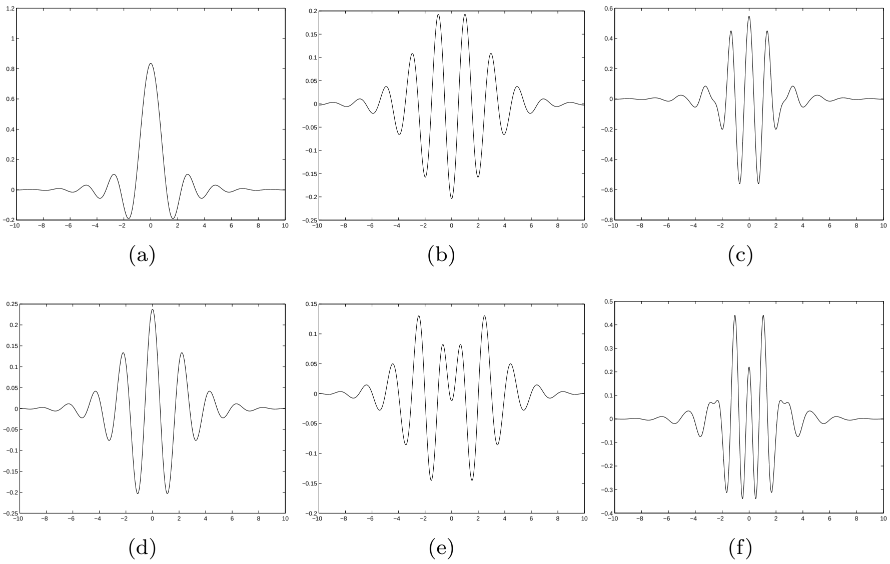


Figure 5: The symmetric but complex-valued generators obtained by the filter $\widetilde{H}_l^{4,2,0}$. Top: (a) Re ψ_0 (b) Re ψ_1 (c) Re ψ_2. Bottom : the corresponding imaginary parts.

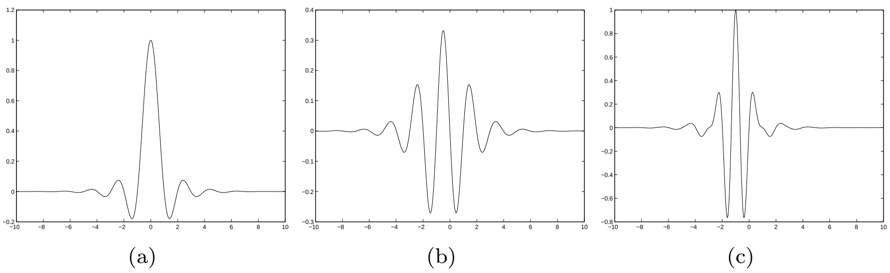


Figure 6: The real-valued symmetric generators obtained by the filter $\widetilde{H}_1^{4,2,1}$ (a) ψ_0 (b) ψ_1 (c) ψ_2

4 Limits

In this section, we consider the limit of the Butterworth multiwavelet frames as n tends to infinity. In particular, we show the Butterworth multiwavelet frames arising from the Butterworth UEP filter system $\{\widetilde{H}_l^{n,K,j}\}_{l=0,1,\cdots,K}$, where

$$\widetilde{H}_l^{n,K,j}(w) = \frac{\sqrt{\binom{K}{l}}(\tan^n \frac{w}{2})^l}{(1+i\tan^n \frac{w}{2})^j(1-i\tan^n \frac{w}{2})^{K-j}} i^l e^{ilw},$$

tends to the classical Shannon wavelet as n tends to infinity.

We define as usual.

$$\widehat{\psi}_0^{n,K,j}(w) = \prod_{m=1}^{\infty} \widetilde{H}_0^{n,K,j}(\frac{w}{2^m}),$$
$$\widehat{\psi}_l^{n,K,j}(w) = \widetilde{H}_l^{n,K,j}(\frac{w}{2})\widehat{\psi}_0^{n,K,j}(\frac{w}{2}),\ l = 1, 2, \cdots, K.$$

The filters defined by

$$H_{SH}^L(w) := \begin{cases} 1, & |w| < \pi/2; \\ 0, & \pi/2 < |w| < \pi, \end{cases}$$
$$H_{SH}^H(w) := \begin{cases} 0, & |w| < \pi/2; \\ 1, & \pi/2 < |w| < \pi, \end{cases}$$

are the low/high-pass filters for the Shannon scaling function φ_{SH} and Shannon wavelet ψ_{SH}, respectively. Note that $\widehat{\varphi}_{SH}(w) = \chi_{[-\pi,\pi]}(w)$ and $\widehat{\psi}_{SH}(w) = \chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}(w)$. As n goes to ∞, we easily check that

$$H_l^{n,K,j}(w) \to \begin{cases} H_{SH}^L(w), & \text{if } l = 0, \\ (-1)^{K-j}e^{iKw}H_{SH}^H(w), & \text{if } l = K, \\ 0, & \text{otherwise.} \end{cases}$$

Precisely, we will show that, as n approaches ∞,

$$\psi_l^{n,K,j} \text{ converges to } \Psi_l := \begin{cases} \varphi_{SH}, & \text{if } l = 0, \\ (-1)^{K-j}e^{iKw}\psi_{SH}, & \text{if } l = K, \\ 0, & \text{otherwise,} \end{cases}$$

uniformly on $\mathbb{R}$ as well as in $L^p(\mathbb{R})$, for $p \geq 2$. The ideas of the proof also appears in [9]. We define

$$H(w) = \begin{cases} 1, & |w| \leq \frac{\pi}{2}, \\ \dfrac{(\cos^6 \frac{w}{2})^{K/2}}{(\cos^6 \frac{w}{2} + \sin^6 \frac{w}{2})^{K/2}}, & \frac{\pi}{2} < |w| \leq \pi \end{cases}$$

for the domination of $\widetilde{H}_l^{n,K,j}$ in the following Lemma.

Lemma 4.1 (a) $|\widetilde{H}_0^{n,K,j}(w)| \le H(w),\ n = 3, 4\cdots.$
(b) $H(w) = \cos^{3K}\frac{w}{2}S(w)$, *and* $\sup_w |S(w)| = 2^{3K/2}$, *where*

$$S(w) = \begin{cases} \dfrac{1}{\cos^{3K}\frac{w}{2}}, & |w| \le \frac{\pi}{2}, \\ \dfrac{1}{(\cos^6\frac{w}{2} + \sin^6\frac{w}{2})^{K/2}}, & \frac{\pi}{2} < |w| \le \pi. \end{cases}$$

Therefore, $\widehat{\varphi}(w) := \prod_{m=1}^{\infty} H(w/2^m)$ *has the decay* $|\widehat{\varphi}(w)| \le C(1+|w|)^{-3/2}$.
(c) $|\widetilde{H}_0^{n,K,j}(w) - 1| \le \begin{cases} 2, & \text{for all } w, \\ \dfrac{2^{K+2}}{\pi}|w|, & |w| \le \pi/2. \end{cases}$

Proof. (a) and (b) are trivial.
For (c), we note that

$$|\widetilde{H}_0^{n,K,j}(w) - 1| \le |\widetilde{H}_0^{n,K,j}(w)| + 1 = \frac{1}{(1+\tan^{2n}\frac{w}{2})^{K/2}} + 1 \le 2.$$

For $|w| \le \pi/2$ and for $l \ge 1$, $(\tan^n \dfrac{|w|}{2})^l \le \tan\dfrac{|w|}{2} \le \dfrac{2}{\pi}|w| \le 1$. Therefore, we have for $|w| \le \pi/2$,

$$\begin{aligned}
|\widetilde{H}_0^{n,K,j}(w) - 1| &= \left|\frac{(1+i\tan^n\frac{w}{2})^j(1-i\tan^n\frac{w}{2})^{K-j} - 1}{(1+i\tan^n\frac{w}{2})^j(1-i\tan^n\frac{w}{2})^{K-j}}\right| \\
&= \left|(1+i\tan^n\frac{w}{2})^j(1-i\tan^n\frac{w}{2})^{K-j} - 1\right| \Big/ (1+\tan^{2n}\frac{w}{2})^{K/2} \\
&\le \left|(1+i\tan^n\frac{w}{2})^j(1-i\tan^n\frac{w}{2})^{K-j} - 1\right| \\
&= \left|\sum_{p=0}^{j}\binom{j}{p}(i\tan^n\frac{w}{2})^p \sum_{q=0}^{K-j}\binom{K-j}{q}(-i\tan^n\frac{w}{2})^q - 1\right| \\
&= \Bigg|\sum_{p=1}^{j}\binom{j}{p}(i\tan^n\frac{w}{2})^p \sum_{q=0}^{K-j}\binom{K-j}{q}(-i\tan^n\frac{w}{2})^q \\
&\quad + \sum_{q=0}^{K-j}\binom{K-j}{q}(-i\tan^n\frac{w}{2})^q - 1\Bigg| \\
&\le \sum_{p=1}^{j}\binom{j}{p}|\tan^n\frac{w}{2}|^p \sum_{q=0}^{K-j}\binom{K-j}{q}|\tan^n\frac{w}{2}|^q \\
&\quad + \sum_{q=1}^{K-j}\binom{K-j}{q}|\tan^n\frac{w}{2}|^q
\end{aligned}$$

$$\leq \sum_{p=1}^{j} \binom{j}{p} \sum_{q=0}^{K-j} \binom{K-j}{q} \frac{2}{\pi}|w| + \sum_{q=1}^{K-j} \binom{K-j}{q} \frac{2}{\pi}|w|$$
$$\leq (2^j 2^{K-j} + 2^{K-j})\frac{2}{\pi}|w|$$
$$\leq \frac{2^{K+2}}{\pi}|w|.$$

□

Lemma 4.2 (a) *For each fixed* w, $\widehat{\psi}_0^{n,K,j}(w) = \prod_{m=1}^{\infty} \widetilde{H}_0^{n,K,j}(w/2^m)$ *converges uniformly on* n.
(b) $\widehat{\psi}_0^{n,K,j}(w) \to \widehat{\varphi}_{SH}(w)$ *pointwise as* $n \to \infty$.

Proof. (a) Fix w and choose m_0 so that $|\frac{w}{2^{m_0}}| \leq \pi/2$. By Lemma 4.1(c),

$$\sum_{m=1}^{\infty} |\widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) - 1| = \sum_{m=1}^{m_0} |\widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) - 1| + \sum_{m=m_0+1}^{\infty} |\widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) - 1|$$
$$\leq 2m_0 + \sum_{m=m_0+1}^{\infty} \frac{2^{k+2}}{\pi}\frac{|w|}{2^j} = 2m_0 + \frac{2^{k+2}}{\pi}\frac{|w|}{2^{m_0}},$$

uniformly on n. Therefore, the product $\psi_0^{n,K,j}(w)$ converges uniformly on n.
(b) Fix w. Given $\epsilon > 0$, by (a) we can choose m_1 (independent of n!) so that

$$|\widehat{\psi}_0^{n,K,j}(w) - \prod_{m=1}^{m_1} \widetilde{H}_0^{n,K,j}(\frac{w}{2^m})| < \epsilon,$$

and

$$|\widehat{\varphi}_{SH}(w) - \prod_{m=1}^{m_0} H_{SH}(\frac{w}{2^m})| < \epsilon.$$

Therefore, we have

$$|\widehat{\psi}_0^{n,K,j}(w) - \widehat{\varphi}_{SH}(w)| \leq |\widehat{\psi}_0^{n,K,j}(w) - \prod_{m=1}^{m_1} \widetilde{H}_0^{n,K,j}(\frac{w}{2^m})|$$
$$+ |\prod_{m=1}^{m_1} \widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) - \prod_{m=1}^{m_1} H_{SH}(\frac{w}{2^m})|$$
$$+ |\prod_{m=1}^{m_1} H_{SH}(\frac{w}{2^m}) - \widehat{\varphi}_{SH}(w)|$$
$$< 2\epsilon + |\prod_{m=1}^{m_1} \widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) - \prod_{m=1}^{m_1} H_{SH}(\frac{w}{2^m})|.$$

Since $\widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) \to H_{SH}(\frac{w}{2^m})$ as $n \to \infty$ for $m = 1, 2, \cdots, m_1$, we can choose n_0 so that

$$|\prod_{m=1}^{m_1} \widetilde{H}_0^{n,K,j}(\frac{w}{2^m}) - \prod_{m=1}^{m_1} H_{SH}(\frac{w}{2^m})| < \epsilon \quad \text{for } n \geq n_0.$$

Therefore, $\widehat{\psi}_0^{n,K,j}(w) \to \widehat{\varphi}_{SH}(w)$ pointwise as $n \to \infty$. □

Now, we can prove our main result in this section.

Theorem 4.3 (a) *For $p \geq 1$ and for $l = 0, 1, \cdots, K$,*

$$||\widehat{\psi}_l^{n,K,j} - \widehat{\Psi}_l||_{L^p(\mathbb{R})} \to 0 \ (n \to \infty).$$

(b) *For $q \geq 2$ and for $l = 0, 1, \cdots, K$, $||\psi_l^{n,K,j} - \Psi_l||_{L^q(\mathbb{R})} \to 0 \ (n \to \infty)$. In particular, $\psi_l^{n,K,j} \to \Psi_l$ uniformly on $\mathbb{R}$.*

Proof. Note that

$$\begin{aligned} |\widehat{\psi}_0^{n,K,j}(w)| &= \prod_{m=1}^{\infty} |\widetilde{H}_0^{n,K,j}(\frac{w}{2^m})| \\ &\leq \prod_{m=1}^{\infty} |H(\frac{w}{2^m})| = |\widehat{\varphi}(w)| \leq C(1 + |w|)^{-3/2} \in L^p, \quad \text{for } p \geq 1. \end{aligned}$$

Since $|\widetilde{H}_l^{n,K,j}(w)| \leq 1$ for $l = 0, 1, 2, \cdots, K$, we have

$$\begin{aligned} |\widehat{\psi}_l^{n,K,j}(w)| &\leq |\widetilde{H}_l^{n,K,j}(\frac{w}{2})||\widehat{\psi}_0^{n,K,j}(\frac{w}{2})| \\ &\leq |\widehat{\varphi}(\frac{w}{2})| \leq C(1 + |\frac{w}{2}|)^{-3/2} \in L^p, \quad \text{for } p \geq 1. \end{aligned}$$

Note that $||f||_{L^q(\mathbb{R})} \leq ||\widehat{f}||_{L^p(\mathbb{R})}$, for $1 \leq p \leq 2$, where q is the conjugate exponent to p. Therefore (a) and (b) follow by the dominated convergence theorem.

Remark 4.4 *We do not know whether Theorem 4.3 is true for the general Butterworth wavelet frame corresponding to the UEP filter $\widetilde{H}_l^{n,K}$ in Theorem 2.1.*

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