CONSTRUCTION OF DIFFERENTIAL OPERATORS HAVING BOCHNER-KRALL ORTHOGONAL POLYNOMIALS AS EIGENFUNCTIONS

K. H. KWON, L. L. LITTLEJOHN, AND G. J. YOON

ABSTRACT. Suppose $\{Q_n\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to the moment functional

$$\tau = \sigma + \nu$$

where σ is a classical moment functional (Jacobi, Laguerre, Hermite) and ν is a point mass distribution with finite support. In this paper, we develop a new method for constructing a differential equation having $\{Q_n\}_{n=0}^{\infty}$ as eigenfunctions.

1. INTRODUCTION

The class of polynomial sequences $\{P_n\}_{n=0}^{\infty}$ that are solutions of real differential equations of the form

$$\sum_{j=0}^{N} a_j(x)y^{(j)}(x) = \lambda y(x) \tag{1.1}$$

where $N \in \mathbb{N}$ and λ is a spectral (eigenvalue) parameter, and are orthogonal with respect to a bilinear form of the type

$$(p, q)_\mu = \int_{\mathbb{R}} pq d\mu, \tag{1.2}$$

where μ is a (possibly signed) Borel measure on the real line $\mathbb{R}$, is the so-called class of Bochner-Krall orthogonal polynomials. In this case, we write $\{P_n\}_{n=0}^{\infty} \in BKS(N)$ and call $\{P_n\}_{n=0}^{\infty}$ a Bochner-Krall sequence of order $\leq N$. It is known that $BKS(2N-1) = \varnothing$ and $BKS(2N) \neq \varnothing$ for each $N \in \mathbb{N}$; however, for even integers, only the classes $BKS(2)$ and $BKS(4)$ are specifically known up to a real or complex linear change of variable. The BKS classification problem of determining the classes $BKS(2N)$ for each $N \in \mathbb{N}$ is of interest and importance in, for example, the area of spectral theory of differential operators since examples from these classes generate unbounded self-adjoint operators. For a general historical account of this classification problem, see [7] and [8].

For later purposes, we list the contents of the classical set $BKS(2)$ up to a real linear change of variable. The determination of $BKS(2)$, up to a complex linear change of variable, can be traced back to work of E. J. Routh [34] in 1885 and later to S. Bochner [4] in 1929 and P. Lesky [28] in 1962. A full account of this classification, in both a real and a complex linear change of variable, can be found in [21].

(C-i) $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$, the Jacobi polynomials, where $-\alpha, -\beta, -(\alpha+\beta+1) \notin \mathbb{N}$. For each $n \in \mathbb{N}_0$, $y = P_n^{(\alpha,\beta)}(x)$ is a solution of the Jacobi differential equation

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y + n(n+\alpha+\beta+1)y = 0.$$

1991 *Mathematics Subject Classification.* 33C45, 34A99.

Key words and phrases. Differential equations, Bochner-Krall orthogonal polynomials, Magnus' conjecture.

KHK is supported by KOSEF(R-01-1999-0000) and GJY is supported by KRF (2002-070-C00004).

Let $\sigma_J^{(\alpha,\beta)}$ denote the canonical orthogonalizing moment functional for these polynomials.

(C-ii) $\{L_n^\alpha\}_{n=0}^\infty$, the Laguerre polynomials, where $-\alpha \notin \mathbb{N}$. For each $n \in \mathbb{N}_0$, $y = L_n^\alpha(x)$ is a solution of the Laguerre differential equation

$$xy'' + (1+\alpha-x)y' + ny = 0.$$

Let σ_L^α denote the canonical orthogonalizing moment functional for these polynomials.

(C-iii) $\{H_n\}_{n=0}^\infty$, the Hermite polynomials. For each $n \in \mathbb{N}_0$, $y = H_n(x)$ is a solution of the Hermite differential equation

$$y'' - 2xy' + 2ny = 0.$$

Let σ_H denote the canonical orthogonalizing moment functional for these polynomials.

(C-iv) $\{y_n^a\}_{n=0}^\infty$, the Bessel polynomials, where $-(a+1) \notin \mathbb{N}$. For each $n \in \mathbb{N}_0$, $y = y_n^a(x)$ is a solution of the Bessel differential equation

$$x^2y'' + ((a+2)x+2)y' - n(n+a+1)y = 0.$$

Let σ_B^a denote the canonical orthogonalizing moment functional for these polynomials.

(C-v) $\{\check{P}_n^{(\alpha,\beta)}\}_{n=0}^\infty$, the twisted Jacobi polynomials, where $-(\alpha+\beta+1) \notin \mathbb{N}$ and $\overline{\alpha} = \beta$. For each $n \in \mathbb{N}_0$, $y = \check{P}_n^{(\alpha,\beta)}(x)$ is a solution of the twisted Jacobi differential equation

$$(1+x^2)y'' + ((\alpha+\beta+2)x + i(\alpha-\beta))y' - n(n+\alpha+\beta+1)y = 0.$$

Let $\sigma_{\check{J}}^{(\alpha,\beta)}$ denote the canonical orthogonalizing moment functional for these polynomials.

(C-vi) $\{\check{H}_n\}_{n=0}^\infty$, the twisted Hermite polynomials. For each $n \in \mathbb{N}_0$, $y = \check{H}_n(x)$ is a solution of the twisted Hermite differential equation

$$y'' + 2xy' - 2ny = 0.$$

Let $\sigma_{\check{H}}$ denote the canonical orthogonalizing moment functional for these polynomials.

Together, the moment functionals listed in (C-i) - (C-vi) are called the *classical moment functionals.* For properties of the polynomials listed in (i)-(iv), see [5] and [36]. For a discussion of the twisted polynomials in (C-v) and (C-vi), see [21] where these polynomials are first introduced.

There are several conjectures in the mathematical literature pertaining to the BKS classification problem. One of these is *Magnus' conjecture* [32] (see also Conjectures 4.3 and 5.3 in [7] and Conjecture 7.1 in [8]) which states that any Bochner-Krall sequence necessarily is orthogonal with respect to a moment functional that is the sum of a classical moment functional plus one or two point masses located at the finite end point(s) of the interval of orthogonality. More specifically, this conjecture asserts that if $\{P_n\}_{n=0}^\infty$ is orthogonal with respect to the bilinear form (1.2) and are solutions of the differential equation (1.1), then

$$(p,q)_\mu = <\sigma, pq> + <\nu, pq>$$

where σ is a classical moment functional and

$$\nu = \sum_{k=0}^{m(a)} c_{k,1}\delta^{(k)}(x-a) + \sum_{k=0}^{m(b)} c_{k,2}\delta^{(k)}(x-b) \tag{1.3}$$

for some non-negative integers $m(a)$ and $m(b)$. In (1.3), we assume that $c_{k,1} = 0$ (respectively, $c_{k,2} = 0$) if $a = -\infty$ (respectively, $b = \infty$). To this extent, the authors in [23] prove, among other results, the following theorem concerning Bochner-Krall sequences of polynomials that are orthogonal with respect to the moment functional

$$\tau = \sigma + \nu, \tag{1.4}$$

where

$$\nu := \sum_{k=1}^{m} \sum_{j=0}^{m_k} c_{k,j} \delta^{(j)}(x - x_k); \tag{1.5}$$

that is to say, the perturbation moment functional ν is a distribution of finite order and finite support.

Theorem 1.1. [23, Theorem 4.2] *Suppose $\tau = \sigma + \nu$ is a quasi-definite moment functional, where σ is a classical moment functional that satisfies the moment equation*

$$(A(x)\sigma)' = B(x)\sigma, \tag{1.6}$$

where $0 \leq \deg(A) \leq 2$ and $\deg(B) = 1$ and where ν is a non-trivial real moment functional defined in (1.5). Suppose $\{Q_n\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to τ and, for each $n \in \mathbb{N}_0$, $y = Q_n(x)$, is a solution of the real differential equation

$$\sum_{j=0}^{2N} a_j(x) y^{(j)}(x) = \lambda y(x) \tag{1.7}$$

for some choice, say $\lambda = \lambda_n$, of the eigenvalue parameter number (see the Remark following this theorem). Then

(i) *$supp(\nu) \subset \{x \in \mathbb{C} \mid A(x) = 0\}$ so that $m \in \{1, 2\}$;*
(ii) *$A(x)$ divides $a_{2N}(x)$;*
(iii) *if $x_0 \in supp(\nu)$ is a zero of order $q \geq 1$ of $a_{2N}(x)$, then x_0 is a zero of order $q - 1$ of $a_{2N-1}(x)$;*
(iv) *the moment functional σ must be either the Jacobi moment functional $\sigma_J^{(\alpha,\beta)}$ or the Laguerre moment functional $\sigma_L^{(\alpha)}$ or the twisted Jacobi moment functional $\sigma_{\tilde{J}}^{(\alpha,\beta)}$; furthermore,*
   (a) *if $\sigma = \sigma_J^{(\alpha,\beta)}$ then either -1 or 1 (or both) belongs to the support of ν. If $1 \in supp(\nu)$ (respectively, $-1 \in supp(\nu)$), then α (respectively, β) must be a non-negative integer; moreover, in this case, the moment functional τ necessarily has the form*

$$\tau = \sigma_J^{(\alpha,\beta)} + \sum_{j=0}^{m(-1)} c_{j,-1} \delta^{(j)}(x+1) + \sum_{j=0}^{m(1)} c_{j,1} \delta^{(j)}(x-1),$$

   where $m(-1)$ and $m(1)$ are non-negative integers. Furthermore, if $-1 \notin supp(\nu)$ (respectively, $1 \notin supp(\nu)$), then $c_{j,-1} = 0$ for $j = 0, \ldots, m(-1)$ (respectively, $c_{j,1} = 0$ for $j = 0, \ldots, m(1)$); otherwise, we have

$$\sum_{j=0}^{m(-1)} |c_{j,-1}| + \sum_{j=0}^{m(1)} |c_{j,1}| \neq 0.$$

   (b) *if $\sigma = \sigma_L^{(\alpha)}$, then $0 \in supp(\nu)$ and α must be a non-negative integer; moreover, in this case, the moment functional τ necessarily has the form*

$$\tau = \sigma_L^{(\alpha)} + \sum_{j=0}^{m(0)} c_{j,0} \delta^{(j)}(x),$$

   where $m(0)$ is a non-negative integer and where $\sum_{j=0}^{m(0)} |c_{j,0}| \neq 0$.

(c) *if* $\sigma = \sigma_{\check{J}}^{(\alpha,\beta)}$, *then* i *and* $-i \in supp(\nu)$, *where* $i = \sqrt{-1}$; *moreover, in this case, the moment functional* τ *necessarily has the form*

$$\tau = \sigma_{\check{J}}^{(\alpha,\beta)} + \sum_{j=0}^{m(i)} c_{j,i}\delta^{(j)}(x+i) + \sum_{j=0}^{m(i)} \overline{c}_{j,i}\delta^{(j)}(x-i),$$

where $m(i)$ *is a non-negative integer and where* $\sum_{j=0}^{m(i)} |c_{j,i}| \neq 0$.

Remark 1.1. *We note that if* $\{Q_n(x)\}_{n=0}^{\infty}$ *is an orthogonal polynomial sequence satisfying* (1.7), *it is well known that necessarily each coefficient* $a_i(x)$ *is a polynomial of degree* $\leq i$. *Writing*

$$a_i(x) = \sum_{j=0}^{i} \ell_{i,j}x^j \quad (0 \leq i \leq 2N),$$

it is also well known that the value of the eigenvalue parameter $\lambda = \lambda_n$, *corresponding to the eigenfunction* $Q_n(x)$, *is necessarily given by*

$$\lambda_n = \ell_{0,0} + n\ell_{1,1} + n(n-1)\ell_{2,2} + \ldots + n(n-1)\cdots(n-2N+1)\ell_{2N,2N} \quad (n \in \mathbb{N}_0).$$

In [23], the authors refine Theorem 1.1 in the case when the perturbation moment functional ν has order zero (that is, each $m_k = 0$ in (1.5)). Before stating this result, we note that the Koornwinder-Jacobi polynomials $\{P_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1,1]$ with respect to the weight distribution

$$w_{\alpha,\beta,c_1,c_2}(x) = (1-x)^{\alpha}(1+x)^{\beta} + M\delta(x-1) + N\delta(x+1)$$

while the Koornwinder-Laguerre polynomials $\{L_n^{\alpha,A}\}_{n=0}^{\infty}$ are orthogonal on the interval $[0,\infty)$ with respect to the weight distribution

$$w_{\alpha,A}(x) = x^{\alpha}e^{-x} + A\delta(x).$$

For further information on these two Koornwinder polynomial sequences, see the contribution [14] where these polynomials are introduced and studied.

Theorem 1.2. [23, Theorem 4.9] *Let* $\{Q_n\}_{n=0}^{\infty}$ *be an orthogonal polynomial sequence with respect to the moment functional*

$$\tau = \sigma + c_1\delta(x-a) + c_2\delta(x-b). \tag{1.8}$$

Then $\{Q_n\}_{n=0}^{\infty} \in BKS(2N)$ *for some* $N \in \mathbb{N}_0$ *if and only if*

(a) *when* $c_1 = c_2 = 0$, *then* σ, *subject to the parameter restrictions listed in* (*C-i*) - (*C-vi*), *is one of the classic moment functionals:*
  (i) *the Jacobi moment functional* $\sigma_J^{(\alpha,\beta)}$
  (ii) *the Laguerre moment functional* $\sigma_L^{(\sigma)}$
  (iii) *the Hermite moment functional* σ_H
  (iv) *the Bessel moment functional* $\sigma_B^{(\alpha,\beta)}$
  (v) *the twisted Jacobi moment functional* $\sigma_{\check{J}}^{(\alpha,\beta)}$
  (vi) *the twisted Hermite moment functional* $\sigma_{\check{H}}$.

(b) *when* $|c_1| + |c_2| \neq 0$, *then* σ *is either the Jacobi moment functional* $\sigma_J^{(\alpha,\beta)}$, *or the Laguerre moment functional* $\sigma_L^{(\alpha)}$, *or the twisted Jacobi moment functional* $\sigma_{\check{J}}^{(\alpha,\beta)}$, *subject to the following constraints:*

(i) *if* $\sigma = \sigma_J^{(\alpha,\beta)}$*, then* $a = -1, b = +1$ *and if* $c_1 \neq 0$ (*respectively,* $c_2 \neq 0$)*, then* $\alpha \in \mathbb{N}_0$ (*respectively,* $\beta \in \mathbb{N}_0$)*. In other words, the polynomials* $\{Q_n\}_{n=0}^{\infty}$ *are Koornwinder-Jacobi polynomials.*

(ii) *if* $\sigma = \sigma_L^{(\sigma)}$*, then* $a = 0$*,* $b = \infty$*,* $c_2 = 0$*, and* $\alpha \in \mathbb{N}_0$*. In other words, the polynomials* $\{Q_n\}_{n=0}^{\infty}$ *are Koornwinder-Laguerre polynomials.*

(iii) *if* $\sigma = \sigma_{\tilde{J}}^{(\alpha,\beta)}$*, then* $\alpha = \beta \in \mathbb{N}_0,$*,* $a = -i$*,* $b = +i$*, and* $c_1 = \overline{c}_2$*. In this case, we call the orthogonal polynomials* $\{Q_n\}_{n=0}^{\infty}$ *twisted Jacobi type polynomials.*

We note that, prior to the publication of [23], J. and R. Koekoek (see [11] and [12]) explicitly computed differential equations for $\{Q_n\}_{n=0}^{\infty}$ in cases (b) (i) and (b) (ii) above. They use a clever combination of properties of certain special functions together with the symbolic program MAPLE and an important new technique called the *inversion formula*, due to Bavinck and Koekoek [2], to compute these differential equations; see also [1] and [13] for further details.

The significance of Theorem 1.2 is that the authors in [23] prove that the *only* polynomials in the Bochner-Krall class that are orthogonal with respect to a moment functional of the form (1.8) are the ones listed in Theorem 1.2.

There are other effective methods, discovered earlier, that have been used to compute differential equations for polynomials in the Bochner-Krall class with respect to weights of the form (1.8). For example, H. L. Krall devised a method, based on the classical Green's formula in differential equations, to compute differential equations; this method was used by Shore [35] and later by Littlejohn and Shore [30] to compute some of the first higher-order differential equations for various Koornwinder-Jacobi and Koornwinder-Laguerre polynomials; this method is known in the literature as *Shore's technique*. Another method was developed by Littlejohn (see [29]) using the weight or symmetry equations to find the differential equation for polynomials orthogonal with respect to a known weight function. This technique relied on the *assumption* that the differential equation could be made Lagrangian symmetric. Later, Kwon and Yoon, in [26], showed in fact that *every* differential equation in the Bochner-Krall class is Lagrangian symmetrizable.

The purpose of this paper is to introduce a new, and effective, way of constructing differential equations in the case when the perturbation ν has *positive* order. This new method is based on earlier results of Kwon and Yoon in [27] and is closely connected to the theory of the Darboux transformation. Significantly less calculations are needed with this method compared to the other techniques described above. After we develop this new method, we consider several examples where this technique is applied.

The contents of this paper are as follows. In Section Two we discuss, and review, several general facts about the calculus of moment functionals and their connections to the Bochner-Krall classification problem. Section Three consists of a brief discussion of the results in [27] and details our new construction technique. Following the results in this section, we discuss three examples in detail to illustrate this new method.

2. Preliminaries

Let $\mathcal{P}$ be the space of all real polynomials in one variable and denote the degree of a polynomial $\pi(x)$ by $deg(\pi)$ with the convention that $deg(0) = -1$. By a polynomial system(PS), we mean a sequence of polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ with $deg(\phi_n) = n$, $n \in \mathbb{N}_0$, the set of non-negative integers.

We call any linear functional σ on $\mathcal{P}$ a *moment functional* and denote its action on a polynomial $\pi(x)$ by $\langle \sigma, \pi \rangle$. For a moment functional σ, we call

$$\sigma_n := \langle \sigma, x^n \rangle, \quad n \in \mathbb{N}_0$$

the *n-th moment* of σ. We say that a moment functional σ is *quasi-definite* (respectively, *positive-definite*) if its moments $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the Hamburger condition

$$\Delta_n(\sigma) := \det[\sigma_{i+j}]_{i,j=0}^{n} \neq 0 \quad (\text{respectively}, \Delta_n(\sigma) > 0) \tag{2.1}$$

for every $n \in \mathbb{N}_0$. Any PS $\{\phi_n(x)\}_{n=0}^{\infty}$ determines a moment functional σ (uniquely up to a non-zero constant multiple), called a *canonical moment functional* of $\{\phi_n(x)\}_{n=0}^{\infty}$, by the conditions

$$\langle \sigma, \phi_0 \rangle \neq 0 \quad \text{and} \quad \langle \sigma, \phi_n \rangle = 0, \quad n \geq 1.$$

Definition 2.1. *A PS $\{P_n(x)\}_{n=0}^{\infty}$ is called an orthogonal polynomial system (OPS) (respectively, a positive-definite OPS) if there is a moment functional σ satisfying*

$$\langle \sigma, P_m P_n \rangle = K_n \delta_{mn}, \qquad (m, n \in \mathbb{N}_0),$$

where $\{K_n\}_{n=0}^{\infty}$ are non-zero (respectively, positive) constants and δ_{mn} is the Kronecker delta function. In this case, we say that $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ and call σ an orthogonalizing moment functional of $\{P_n(x)\}_{n=0}^{\infty}$.

Due to the representation theorems for the moment problem by Boas [3] and Duran [6], any moment functional σ has an integral representation of the form

$$\langle \sigma, \pi \rangle = \int_{-\infty}^{\infty} \pi(x) d\mu(x) = \int_{-\infty}^{\infty} \pi(x)\phi(x) dx \quad (\pi \in \mathcal{P}),$$

where μ is a finite, signed Borel measure on $\mathbb{R}$ and $\phi(x)$ is a smooth, rapidly decaying function in the Schwartz space $\mathcal{S}(\mathbb{R})$. Hence, for any *OPS* $\{P_n(x)\}_{n=0}^{\infty}$, there is a distribution $w(x)$ relative to which $\{P_n(x)\}_{n=0}^{\infty}$ is orthogonal. In this case, we call $w(x)$ a *distributional orthogonalizing weight* for $\{P_n(x)\}_{n=0}^{\infty}$.

For a moment functional σ, a polynomial $\pi(x)$, we let σ' and $\pi\sigma$ be the moment functionals defined by

$$\langle \sigma', \phi \rangle = -\langle \sigma, \phi' \rangle \quad \text{and} \quad \langle \pi\sigma, \phi \rangle = \langle \sigma, \pi\phi \rangle \quad \text{for } \phi \in \mathcal{P}.$$

The following results are immediate consequences of these definitions.

Lemma 2.1. ([19] *and* [22]) *Let σ and τ be moment functionals. Then for a polynomial $\pi(x)$ and a real number λ, we have*
(i) Leibniz' rule: $(\pi(x)\sigma)' = \pi'(x)\sigma + \pi(x)\sigma'$;
(ii) $\sigma' = 0$ if and only if $\sigma = 0$.
If σ is quasi-definite and $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ, then
(iii) $\pi(x)\sigma = 0$ if and only if $\pi(x) = 0$.
(iv) $\langle \tau, P_n \rangle = 0, \quad n \geq k+1$ for some integer $k \geq 0$ if and only if $\tau = \phi(x)\sigma$ for some polynomial $\phi(x)$ of degree $\leq k$.

Consider a linear differential equation :

$$L_N[y](x) = \sum_{i=1}^{N} \ell_i(x) y^{(i)}(x) = \sum_{i=1}^{N} \sum_{j=0}^{i} \ell_{i,j} x^j y^{(i)}(x) = \lambda_n y(x), \tag{2.2}$$

where $\ell_{i,j}$ are real constants and $\lambda_n = \ell_{11} n + \cdots + \ell_{NN} n(n-1)\cdots(n-N+1)$. Then necessary and sufficient conditions for an *OPS* to satisfy the differential equation (2.2) were found first by Krall [17], of which another simpler proof can be found in [22].

We call an *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ a *Bochner-Krall OPS* (*BKOPS*) of order $\leq N$ if $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a differential equation (2.2) of order N.

Proposition 2.2. (*see* [16], [17], [22] *and* [31]) *Let* $\{P_n(x)\}_{n=0}^{\infty}$ *be an OPS relative to* σ. *Then the following statements are equivalent.*

(i) $\{P_n(x)\}_{n=0}^{\infty}$ *is a BKOPS satisfying the differential equation* (2.2);

(ii) The moments $\{\sigma_n\}_{n=0}^{\infty}$ *of* σ *satisfy* $r := [\frac{N+1}{2}]$ *recurrence relations :*

$$R_k[\sigma] = \sum_{i=2k+1}^{N} (-1)^i \binom{i-k-1}{k} (\ell_k \sigma)^{(i-2k-1)} = 0 \quad (k = 0, 1, \cdots, r-1); \tag{2.3}$$

(iii) σ *satisfies* $N+1$ *functional equations*:

$$\sum_{i=k}^{N} (-1)^i \binom{i}{k} (\ell_i \sigma)^{(i-k)} = \ell_k \sigma \quad (k = 0, 1, \cdots, N). \tag{2.4}$$

Furthermore, if any of the above equivalent conditions holds, then $N = 2r$ *must be even.*

Proof. See Theorem 2.4 in [22]. ☐

When σ is a classical moment functional ([20, 33]) satisfying (1.6), then it is well known that an *OPS* $\{P_n(x)\}_{n=0}^{\infty}$ relative to σ satisfies the second-order differential equation

$$A(x)P_n''(x) + B(x)P_n'(x) = (\frac{1}{2}n(n-1)A''(x) + nB'(x))P_n(x) \quad (n \in \mathbb{N}_0). \tag{2.5}$$

By iterating an equation (2.5), we can see that any classical orthogonal polynomials may satisfy differential equations (2.2) for any $N \geq 2$. In [24, Proposition 1], the converse has been shown :

Theorem 2.3. *Assume that* $\{P_n(x)\}_{n=0}^{\infty}$ *is a classical OPS satisfying the differential equation* (2.5). *If* $\{P_n(x)\}_{n=0}^{\infty}$ *also satisfies a differential equation* (2.2) *of order* $N = 2r$ $(r \geq 1)$, *then* $L_N[\cdot]$ *is a linear combination of iterations of the differential equation* (2.5).

We now consider a point-mass perturbation $\tau := \sigma + \nu$ of a classical moment functional σ at finitely many points, where

$$\nu = \sum_{k=1}^{m} \sum_{j=0}^{m_k} c_{k,j} \delta^{(j)}(x - x_k) \tag{2.6}$$

is a distribution with finite support $\{x_k\}_{k=1}^{m}$ in $\mathbb{C}$ and $c_{k,j} \in \mathbb{C}$.

Lemma 2.4. ([23]) *Let* σ *be a quasi-definite moment functional. If for some polynomial* $\pi(x), \pi(x)\sigma = \nu$, *where* ν *is as in* (2.6), *then* $\pi(x) \equiv 0$ *and* $\nu \equiv 0$.

Proof. Let $\phi(x) = \prod_{k=1}^{m}(x - x_k)^{m_k+1}$. Then $\phi(x)\nu = 0$ so that $\phi(x)\pi(x)\sigma = \phi(x)\nu = 0$. Hence, by Lemma 2.1, $\phi(x)\pi(x) \equiv 0$ so that $\pi(x) \equiv 0$ and $\nu \equiv 0$. ☐

3. Construction of differential operators

We want to construct a differential operator in (2.2) having as eigenfunctions an OPS $\{Q_n(x)\}_{n=0}^{\infty}$ orthogonal to a moment functional τ of the form

$$\tau = \sigma + \nu \tag{3.1}$$

where σ is a classical moment functional and ν a distribution with finite support. It is, however, open that an OPS in the Koornwinder class is a BKOPS.

From now on, we let $\{P_n(x)\}_{n=0}^\infty$ be a monic OPS relative to σ and

$$L_0[\cdot] = \sum_{i=0}^k a_i(x)D^i = \sum_{i=0}^k \sum_{j=0}^i a_{ij}x^jD^i, \quad (D = d/dx) \tag{3.2}$$

a linear differential operator of order k with polynomial coefficients $a_i(x)$, $(a_k(x) \not\equiv 0)$. For non-negative integers $n \geq 0$, let

$$\alpha_n := \sum_{i=0}^k a_{ii}(n+r)_{(i+r)}, \quad n \geq 0, \tag{3.3}$$

where

$$n_{(i)} = \begin{cases} 1, & \text{if} \quad i = 0, \\ n(n-1)\cdots(n-i+1), & \text{if} \quad i \geq 1. \end{cases}$$

Then for each $n \geq 0, L_0[P^{(r)}_{n+r}(x)]$ is either 0 if $\alpha_n = 0$ or a polynomial of degree n with leading coefficient α_n if $\alpha_n \neq 0$.

Theorem 3.1. *Let $\{Q_n(x)\}_{n=0}^\infty$ be a monic OPS relative to τ. Then for an integer $r \geq 0$, $\{Q_n(x)\}_{n=0}^\infty$ satisfies the relations :*

$$L_0[P^{(r)}_{n+r}(x)] = \alpha_n Q_n(x), \quad n \geq 0 \tag{3.4}$$

if and only if there are $k+r+1$ polynomials $\{b_i(x)\}_{i=0}^{k+r}$ with deg$(b_i) \leq i+r$ satisfying

$$\sum_{i=j}^{k+r}(-1)^i\binom{i}{j}(a_{i-r}(x)\tau)^{(i-j)} = b_j(x)\sigma, \quad 0 \leq j \leq k+r \tag{3.5}$$

where $a_i(x) = 0$ for $i < 0$.

Proof. Assume that $\{Q_n(x)\}_{n=0}^\infty$ satisfies (3.4). First,

$$< \tau, \alpha_n Q_n(x) >=< \sum_{i=0}^{k+r}(-1)^i(a_{i-r}(x)\tau)^{(i)}, P_{n+r}(x) >= 0, \quad n \geq 1$$

so that by Lemma 2.1 (iv), there is a polynomial $b_0(x)$ of degree $\leq r$, with which (3.5) holds for $j = 0$. Assume now that there are polynomials $\{b_j(x)\}_{j=0}^\ell$ $(0 \leq \ell < k+r)$ with $\deg(b_j) \leq j+r$ and (3.5) holds for $0 \leq j \leq \ell$. Then for $n \geq \ell+2$

$$\begin{aligned} 0 = &< \tau,\ \alpha_n Q_{\ell+1}Q_n >=< \sum_{i=0}^{k+r}(-1)^i(Q_{\ell+1}a_{i-r}\tau)^{(i)},\ P_{n+r} > \\ &= < \sum_{j=0}^{k+r} Q^{(j)}_{\ell+1}\sum_{i=j}^{k+r}(-1)^i\binom{i}{j}(a_{i-r}\tau)^{(i-j)},\ P_{n+r} > \\ &= Q^{(\ell+1)}_{\ell+1} < \sum_{i=\ell+1}^{k+r}(-1)^i\binom{i}{\ell+1}(a_{i-r}\tau)^{(i-\ell-1)},\ P_{n+r} > + < \sigma,\ (\sum_{j=0}^{\ell} Q^{(j)}_{\ell+1}b_j)P_{n+r} > \\ &= Q^{(\ell+1)}_{\ell+1} < \sum_{i=\ell+1}^{k+r}(-1)^i\binom{i}{\ell+1}(a_{i-r}\tau)^{(i-\ell-1)},\ P_{n+r} > \end{aligned}$$

so that

$$< \sum_{i=\ell+1}^{k+r} (-1)^i \binom{i}{\ell+1} (a_{i-r}\tau)^{(i-\ell-1)}, P_{n+r} >= 0, \quad n \geq \ell+2.$$

Hence, again by Lemma 2.1 (iv), there is a polynomial $b_{\ell+1}(x)$ of degree $\leq \ell+r+1$ with which (3.5) holds for $j = \ell+1$, which proves (3.5).

Conversely, we assume that there are polynomials $\{b_j(x)\}_{j=0}^{k+r}$ with $\deg(b_j) \leq j+r$ satisfying (3.5). Then for each $n \geq 0$, $L_0[P_{n+r}^{(r)}](x)$ is a polynomial of degree $\leq n$. Expand $L_0[P_{n+r}^{(r)}](x)$ in terms of $\{Q_j(x)\}_{j=0}^{n}$;

$$L_0[P_{n+r}^{(r)}](x) = \alpha_n Q_n(x) + \sum_{j=0}^{n-1} c_j Q_j(x).$$

Then

$$\begin{aligned} c_j < \tau, Q_j^2 > &=< \tau, L_0[P_{n+r}^{(r)}]Q_j > \\ &=< \sum_{i=0}^{k+r} (-1)^i (Q_j a_{i-r}\tau)^{(i)}, P_{n+r} > \\ &=< \sum_{i=0}^{k+r} (-1)^i \sum_{m=0}^{i} \binom{i}{m} Q_j^{(m)} (a_{i-r}\tau)^{(i-m)}, P_{n+r} > \\ &=< \sigma, (\sum_{m=0}^{k+r} b_m Q_j^{(m)}) P_{n+r} >= 0, \quad 0 \leq j \leq n-1 \end{aligned}$$

so that $c_j = 0,\ j = 0, 1, 2, \cdots, n-1$. Thus $\{Q_n(x)\}_{n=0}^{\infty}$ satisfies (3.4), which completes the proof. □

Remark 3.1. *It was shown in* [27] *that (*3.5*) is equivalent to the relations :*

(3.6) $$\sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (b_i(x)\sigma)^{(i-j)} = a_{j-r}(x)\tau, \quad 0 \leq j \leq k+r.$$

Theorem 3.1 implies that if $\{Q_n(x)\}_{n=0}^{\infty}$ is an OPS satisfying (3.4), then the operator $L_0[\cdot]$ gives rise to a bispectral operator $L := L_2 \circ L_1$ such that

$$\begin{cases} L_2 \circ L_1[P_n](x) &= \alpha_{n-r}\beta_{n-r} P_n(x), \quad n \geq 0 \\ L_1 \circ L_2[Q_n](x) &= \alpha_n \beta_n Q_n(x), \quad n \geq 0 \end{cases}$$

where $L_1[\cdot] = L_0 \circ D^r$ and

$$L_2[\cdot] = \sum_{i=0}^{k+r} b_i(x) D^i \quad (deg(b_i) \leq i+r)$$

for polynomials $b_i(x) = \sum_{j=0}^{i+r} b_{ij} x^j$ given in (3.5), and the constants β_n are given as

(3.7) $$\beta_n := \sum_{j=0}^{k+r} b_{j,j+r} n_{(j)}, \quad n \geq 0.$$

More precisely, we have the followings.

Theorem 3.2. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a monic OPS satisfying (3.4) and $\{b_i(x)\}_{i=0}^{k+r}$ the polynomials satisfying (3.5). Then for the linear differential operator $L_2[\cdot]$ defined*

$$L_2[\cdot] = \sum_{i=0}^{k+r} b_i(x) D^i \quad (deg(b_i) \leq i + r),$$

$\{Q_n(x)\}_{n=0}^{\infty}$ satisfies

$$L_2[Q_n(x)] = \sum_{i=0}^{k+r} b_i(x) Q_n^{(i)}(x) = \beta_n P_{n+r}(x), \quad n \geq 0 \tag{3.8}$$

so that

$$\begin{cases} L_2 \circ L_1[P_n](x) & = \alpha_{n-r}\beta_{n-r}P_n(x), \quad n \geq 0 \\ L_1 \circ L_2[Q_n](x) & = \alpha_n \beta_n Q_n(x), \quad n \geq 0 \end{cases} \tag{3.9}$$

where $L_1[\cdot] := L_0 \circ D^r$.

Proof. By the same argument used in the proof of the necessity of Theorem 3.1, it is easily shown that for each $n \geq 0$, $L_2[Q_n(x)] = \sum_{j=0}^{k+r} b_j(x) Q_n^{(j)}(x)$ is either a polynomial of degree exactly $n + r$ or identically zero. We now define a PS $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ by

$$\tilde{P}_n(x) := \begin{cases} L_2[Q_{n-r}](x), & \text{if } n \geq r \text{ and } \beta_{n-r} \neq 0, \\ P_n(x), & \text{otherwise.} \end{cases}$$

Then $\deg(\tilde{P}_n) = n,\ n \geq 0$ by (3.7), and by using (3.6) we can show easily that

$$< \sigma, \tilde{P}_n \tilde{P}_m >= 0, \quad \text{for} \quad n \neq m,$$

which shows that $\{\tilde{P}_n(x)\}_{n=0}^{\infty}$ is an OPS relative to σ. Hence,

$$L_2[Q_{n-r}](x) = \beta_{n-r} P_n(x), \quad n \geq r$$

for β_n given in (3.7). Finally,

$$L_2 \circ L_1[P_n](x) = L_2 \circ L_0[P_n^{(r)}](x) = \begin{cases} 0, & \text{if } \ 0 \leq n \leq r-1 \\ L_2[Q_{n-r}](x) = \alpha_{n-r}\beta_{n-r}P_n(x), & \text{if } \ n \geq r \end{cases}$$

and

$$L_1 \circ L_2[Q_n](x) = \beta_n L_1[P_{n+r}](x) = \alpha_n \beta_n Q_n(x), \quad n \geq 0,$$

which proves (3.9). □

Without the information for the orthogonalizing weight or measure of an OPS $\{Q_n(x)\}_{n=0}^{\infty}$, we have no way other than checking all the Hankel determinants given in (2.1) in order to check the positive-definiteness of $\{Q_n(x)\}_{n=0}^{\infty}$. But it is complicated and cumbersome. So the following characterization of positive-definiteness of such an OPS $\{Q_n(x)\}_{n=0}^{\infty}$ as given in (3.4) is useful and efficient.

Theorem 3.3. *Let $\{Q_n(x)\}_{n=0}^{\infty}$ be a monic OPS relative to τ satisfying (3.4). Assume that σ is a positive-definite moment functional. If $\alpha_n \neq 0,\ n \geq 0$ for α_n in (3.3), then $\{Q_n(x)\}_{n=0}^{\infty}$ is a positive-definite OPS if and only if*

$$\alpha_n \beta_n > 0, \quad n \geq 0$$

for α_n in (3.3) and β_n in (3.7).

Proof. Using the relations in (3.5), we have

$$\begin{aligned} < \tau, Q_n^2 > &= \alpha_n^{-1} < \sum_{j=0}^{k+r} Q_n^{(j)} \sum_{i=j}^{k+r} (-1)^i \binom{i}{j} (a_{i-r}\tau)^{(i-j)},\ P_{n+r} > \\ &= \alpha_n^{-1} < \sigma,\ (\sum_{j=0}^{k+r} Q_n^{(j)} b_j) P_{n+r} > \\ &= \alpha_n^{-1} \beta_n\ < \sigma, x^{n+r} P_{n+r} >, \quad n \geq 0. \end{aligned}$$

Since $\langle \sigma, x^{n+r}P_{n+r}\rangle > 0,\ n \geq 0$, it follows that $\{Q_n(x)\}_{n=0}^{\infty}$ is positive-definite if and only if $\alpha_n\beta_n > 0,\ n \geq 0$, which proves the theorem. □

Remark 3.2. *Theorem 3.1 and 3.2 in* [27] *were obtained under the assumption that* $\alpha_n \neq 0,\ n \geq 0$ *for* α_n *in (*3.3*). Note that for each* $n \geq 0$,

$$< \tau, \alpha_n Q_n^2 > = < \tau, L_0[P_{n+r}^{(r)}] Q_n > = < \sigma, \beta_n P_{n+r}^2 >$$

which implies that $\alpha_n \neq 0$ *if and only if* $\beta_n \neq 0$.

4. Illustration

We want to construct a differential operator in (2.2) having as eigenfunctions an OPS $\{Q_n(x)\}_{n=0}^{\infty}$ orthogonal relative to a moment functional τ of the form

$$\tau = \sigma + \nu \tag{4.1}$$

where σ is a classical moment functional and ν a distribution with finite support. It is, however, open that an OPS in the Koornwinder class is a BKOPS.

In this situation, Theorem 3.1 suggests a method to construct a differential operator having a sequence of eigenpolynomials orthogonal relative to τ as in (4.1). We illustrate Theorem 3.1 with some examples. In particular, this result give a useful method to construct such differential operators.

In the remainder of this section, we let $\{P_n(x)\}_{n=0}^{\infty}$ be a classical monic OPS relative to σ and $\{Q_n(x)\}_{n=0}^{\infty}$ a monic OPS relative to τ in (4.1).

Example 1. The Krall-Laguerre polynomials I.

Let $\sigma = \sigma_L^{(\alpha)}$ be the Laguerre moment functional, which satisfies the moment equation

$$(x\sigma)' = (\alpha + 1 - x)\sigma. \tag{4.2}$$

Assume that there are polynomials $a_2(x) = ax^2 + bx + c,\ a_1(x) = rx + s$, and a constant a_0 such that

$$L_0[P_n(x)] = (ax^2 + bx + c)P_n''(x) + (rx + s)P_n'(x) + a_0 P_n(x) = \alpha_n Q_n(x), \quad n \geq 0, \tag{4.3}$$

where $\alpha_n = a(n^2 - n) + rn + a_0$. Then Theorem 3.1 implies that there are polynomials $b_2(x) = b_{22}x^2 + b_{21}x + b_{20}, b_1(x) = b_{10}x + b_{10}$ and a constant b_0 such that

$$a_2(x)\tau = b_2(x)\sigma, \tag{4.4}$$

$$2(a_2(x)\tau)' - a_1(x)\tau = b_1(x)\sigma, \tag{4.5}$$

$$(a_2(x)\tau)'' - (a_1(x)\tau)' - a_0(x)\tau = b_0\sigma. \tag{4.6}$$

And $\{P_n(x)\}_{n=0}^{\infty}$ is obtained from $\{Q_n(x)\}_{n=0}^{\infty}$ via the differential operator $M_0[\cdot]$:

$$M_0[Q_n](x) := b_2(x)Q_n''(x) + b_1(x)Q_n'(x) + b_0Q_n(x) = \beta_nP_n(x), \quad n \geq 0$$

where $\beta_n = b_{22}(n^2 - n) + b_{11}n + b_0$. Since $\{P_n(x)\}_{n=0}^{\infty}$ satisfies the differential equation $M_0 \circ L_0[P_n](x) = \alpha_n\beta_nP_n(x)$, Theorem 2.3 implies that $a_2(x)b_2(x) = Cx^2$ for some non-zero constant C. Then there are three cases and we consider only the case when $a_2(x) = b_2(x) = x$. Solve the relation (4.4) and then there is a constant λ such that

$$\tau = \sigma + \lambda\delta(x). \tag{4.7}$$

To avoid the trivial case, we assume that $\lambda \neq 0$. By substituting (4.2) into the left expression in (4.5), we have that

$$\begin{aligned} 2(a_2(x)\tau)' - a_1(x)\tau &= 2(x\sigma)' - (rx + s)\sigma - \lambda(rx + s)\delta(x) \\ &= 2(\alpha + 1 - x)\sigma - (rx + s)\sigma - s\lambda\delta(x) \\ &= (2\alpha + 2 - 2x - rx - s)\sigma - s\lambda\delta(x). \end{aligned}$$

Hence (4.5) yields the relation

$$(2\alpha + 2 - (r + 2)x - b_1(x))\sigma = s\lambda\delta(x),$$

and Lemma 2.4 implies

$$a_1(x) = rx \quad \text{and} \quad b_1(x) = 2(\alpha + 1) - (r + 2)x.$$

Now we consider the moment equation in (4.6). Using (4.2), we expand (4.6) as

$$\begin{aligned} b_0\sigma &= [(x\sigma)']' - r(x\sigma)' + a_0\sigma + a_0\lambda\delta(x) \\ &= (\alpha + 1)\sigma' - (r + 1)(\alpha + 1 - x)\sigma + a_0\sigma + a_0\lambda\delta(x). \end{aligned}$$

Multiplying by x and implying the relation $x\sigma' = (\alpha - x)\sigma$, we get the identity

$$(\alpha + 1)(\alpha - x) - (r + 1)(\alpha + 1 - x)x + a_0x = b_0x,$$

which shows that $\alpha(\alpha + 1) = 0$ and $r = -1$. Since α is a non-negative integer by Theorem 1.1, we have that $\alpha = 0$.

On the other hand, $\sigma_L^{(0)}$ is represented with the weight function e^{-x} on $[0, \infty)$, we have the explicit expression for the derivative of $\sigma_L^{(0)}$

$$\sigma' = (\sigma_L^{(0)})' = -\sigma + \delta(x). \tag{4.8}$$

Using (4.8), we simplify (4.6) into

$$-\sigma + \delta(x) + a_0\sigma + a_0\lambda\delta(x) = b_0\sigma.$$

By implying Lemma 2.4, we have

$$r = -1, \quad a_0\lambda + 1 = 0, \quad b_0 = a_0 - 1.$$

Hence we have shown that $\{Q_n(x)\}_{n=0}^{\infty}$ is orthogonal with respect to $\tau = \sigma_L^{(0)} + \lambda\delta(x)$ and that

$$L_0[P_n(x)] = xP_n''(x) - xP_n'(x) - \frac{1}{\lambda}P_n(x) = -(n + \frac{1}{\lambda})Q_n(x), \quad n \geq 0$$

and

$$M_0[Q_n(x)] = xQ_n''(x) - (x - 2)Q_n'(x) - (1 + \frac{1}{\lambda})Q_n(x) = -(n + 1 + \frac{1}{\lambda})P_n(x), \quad n \geq 0,$$

hence $\{Q_n(x)\}_{n=0}^\infty$ satisfies the fourth order differential equation

$$\begin{aligned} L_0 \circ M_0[Q_n(x)] = &\ x^2 Q_n^{(4)}(x) - 2x(x-2)Q_n^{(3)}(x) + x(x-6-\frac{2}{\lambda})Q_n''(x) \\ &+ 2[(1+\frac{1}{\lambda})x - \frac{1}{\lambda}]Q_n'(x) + \frac{1}{\lambda}(1+\frac{1}{\lambda})Q_n(x) \\ = &\ (n+\frac{1}{\lambda})(n+1+\frac{1}{\lambda})Q_n(x) \end{aligned}$$

and $\{P_n(x)\}_{n=0}^\infty$ satisfies the fourth order differential equation

$$\begin{aligned} M_0 \circ L_0[P_n(x)] = &\ x^2 P_n^{(4)}(x) - 2x(x-2)P_n^{(3)}(x) + [x^2 - (6+\frac{2}{\lambda})x + 2]P_n''(x) \\ &+ 2(1+\frac{1}{\lambda})(x-1)P_n'(x) + \frac{1}{\lambda}(1+\frac{1}{\lambda})P_n(x) \\ = &\ (n+\frac{1}{\lambda})(n+1+\frac{1}{\lambda})P_n(x). \end{aligned}$$

The operator $M_0 \circ L_0$ is expressed as

$$M_0 \circ L_0 = L^2 + (1+\frac{2}{\lambda})L + \frac{1}{\lambda}(1+\frac{1}{\lambda})$$

for the Laguerre operator $L = xD^2 + (1-x)D$ with $D = d/dx$. This is the same result as that in [18] or [9]. Note that Theorem 3.3 shows that $\tau = \sigma_L^{(0)} + \lambda\delta(x)$ is a positive-definite moment functional if and only if λ is positive.

Remark 4.1. *F. Grünbaum and L. Haine* [9] *gave the special condition that* $\alpha = 0$*, in order to obtain the same result. The reason for the specialization seems due to the fact that the Krall-Laguerre polynomials satisfying a fourth-order differential equation were found before their work. (see,* [18]*).*

Example 2. Krall-Laguerre polynomials II

Let τ be a point mass perturbation of the Laguerre moment functional $\sigma = \sigma_L^{(0)}$,

$$\tau = \sigma_L^{(0)} + \lambda\delta(x) + \mu\delta'(x) \tag{4.9}$$

for some constants λ and $\mu \neq 0$. Assume that there are polynomials $\{a_i(x)\}_{i=0}^6$ with $deg(a_i) \leq i$ such that

$$L_0[P_n(x)] = \sum_{i=0}^{6} a_i(x)P_n^{(i)}(x) = \alpha_n Q_n(x), \quad n \geq 0 \tag{4.10}$$

where $\{Q_n(x)\}_{n=0}^\infty$ is the monic OPS relative to τ. Theorem 3.1 implies that there are polynomials $\{b_i(x)\}_{i=0}^6$ with $deg(b_i) \leq i$ such that σ and τ satisfy the equations

$$\sum_{i=j}^{6} (-1)^i \binom{i}{j} (a_i(x)\tau)^{(i-j)} = b_j(x)\sigma, \quad j = 0, 1, 2, \cdots, 6 \tag{4.11}$$

and

$$M_0[Q_n(x)] = \sum_{i=0}^{6} b_i(x)Q_n^{(i)}(x) = \beta_n P_n(x), \quad n \geq 0. \tag{4.12}$$

Since $\{P_n(x)\}_{n=0}^{\infty}$ satisfy the 12th order differential equation $M_0 \circ L_0[P_n](x) = \alpha_n \beta_n P_n(x)$, Theorem 2.3 implies that $a_6(x)b_6(x) = Cx^6$ for some constant C. The relation (4.9) yields that $a_6(x)$ and $b_6(x)$ are of the same degree, so we may assume that $a_6(x) = b_6(x) = x^3$. By implying the relations

$$[\sigma_L^{(0)}]^{(k)} = (-1)^k \sigma_L^{(0)} + \sum_{j=0}^{k-1} (-1)^{k-1-j} \delta^{(j)}(x) \tag{4.13}$$

and Lemma 2.4 to equations in (4.11), and then equating coefficients, we obtain

$$a_5(x) = (Ax + B)x^2, a_4(x) = (Cx + D)x^2, \quad a_3(x) = (Ex + F)x^2, \quad a_2(x) = (Gx + H)x$$

for some constants. And let $a_1(x) = Kx + L$. Using (4.13), we solve the equations in (4.11) and we have that

$$\begin{aligned}
a_5(x) &= -3x^3 + 5x^2,\\
a_4(x) &= 3x^3 + Dx^2,\\
a_3(x) &= -x^3 - (2D + 15)x^2,\\
a_2(x) &= (D + 10)x^2 - \frac{10}{\mu}x,\\
a_1(x) &= \frac{10}{\mu}x + \frac{6}{\mu},\\
a_0 &= \frac{6D + 72}{\mu},
\end{aligned}$$

where $D = \frac{3\lambda}{4\mu} - \frac{53}{4}$ and the constant λ is determined by the relation

$$5\mu^2 - 18\lambda\mu + 9\lambda^2 - 32\mu = 0,$$

for a free parameter μ ($\mu \leq -8$ or $\mu \geq 0$). Also we have

$$\begin{aligned}
b_5(x) &= -3x^3 + 13x^2,\\
b_4(x) &= 3x^3 + (D - 20)x^2 + 40x,\\
b_3(x) &= -x^3 + (-2D + 1)x^2(8D + 20)x + 20,\\
b_2(x) &= (D + 6)x^2 - (12D + \frac{10}{\mu} + 102)x + 12D + 120,\\
b_1(x) &= (4D + \frac{10}{\mu} + 42)x - 12D - \frac{26}{\mu} - 138,\\
b_0(x) &= 2D + \frac{16}{\mu} + a_0 + 24.
\end{aligned}$$

Hence for $n \geq 0$, we get

$$\begin{aligned}
\alpha_n &= -n(n-1)(n-2) + (D + 10)n(n-1) + \frac{10}{\mu}n + \frac{6D + 72}{\mu},\\
\beta_n &= -n(n-1)(n-2) + (D + 6)n(n-1) + (4D + \frac{10}{\mu} + 42)n + 2D + \frac{16}{\mu} + a_0 + 24.
\end{aligned}$$

In this case, the 5-th Hankel determinant $\Delta_5(\tau) = \det[\tau_{i+j}]_{i,j=0}^{5}$ of τ is

$$\Delta_5(\tau) = 1 + 6\lambda + 30\mu - 105\mu^2.$$

	$\mu=-10$		$\mu=0.5$		$\mu=10$	
	α_n	β_n	α_n	β_n	α_n	β_n
$n=0$	0.16584	−1.98695	18.73863	53.86174	0.10249	2.04413
$n=1$	−.83416	−10.09252	38.73863	74.10795	1.10249	−2.27259
$n=2$	−6.38695	−30.75088	57.86174	85.47727	−1.55587	−18.24766
$n=3$	−22.49252	−69.96203	70.10795	81.96969	−13.87259	−51.88110
$n=4$	−55.15088	−133.7296	69.47727	57.58522	−41.84766	−109.17290
$n=5$	−110.36203	−228.04268	49.96969	6.32385	−91.48110	−196.12305
$n=6$	−194.12596	−358.91218	5.58522	−77.81441	−168.77290	−318.73157
$n=7$	−312.44268	−532.33447	−69.67615	−200.82956	−279.72305	−482.99844
$n=8$	−471.31218	−754.30955	−181.81441	−368.72161	−430.33157	−694.92367
$n=9$	−676.73447	−1030.8374	−336.82956	−587.49056	−626.59844	−960.50726

TABLE 1. Table of the values of α_n and β_n for $\lambda=\mu+\frac{2}{3}\sqrt{\mu^2+8\mu}$

	$\mu=-10$		$\mu=0.5$		$\mu=10$	
	α_n	β_n	α_n	β_n	α_n	β_n
$n=0$	0.43416	−2.61305	−30.73863	−3.86174	−0.70249	−1.44413
$n=1$	−0.56584	−12.50748	−10.73863	−0.10795	0.29751	−11.12741
$n=2$	−7.01305	−35.84912	0.13826	−13.47727	−5.04413	−35.15234
$n=3$	−24.90748	−78.63797	−4.10795	−49.96969	22.72741	−79.51890
$n=4$	−60.24912	−146.87404	−29.47727	−115.58522	−58.75234	−150.22710
$n=5$	−119.03797	−246.55732	−81.96969	−216.32385	−119.11890	−253.27695
$n=6$	−207.27404	−383.68782	−167.58522	−358.18559	−209.82710	−394.66843
$n=7$	−330.95732	−564.26553	−292.32385	−547.17044	−336.87695	−580.40156
$n=8$	−496.08782	−794.29045	−462.18559	−789.27839	−506.26843	−816.47633
$n=9$	−708.66553	−1079.76258	−683.17044	−1090.50944	−724.00156	−1108.89274

TABLE 2. Table of the values of α_n and β_n for $\lambda=\mu-\frac{2}{3}\sqrt{\mu^2+8\mu}$

This yields that in order for τ to be positive-definite on $[0,\infty)$, $\mu<1$. However, $\langle\tau,x\rangle=1-\mu$ should be positive because $p(x)=x$ is non-negative on $[0,\infty)$. Hence, τ is not positive-definite on $[0,\infty)$. We can see from the tables below that there exists an integer $n\geq 0$ such that $\alpha_n\beta_n<0$.

Example 3. The Krall-Legendre polynomials.

Let τ be a point mass perturbation of the Legendre moment functional $\sigma=\sigma_J^{(0,0)}$,

$$\tau=\sigma_J^{(0,0)}+\lambda_1\delta(x-1)+\lambda_2\delta'(x-1)+\mu_1\delta(x+1)+\mu_2\delta(x+1), \tag{4.14}$$

for some constants λ_i and μ_i. Here we assume $\lambda_2\neq 0$. We attempt to construct an 8th order differential operator having $\{Q_n(x)\}_{n=0}^{\infty}$ as eigenfunctions. Assume that there are polynomials

$\{a_i(x)\}_{i=0}^4,\ deg(a_i) \le i$ and $\{b_i(x)\}_{i=0}^4,\ deg(b_i) \le i$ satisfying

$$a_4(x)\tau = b_4(x)\sigma, \tag{4.15}$$

$$4(a_4(x)\tau)' - a_3(x)\tau = b_3(x)\sigma, \tag{4.16}$$

$$6(a_4(x)\tau)'' - 3(a_3(x)\tau)' + a_2(x)\tau = b_2(x)\sigma, \tag{4.17}$$

$$4(a_4(x)\tau)''' - 3(a_3(x)\tau)'' + 2(a_2(x)\tau)' - a_1(x)\tau = b_1(x)\sigma, \tag{4.18}$$

$$(a_4(x)\tau)^{(iv)} - (a_3(x)\tau)''' + (a_2(x)\tau)'' - (a_1(x)\tau)' + a_0(x)\tau = b_0(x)\sigma. \tag{4.19}$$

In this case, we may assume that

$$a_4(x) = b_4(x) = (x^2-1)^2.$$

Using the identity

$$\sigma' = \delta(x+1) - \delta(x-1), \tag{4.20}$$

we can easily show the relations

$$[(x^2-1)^2\tau]' = 4x(x^2-1)\sigma, \tag{4.21}$$

$$[(x^2-1)^2\tau]'' = (12x^2-4)\sigma, \tag{4.22}$$

$$[(x^2-1)^2\tau]''' = 24x\sigma + 8\delta(x+1) - 8\delta(x-1), \tag{4.23}$$

$$[(x^2-1)^2\tau]^{(iv)} = 24\sigma - 24\delta(x+1) - 24\delta(x-1) + 8\delta'(x+1) - 8\delta'(x-1). \tag{4.24}$$

The relations (4.16-4.18) yield that $a_3(x), a_2(x)$, and $a_1(x)$ are written as

$$a_3(x) = (Ax+B)(x-1)^2,\ a_2(x) = C(x-1)^2,\ a_1(x) = D(x-1).$$

Using the relations (4.20-4.24), we solve the equations (4.15-4.19) that we have

$$\tau = \sigma - \frac{64}{3}\delta(x-1) + \frac{128}{3}\delta'(x-1) + \frac{8}{3}\delta(x+1)$$

and

$$\begin{aligned} a_3(x) &= 3(x+1)(x-1)^2, & b_3(x) &= (13x+3)(x^2-1) \\ a_2(x) &= 0, & b_2(x) &= 45x^2+18x-15 \\ a_1(x) &= \frac{3}{4}(x-1), & b_1(x) &= \frac{165}{4}x + \frac{75}{4} \\ a_0(x) &= -\frac{9}{16}, & b_0(x) &= \frac{75}{16}. \end{aligned}$$

So in this case, $\{Q_n(x)\}_{n=0}^\infty$ is quasi-definite but not positive-definite.

References

[1] H. Bavinck, *A direct approach to Koekoek's differential equation for generalized Laguerre polynomials,* Acta. Math. Hungar. 66(1995), 247-253.
[2] H. Bavinck and R. Koekoek, *On a difference equation for generalizations of Charlier polynomials,* J. Approx. Theory 81(1995), 195-206.
[3] R. P. Boas, *The Stieltjes moment problem for functions of bounded variation,* Bull. Amer. Math. Soc. 45 (1939), 399–404.
[4] S. Bochner, *Über Sturm-Liouvillesche Polynomsysteme*, Math. Z. 29 (1929), 730–736.
[5] T.S. Chihara, **An Introduction to Orthogonal Polynomials**, Gordon and Breach, New York, 1977.
[6] A. J. Duran, *The Stieltjes moment problem for rapidly decreasing functions*, Proc. Amer. Math. Soc. 107(3) (1989), 731–741.

[7] W. N. Everitt and L. L. Littlejohn, *Orthogonal polynomials and spectral theory: a survey,* in Orthogonal polynomials and their applications (Eds: C. Brezinski, L. Gori, A. A. Ronveaux), IMACS Annals on Computing and Applied Mathematics, Vol. 9, J. C. Baltzer AG Publishers, 1991, 21-55.
[8] W. N. Everitt, K. H. Kwon, L. L. Littlejohn, and R. Wellman, *Orthogonal polynomial solutions of linear ordinary differential equations,* J. Comput. Appl. Math. 133(2001), 85-109.
[9] F. A. Grünbaum and L. Haine, *Bispectral Darboux transformation: An extention of the Krall polynomials*, IMRN (Internat. Mat. Res. Notices) 8 (1997), 359–392.
[10] F. A. Grünbaum, L. Haine, and E. Horozov, *Some functions that generalize the Krall-Laguerre polynomials*, J. Comput. Appl. Math. 106 (1999), 271–297.
[11] J. Koekoek and R. Koekoek, *On a differential equation for Koornwinder's generalized Laguerre polynomials*, Proc. Amer. Math. Soc. 112 (1991), 1045–1054.
[12] J. Koekoek and R. Koekoek, *Differential equations for generalized Jacobi polynomials*, J. Comput. Appl. Math. 126 (2000), 1–31.
[13] J. Koekoek and R. Koekoek, *The Jacobi inversion formula,* Complex Variables 39(1999), 1-18.
[14] T. H. Koornwinder, *Orthogonal polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$, Canad. Math. Bull. 27(2) (1984), 205–214.
[15] A. M. Krall, *Orthogonal polynomials satisfying fourth order differential equations*, Proc. Royal Soc. Edin. 87(A) (1981), 271–288.
[16] A. M. Krall and L. L. Littlejohn, *On the classification of differential equations having orthogonal polynomial solutions II*, Ann. Mat. Pura Appl. 149 (1987), 77–102.
[17] H. L. Krall, *Certain differential equations for Tchebycheff polynomials*, Duke Math. J. 4 (1938), 705–718.
[18] H. L. Krall, *On orthogonal polynomials satisfying a certain fourth order differential equation*, The Pennsylvania State College Studies No.6, The Penn. State college, Pa., 1940.
[19] K. H. Kwon, D. W. Lee, F. Marcellán and S. B. Park, *On kernel polynomials and self perturbation of orthogonal polynomials*, Ann. Mat. Pura Appl. 180(2) (2001), 127–146.
[20] K. H. Kwon, J. K. Lee and B. H. Yoo, *Characterizations of classical orthogonal polynomials*, Results in Math. 24 (1993), 119–128.
[21] K. H. Kwon and L. L. Littlejohn, *Classification of classical orthogonal polynomials*, J. Korean Math. Soc. 34 (1997), 973–1008.
[22] K. H. Kwon, L. L. Littlejohn, and B. H. Yoo, *Characterizations of orthogonal polynomials satisfying differential equations*, SIAM J. Math. Anal. 25(3) (1994), 976–990.
[23] K. H. Kwon, L. L. Littlejohn, and G. J. Yoon, *Orthogonal Polynomial Solutions of Spectral Type Differential Equations: Magnus' Conjecture*, J. Approx. Theory 112(2) (2001), 189–215.
[24] K. H. Kwon, B. H. Yoo, and G. J. Yoon, *A characterization of Hermite polynomials*, J. Comput. Appl. Math. 78 (1997), 295–299.
[25] K. H. Kwon, B. H. Yoo, and G. J. Yoon, *A note on Magnus' conjecture*, Intern. Math. Journal, 3(2) (2003), 143–156.
[26] K. H. Kwon and G. J. Yoon, *Symmetrizability of differntial equations having orthogonal polynomial solutions*, J. Comput. Appl. Math. 83 (1997), 257–268.
[27] K. H. Kwon and G. J. Yoon, *Generalized Hahn's theorem*, J. Comput. Appl. Math. 116 (2000), 243–262.
[28] P. Lesky, *Die Charakterisierung der klassischen orthogonalen Polynome durch Sturm-Liouvillesche Differentialgleichungen,* Arch. Rational Mech. Anal. 10(1962), 341-352.
[29] L. L. Littlejohn, *An application of a new theorem on orthogonal polynomials and differential equations,* Quaestiones Math. 10(1986), 49-61.
[30] L. L. Littlejohn and S. D. Shore, *Nonclassical orthogonal polynomials as solutions to second order differential equations*, Canad. Math. Bull. 25(3)(1982), 291-295.
[31] L. L. Littlejohn, *On the classification of differential equations having orthogonal polynomial solutions*, Ann. Mat. Pura Appl. 138 (1984), 35–53.
[32] A. P. Magnus, *Open problems*, **Orthogonal Polynomials and Their Applications** (Editors: C. Brezinski et al), J. C. Balter AG, Scientific Publishing Co, IMACS, 1991, 417–419.
[33] P. Maroni, *Variations around classical orthogonal polynomials. Connected problems*, J. Comput. Appl. Math. 48 (1993), 133–155.
[34] E. J. Routh, *On some properties of certain solutions of a differential equation of the second order,* Proc. Lond. Math. Soc. 16(1885), 245-261.

[35] S. D. Shore, *On the second order differential equation which has orthogonal polynomial solutions,* Bull. Calcutta Math. Soc. 56(1964), 195-198.

[36] G. Szegö, *Orthogonal polynomials,* 4th edition, Amer. Math. Soc. Colloquium Publications, Vol. 23, AMS, Providence, R.I., 1978.

Division of Applied Mathematics, KAIST, Taejon 305-701, Korea
E-mail address: khkwon@amath.kaist.ac.kr

Department of Mathematics and Statistics, Utah State University, Logan, Utah, 84322-3900
E-mail address: lance@math.usu.edu

Division of Applied Mathematics, KAIST, Taejon 305-701, Korea
E-mail address: ykj@amath.kaist.ac.kr