Teutoburgan Split Systems

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Abstract

In this paper we consider the tight-span of a finite metric space. This is a polytopal complex with interesting combinatorial and geometrical properties that was independently discovered and studied by J. Isbell [14], A. Dress [4], and M. Chrobak and L. Larmore [2]. Although the tight-span is quite difficult to understand in general, for certain types of metrics much can be said about its structure. For example, in a series of papers [6, 7, 9] various results were proven concerning the structure of the tight-span of a totally split-decomposable metric, which resulted in an explicit description of the tight-span for a large subclass of such metrics. In [9] we introduced the concept of a cell-decomposable metric, and showed that the tight-span of such a metric is comprised of smaller, easier to understand tight-spans. Here we show that totally split-decomposable metrics are cell-decomposable and, moreover, that the cells in the tight-span of a totally split-decomposable metric are zonotopes that are polytope isomorphic to either hypercubes or rhombic dodecahedra.

1 Introduction

In this paper X will denote a finite set with $|X| \geq 2$. Given a metric d on X, i.e. a symmetric map $d : X \times X \to \mathbb{R}$ that vanishes precisely on the diagonal and satisfies the usual triangle inequality, we can canonically associate a polytopal complex $T(d)$ to d as follows (see Section 2 for definitions concerning polytopes and polytopal complexes). Let $\mathbb{R}^X$ denote the set of functions that map X to $\mathbb{R}$. To the metric d associate the polyhedron

$$P(d) = \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\},$$

and let $T(d)$ consist of the bounded faces of $P(d)$. The complex $T(d)$ is known as the *tight-span* of d. It was independently discovered and studied by J. Isbell [14], A. Dress [4], and by M. Chrobak and L. Larmore [2]. More recently, it has also been seen to arise naturally in the context of *tropical geometry*. In particular, it was shown in [3] that the tight-span of a metric is a *tropical polytope*, and that any tropical polytope which is fixed under a certain canonical involution defined on the set of all tropical polytopes must in fact be the tight-span of a metric.

Due to its fundamental nature, it is of interest to describe the structure of the tight-span of various types of metrics. Several results of this nature have been presented in the literature. For example, an explicit description of the structure of the tight-span of a metric on five or less points was given in [1, 4]. Moreover, various characterizations for metrics having "tree-like" tight-spans were given in [4], some of which were subsequently extended in a series of papers [6, 7, 9] on the structure of the tight-span of a *totally split-decomposable* metric (see below for the definition of this latter type of metric).

To better understand the structure of a the tight-span, in [12] we introduced the concept of *cell-decomposable* metrics, the definition of which we now recall. It can be shown (cf. [4, 14]) that if d is a metric on X, then the map $d_\infty : \mathbb{R}^X \times \mathbb{R}^X \to \mathbb{R}^{\geq 0}$ defined, for $f, g \in \mathbb{R}^X$, by

$$d_\infty(f, g) = \max_{x \in X} |f(x) - g(x)|,$$

restricts to give a metric on on $P(d)$ and $T(d)$, and that the map

$$\Psi : X \to T(d) : y \mapsto (h_y : X \to \mathbb{R} : x \mapsto d(x, y))$$

is an embedding of the metric space (X, d) into $(T(d), d_\infty)$, i.e. Ψ is an injection with $d_\infty(\Psi(x), \Psi(y)) = d(x, y)$ holding for all $x, y \in X$. Now, given a cell C of $T(d)$ (i.e. a bounded face of $P(d)$) and some $x \in X$, we call a (necessarily unique) element $g \in C$ a *gate in C for x* if, for all $h \in C$,

$$d_\infty(h_x, h) = d_\infty(h_x, g) + d_\infty(g, h).$$

Moreover, we say that C is *X-gated* if there is a gate in C for each $x \in X$. In [12, Theorem 1.1] we proved that if a cell C in $T(d)$ with non-zero dimension is X-gated, then

(i) the metric obtained by restricting d_∞ to the set $G(C)$, consisting of the gates in C for all of the elements in X, is antipodal (recall that a metric d on a finite set Y is *antipodal* if there is an involution $\sigma : Y \to Y$, mapping each element y in Y to its *antipode* $\overline{y}$ with $d(y, z) + d(z, \overline{y}) = d(y, \overline{y})$ holding for all $z \in Y$), and

(ii) the tight-span of the metric $d_\infty|_{G(C)}$ is a polytope that is polytope isomorphic to C.

Moreover, in [11] we described some structural properties of the tight-span of an antipodal metric. Thus, if every cell in $T(d)$ is X-gated, in which case d is called *cell-decomposable*, it is in principle possible to deduce the structure of all of its cells. In this paper we apply this strategy to deduce the structure of the cells of the tight-span of a totally split-decomposable metric. But, before stating our main result, we recall the definition of such a metric.

A *split* of X is a bipartition of X, and a set $\mathcal{S}$ of splits of X is a *split system* (on X). Denote the split system consisting of all possible splits of X by $\mathcal{S}(X)$. For every $x \in X$ and any split S of X, we denote by $S(x)$ the element of S that contains x, and by $\overline{S}(x)$ the complement of $S(x)$. To avoid certain non-essential technicalities, in this paper we will assume that all split systems are non-empty and, for $\mathcal{S}$ a split system on X, that, for all $x \neq y$, there exists some split $S \in \mathcal{S}$ with $S(x) \neq S(y)$. A *weighting* on a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ is a map $\alpha : \mathcal{S} \to \mathbb{R}^{>0} : S \mapsto \alpha_S = \alpha(S)$, and such a pair $(\mathcal{S}, \alpha)$ is called a *weighted split system (on X)*. We call a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ *weakly compatible* if there exist no three distinct splits $S_1, S_2, S_3 \in \mathcal{S}$ and four elements $x_0, x_1, x_2, x_3 \in X$ such that

$$S_j(x_i) = S_j(x_0) \text{ if and only if } i = j. \tag{1}$$

Now, a metric d on X is called *totally split-decomposable* if there exists a weighted split system $(\mathcal{S}, \alpha)$ on X with

$$d = d_{\mathcal{S},\alpha} = \sum_{S \in \mathcal{S}} \alpha_S \delta_S,$$

where, for any split $S \in \mathcal{S}(X)$ and all $x, y \in X$,

$$\delta_S(x, y) = \begin{cases} 1 \text{ if } S(x) \neq S(y), \\ 0 \text{ else.} \end{cases}$$

Note that if d is such a metric, then it follows by results in [1] that if $d = d_{\mathcal{S}',\alpha'}$ for some weakly compatible split system $\mathcal{S}'$ and weighting α' on $\mathcal{S}'$, then $\mathcal{S}' = \mathcal{S}$ and $\alpha' = \alpha$. Totally split-decomposable metrics were introduced in [1]. Besides having mathematical interest, such metrics play a useful role in phylogenetic analysis (cf. e.g. [10, 13]).

We now state our main result. Recall that a *zonotope* is a centrally symmetric polytope, that is, a polytope P in $\mathbb{R}^n$ containing a point c called the *centre* of P such that $c + x \in P$ if and only if $c - x \in P$, for $x \in \mathbb{R}^n$.

Theorem 1.1 *Suppose that d is a totally split-decomposable metric. Then the following statements hold:*

(i) d is cell-decomposable.

(ii) Every cell in $T(d)$ is a zonotope that is polytope isomorphic either to a hypercube or to the rhombic dodecahedron.

Note that in [12] we showed that an antipodal metric is totally split-decomposable if and only if it is cell-decomposable (in case $|X| \geq 4$), and conjectured that this result held for general metrics. Part (i) of the last theorem implies that the 'only if' direction of this conjecture holds.

The rest of the paper is organized as follows. In the next section we present some preliminaries. In particular, we review the definition and some results concerning a polytopal complex that can be associated to a weighted split system $(\mathcal{S}, \alpha)$ called the *Buneman complex*, and denoted $B(\mathcal{S}, \alpha)$. This complex was introduced in [5] and subsequently used in [9] to deduce several properties of the tight-span of a totally split-decomposable metric. The Buneman complex will be key in proving our main results.

In Section 3 we present some new results concerning the Buneman complex. In the following section we consider a map, which we call κ, that was introduced in [6] to relate $B(\mathcal{S}, \alpha)$ and $T(d_{\mathcal{S},\alpha})$. In particular, in Theorem 4.3 we characterize split systems $\mathcal{S}$ for which $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$ holds for any weighting α on $\mathcal{S}$. We call such split systems *Teutoburgan*. The class of Teutoburgan split systems is rich; for example, as we shall see, weakly compatible split systems and so-called *antipodal* split systems [8] are all Teutoburgan. Theorem 4.3 complements the main result of [6] in which it was proven that $T(d_{\mathcal{S},\alpha}) \subseteq \kappa(B(\mathcal{S}, \alpha))$ if and only if $\mathcal{S}$ is weakly compatible.

In Section 5, we prove a result that will be key in proving our main theorem (Corollary 5.2), which states that if $\mathcal{S}$ is Teutoburgan, then the map κ induces an injection from the set of maximal cells of $B(\mathcal{S}, \alpha)$ into set of the maximal cells of $T(d_{\mathcal{S},\alpha})$. As a consequence of this result and [6, Theorem 3.1], we prove in Section 6 that if $\mathcal{S}$ is weakly compatible, then κ induces a bijection between the set of maximal cells of $B(\mathcal{S}, \alpha)$ and the set of maximal cells of $T(d_{\mathcal{S},\alpha})$. In addition, we present some new characterizations of weakly compatible split systems. In the final section we use our various results together with some results from [11] and [12] to prove Theorem 1.1.

2 Preliminaries

In this section, we review some properties of the Buneman complex and the tight-span. We begin by recalling some basic definitions concerning polytopes and polytopal complexes.

2.1 Polytopal complexes

We follow [15] and [16]. A *polyhedron* in $\mathbb{R}^n$, $n \in \mathbb{N}$, is the intersection of a finite collection of halfspaces in $\mathbb{R}^n$ and a *polytope* is a bounded polyhedron. A *face* of a polyhedron P is the empty-set, P itself, or the intersection of P with a supporting hyperplane and, if $\dim(P) = d$, i.e. P is d-dimensional, then its 0-dimensional faces are called its *vertices*. The collection of all faces of a polytope forms a lattice with respect to the ordering given by set inclusion, and we say that two polytopes are *polytope isomorphic* if their face-lattices are isomorphic. A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra (which we call *cells*) such that each face of a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$, and the intersection of two members of $\mathcal{C}$ is a face of each. If all of the cells in $\mathcal{C}$ are polytopes, we call $\mathcal{C}$ a *polytopal complex*. Given a polyhedral complex $\mathcal{C}$, we will not usually distinguish between $\mathcal{C}$ and its *underlying set* $\bigcup_{C \in \mathcal{C}} C$. For any c in the underlying set of $\mathcal{C}$, we let $[c]$ denote the minimal cell in $\mathcal{C}$ (under inclusion of cells), that contains c. Also, if c is in the underlying set of $\mathcal{C}$ and C is a cell in $\mathcal{C}$ with $C = [c]$, then we say that c is a *generator* of C.

2.2 The Buneman complex

We begin by recalling some further definitions concerning splits and split systems. Recall that X is a finite set. For every proper non-empty subset $A \subseteq X$, we denote the split $\{A, \overline{A}\}$ by S_A. Given a split system $\mathcal{S} \subseteq \mathcal{S}(X)$, we define its underlying set $\mathcal{U}(\mathcal{S})$ by

$$\mathcal{U}(\mathcal{S}) = \bigcup_{S \in \mathcal{S}} S = \{A \subseteq X \mid \text{ there exists } S \in \mathcal{S} \text{ with } A \in S\}.$$

We call two distinct splits $S, S' \in \mathcal{S}(X)$ *compatible* if there exists some $A \in S$ and some $A' \in S'$ with $A \cap A' = \emptyset$, otherwise we call S and S' *incompatible*. We call a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ *incompatible* if every pair of distinct splits in $\mathcal{S}$ is incompatible. We also define any split system with cardinality one to be incompatible.

Now, given any map $\phi : \mathcal{U}(\mathcal{S}) \to \mathbb{R}$, we define

$$supp(\phi) = \{A \in \mathcal{U}(\mathcal{S}) \mid \phi(A) \neq 0\},$$

and put

$$\mathcal{S}(\phi) = \{S \in \mathcal{S} : S \subseteq supp(\phi)\}.$$

Given a weighted split system $(\mathcal{S}, \alpha)$ on X, put

$$H(\mathcal{S}, \alpha) = \{\phi \in \mathbb{R}^{\mathcal{U}(\mathcal{S})} \,:\, \phi(A) \geq 0 \text{ and } \phi(A) + \phi(\overline{A}) = \frac{\alpha_{S_A}}{2} \text{ for all } A \in \mathcal{U}(\mathcal{S})\}.$$

It is straight-forward to check that this is a polytope in $\mathbb{R}^{\mathcal{U}(\mathcal{S})}$ that is polytope isomorphic to an $|\mathcal{S}|$-dimensional hypercube. The subset $B(\mathcal{S}, \alpha)$ of $H(\mathcal{S}, \alpha)$ defined by

$$B(\mathcal{S}, \alpha) = \{\phi \in H(\mathcal{S}, \alpha) \,:\, A_1, A_2 \in supp(\phi) \text{ and } A_1 \cup A_2 = X \Rightarrow A_1 \cap A_2 = \emptyset\}$$

is a polytopal complex called the *Buneman complex* associated to $(\mathcal{S}, \alpha)$. This complex was introduced in [5] – see also [6] (note in the definition that we present for $H(\mathcal{S}, \alpha)$, we have introduced a factor of $\frac{1}{2}$ for scaling purposes).

It can be shown that the map $d_1 : \mathbb{R}^{\mathcal{U}(\mathcal{S})} \times \mathbb{R}^{\mathcal{U}(\mathcal{S})} \to \mathbb{R}^{\geq 0}$ defined, for all $\phi, \phi' \in \mathbb{R}^{\mathcal{U}(\mathcal{S})}$, by

$$d_1(\phi, \phi') = \sum_{A \in \mathcal{U}(\mathcal{S})} |\phi(A) - \phi'(A)|$$

restricts to give a metric on both $H(\mathcal{S}, \alpha)$ and $B(\mathcal{S}, \alpha)$, and that the map $\Phi : X \to B(\mathcal{S}, \alpha)$ defined, for $x \in X$, by

$$\Phi(x) = \phi_x : \mathcal{U}(\mathcal{S}) \to \mathbb{R}^{\geq 0} : A \mapsto \begin{cases} \frac{\alpha_{S_A}}{2} & \text{if } x \notin A, \\ 0 & \text{else,} \end{cases}$$

is an embedding of $(X, d_{\mathcal{S},\alpha})$ into $(B(\mathcal{S}, \alpha), d_1)$ [5, Section 2]. We will make use of the following results:

(B1) [5, Section 2] If $\phi \in B(\mathcal{S}, \alpha)$, then

$$[\phi] = \{\psi \in H(\mathcal{S}, \alpha) \mid supp(\psi) \subseteq supp(\phi)\}.$$

(B2) [5, Lemma 5.2] For all $\mathcal{S}' \subseteq \mathcal{S}$ the (restriction) map

$$B(\mathcal{S}, \alpha) \to B(\mathcal{S}', \alpha) : \phi \mapsto \phi|_{\mathcal{S}'}$$

is surjective.

(B3) Using (B2) it can be shown that if $\mathcal{S}' \subseteq \mathcal{S}$ is a maximal incompatible split system, then there exists a unique maximal cell C in $B(\mathcal{S}, \alpha)$ with $\mathcal{S}(\phi) = \mathcal{S}'$, for any generator ϕ of C.

(B4) Using (B2) it can be shown that if $\phi \in B(\mathcal{S}, \alpha)$ with $[\phi]$ a maximal cell of $B(\mathcal{S}, \alpha)$, then $\mathcal{S}(\phi)$ is a maximal incompatible split system in $\mathcal{S}$.

(B5) [5, Section 2] If C is a cell in $B(\mathcal{S}, \alpha)$ and ϕ is any generator of C, then $\dim(C) = |\mathcal{S}(\phi)|$.

2.3 The tight-span

Suppose that d is a metric on X. Given $f \in P(d)$, we define a graph $K(f)$ with vertex set X and edge set consisting of those subsets $\{x, y\}$ of X with $f(x)+f(y) = d(x, y)$. Proofs for the following statements can be found in [4]:

(TS1) If $f \in T(d)$, then

$$[f] = \{g \in T(d) \, : \, K(f) \subseteq K(g)\}.$$

(TS2) If $f \in P(d)$, then $f \in T(d)$ if and only if for all $x \in X$ there is some $y \in X$ distinct from x with $\{x, y\}$ an edge of $K(f)$.

(TS3) If $f \in T(d)$ and $f(y) = 0$ for some $y \in X$, then $f = h_y$.

3 Gates in the Buneman complex

In this section, we prove some results concerning the Buneman complex. Suppose that $(\mathcal{S}, \alpha)$ is any weighted split system on X. In direct analogy with definition of X-gated cells in the tight-span presented in the introduction, we say that a cell C in the Buneman complex $B(\mathcal{S}, \alpha)$ is *X-gated* if for every $x \in X$ there is a *gate for x in C*, i.e. an element γ in C with

$$d_1(\phi_x, \psi) = d_1(\phi_x, \gamma) + d_1(\gamma, \psi)$$

holding for all $\psi \in C$. Now, for any $x \in X$ and any generator ϕ of C, define

$$\gamma_C^x = \gamma^x : \mathcal{U}(\mathcal{S}) \to \mathbb{R}^{\geq 0} : A \mapsto \begin{cases} \phi_x(A) \text{ if } A \in \mathcal{U}(\mathcal{S}(\phi)), \\ \phi(A) \text{ else.} \end{cases}$$

We first prove that γ^x is a gate for x in C.

Lemma 3.1 *Suppose $(\mathcal{S},\alpha)$ is a weighted split system on X, C is any cell in $B(\mathcal{S},\alpha)$, and $\phi \in B(\mathcal{S},\alpha)$ is any generator of C. Then the following statements hold.*
(i) If $A \in \mathcal{U}(\mathcal{S}-\mathcal{S}(\phi))$ and $\psi \in C$, then $\phi(A) \in \{0, \frac{\alpha_{S_A}}{2}\}$ and $\psi(A)=\phi(A)$.
(ii) For any $x \in X$, the map γ^x defined above is a gate for x in C.
(iii) For all $x, y \in X$,

$$d_1(\gamma^x,\gamma^y) = \sum_{S\in\mathcal{S}(\phi)} \alpha_S \delta_S(x,y).$$

Proof: Suppose $x, y \in X$, C is a cell in $B(\mathcal{S},\alpha)$, and $\phi \in B(\mathcal{S},\alpha)$ is a generator of C. It is straight-forward to see that (i) – (iii) all hold in case C is a vertex. So, without loss of generality, we will assume $\dim(C) > 0$. In particular, by (B5) $\mathcal{S}(\phi) \neq \emptyset$.
(i): Suppose $A \in \mathcal{U}(\mathcal{S}-\mathcal{S}(\phi))$. Then $|\{A,\overline{A}\} \cap supp(\phi)| = 1$, by the definition of $\mathcal{S}(\phi)$. Hence $\phi(A) \in \{0, \frac{\alpha_{S_A}}{2}\}$, as $\phi \in H(\mathcal{S},\alpha)$. Now suppose $\psi \in C$. By (B1) it follows that $|\{A,\overline{A}\} \cap supp(\psi)| \leq |\{A,\overline{A}\} \cap supp(\phi)| = 1$ and so, again by (B1), $\psi(A)=\phi(A)$.
(ii): By (B1) and the definition of γ^x, ϕ_x, and $\mathcal{S}(\phi)$, it follows that γ^x is contained in C. Now suppose ψ is any element of C. By (i) and the definition of γ^x,

$$\begin{aligned} d_1(\phi_x,\psi) &= \sum_{A\in\mathcal{U}(\mathcal{S})} |\phi_x(A)-\psi(A)| \\ &= \sum_{A\in\mathcal{U}(\mathcal{S}-\mathcal{S}(\phi))} |\phi_x(A)-\psi(A)| + \sum_{A\in\mathcal{U}(\mathcal{S}(\phi))} |\phi_x(A)-\psi(A)| \\ &= \sum_{A\in\mathcal{U}(\mathcal{S}-\mathcal{S}(\phi))} |\phi_x(A)-\gamma^x(A)| + \sum_{A\in\mathcal{U}(\mathcal{S}(\phi))} |\gamma^x(A)-\psi(A)| \\ &= d_1(\phi_x,\gamma^x)+d_1(\gamma^x,\psi). \end{aligned}$$

Hence γ^x is a gate for x in C.
(iii): By (i), (ii), and the definition of ϕ_z and γ^z, $z \in X$,

$$\begin{aligned} d_1(\gamma^x,\gamma^y) &= \sum_{A\in\mathcal{U}(\mathcal{S}(\phi))} |\phi_x(A)-\phi_y(A)| \\ &= \sum_{S\in\mathcal{S}(\phi)} \phi_y(S(x)) + \sum_{S\in\mathcal{S}(\phi)} |\frac{\alpha_S}{2} - \phi_y(\overline{S}(x))| \\ &= \sum_{S\in\mathcal{S}(\phi)} \alpha_S\delta_S(x,y). \end{aligned}$$

■

In view of the last lemma, it follows that the Buneman complex $B(\mathcal{S},\alpha)$ is X-*gated* i.e., every cell in $B(\mathcal{S},\alpha)$ is X-gated, for all weightings α on $\mathcal{S}$.

Now, for C a cell of $B(\mathcal{S},\alpha)$ with $\dim(C) > 0$, let

$$\Gamma(C) = \{\gamma_C^x : x \in X\},$$

noting that this is clearly a finite set. For later use, we now consider what can be deduced for cells C of the Buneman complex for which the metric space $(\Gamma(C), d_1|_{\Gamma(C)})$ is antipodal (see the introduction for the definition of this term). In this case, we will say that C is *antipodal* X-*gated* and, for $x, y \in X$, we call γ^x the *antipode* of γ^y in C if γ^x is the antipode of γ^y in $(\Gamma(C), d_1|_{\Gamma(C)})$.

Proposition 3.2 *Suppose* $(\mathcal{S},\alpha)$ *is a weighted split system on* X*,* C *is a cell in* $B(\mathcal{S},\alpha)$ *with* $\dim(C) > 0$*, and* ϕ *is any generator of* C*. Suppose in addition that* $(\Gamma(C), d_1|_{\Gamma(C)})$ *is antipodal, and* $x, y \in X$ *distinct. Then the following statements are equivalent.*
(i) γ^x *is the antipode of* γ^y *in* C*.*
(ii) $S(x) \neq S(y)$ *for all* $S \in \mathcal{S}(\phi)$*.*
(iii) $d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \psi) + d_1(\psi, \gamma^y)$ *for all* $\psi \in C$*.*
(iv) For all $S \in \mathcal{S}(\phi)$ *and all* $\psi \in C$*,*
$\alpha_S = \sum_{A \in S} |\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)|$.
(v) $d_1(\gamma^x, \gamma^y) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S$.

Proof: (i) ⇒ (ii): Suppose γ^x is the antipode of γ^y in C, and that there is some $S_0 \in \mathcal{S}(\phi)$ with $S_0(x) = S_0(y)$. Let $z \in \overline{S_0}(x)$. Since γ^x is the antipode of γ^y,

$$d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \gamma^z) + d_1(\gamma^z, \gamma^y),$$

and so, by Lemma 3.1, $\alpha_S \delta_S(x, y) = \sum_{A \in S} |\phi_x(A) - \phi_z(A)| + |\phi_z(A) - \phi_y(A)|$, for all $S \in \mathcal{S}(\phi)$. Thus,

$$\begin{aligned} 0 &= \alpha_{S_0} \delta_{S_0}(x, y) \\ &= \sum_{A \in S_0} |\phi_x(A) - \phi_z(A)| + |\phi_z(A) - \phi_y(A)| \\ &= 2(|\phi_x(S_0(x)) - \phi_z(S_0(x))| + |\phi_z(S_0(x)) - \phi_y(S_0(x))|) \\ &= 4\phi_z(S_0(x)), \end{aligned}$$

and so $\phi_z(S_0(x)) = 0$. Thus $z \in S_0(x)$, a contradiction.
(ii) $\Rightarrow$ (iv): Suppose $S \in \mathcal{S}(\phi)$ and $\psi \in C$. Then

$$\begin{aligned}\alpha_S &= \psi(S(x)) + \frac{\alpha_S}{2} - \psi(S(x)) + \frac{\alpha_S}{2} - \psi(\overline{S}(x)) + \psi(\overline{S}(x)) \\ &= \sum_{A \in S} |\phi_x(A) - \psi(A)| + |\phi_y(A) - \psi(A)|.\end{aligned}$$

(iv) $\Rightarrow$ (iii): Suppose $\psi \in C$. By Lemma 3.1 (i), $\gamma^x(A) = \gamma^y(A) = \phi(A) = \psi(A)$ holds for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. (iii) now follows.
(iii) $\Rightarrow$ (i): This is clear since $\Gamma(C) \subseteq C$.
(ii) $\Rightarrow$ (v): This follows by Lemma 3.1(iii).
(v) $\Rightarrow$ (iv): Suppose $\psi \in C$ and that there exists some $S \in \mathcal{S}(\phi)$ with $S(x) = S(y)$. Then $\sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\phi_x(A) - \phi_y(A)| = d_1(\gamma^x, \gamma^y) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S$, and so there must be some $S' \in \mathcal{S}(\phi)$ with

$$\alpha_{S'} < \sum_{A \in S'} |\phi_x(A) - \phi_y(A)| = 2|\phi_x(S(x)) - \phi_y(S(x))| = \alpha_{S'},$$

which is impossible. ∎

Corollary 3.3 *Suppose that the conditions stated in the last proposition all hold and that in addition the cell C is maximal. Then the following statements hold.*
(i) If $S(x) \neq S(y)$ for all $S \in \mathcal{S}(\phi)$, then $\phi(S'(x)) = 0$ for all $S' \in \mathcal{S} - \mathcal{S}(\phi)$ with $S'(x) = S'(y)$.
(ii) $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$ if and only if γ^x is the antipode of γ^y in C.
(iii) Suppose $x_1, x_2, y_1, y_2 \in X$ and that the antipode of γ^{y_i} in C is γ^{x_j} for all $i, j \in \{1, 2\}$. Then $d_{\mathcal{S},\alpha}(x_1, y_1) + d_{\mathcal{S},\alpha}(x_2, y_2) = d_{\mathcal{S},\alpha}(x_1, y_2) + d_{\mathcal{S},\alpha}(x_2, y_1)$.

Proof: (i): Suppose $S' \in \mathcal{S} - \mathcal{S}(\phi)$ with $S'(x) = S'(y)$. Since $\mathcal{S}(\phi)$ is maximal incompatible by (B4), there must exist some $S \in \mathcal{S}(\phi)$ which is compatible with S'. As $y \in S'(x)$ and $S(x) \neq S(y)$ by assumption, either $S(x) \cup S'(x) = X$ or $S(y) \cup S'(x) = X$. Since $S(x), S(y) \in supp(\phi)$ and $S(x) \cap S'(x) \neq \emptyset \neq S(y) \cap S'(y) = S(y) \cap S'(x)$, it follows that $\phi(S'(x)) = 0$.

(ii): Suppose $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$. Then clearly $d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \phi) + d_1(\phi, \gamma^y)$. Hence, by Proposition 3.2, γ^x is the antipode of γ^y in C.

Conversely, suppose that γ^x is the antipode of γ^y in C. By (i)

$$\alpha_S \delta_S(x, y) = 2(\phi(S(x)) + |\phi_y(S(x)) - \phi(S(x))|)$$

for all $S \in \mathcal{S} - \mathcal{S}(\phi)$. Now using Lemma 3.1 and Proposition 3.2, it is straight-forward to check that

$$\begin{aligned} d_1(\phi_x, \phi_y) &= \sum_{S \in \mathcal{S}} \alpha_S \delta_S(x, y) \\ &= \sum_{S \in \mathcal{S}(\phi)} \alpha_S + \sum_{S \in \mathcal{S} - \mathcal{S}(\phi)} \alpha_S \delta_S(x, y) \\ &= d_1(\gamma^x, \gamma^y) + \sum_{S \in \mathcal{S} - \mathcal{S}(\phi)} 2(\phi(S(x)) + |\phi_y(S(x)) - \phi(S(x))|) \\ &= d_1(\gamma^x, \gamma^y) + \sum_{A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))} (|\phi_x(A) - \phi(A)| + |\phi_y(A) - \phi(A)|) \\ &= d_1(\gamma^x, \gamma^y) + d_1(\phi_x, \gamma^x) + d_1(\gamma^y, \phi_y) \end{aligned}$$

holds. But by Proposition 3.2, $d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \phi) + d_1(\phi, \gamma^y)$. It immediately follows that $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$.

(iii): Using (i) and Proposition 3.2 it is straight-forward to show that

$$d_{\mathcal{S},\alpha}(x_i, y_j) = d_1(\phi_{x_i}, \gamma^{x_i}) + d_1(\gamma^{x_i}, \gamma^{y_j}) + d_1(\gamma^{y_j}, \phi_{y_j})$$

holds for all $i, j \in \{1, 2\}$. But by uniqueness of gates, $\gamma^{x_1} = \gamma^{x_2}$ and $\gamma^{y_1} = \gamma^{y_2}$ and (iii) now easily follows. ∎

4 Teutoburgan split systems

Given a weighted split system $(\mathcal{S}, \alpha)$ on X, define a map

$$\kappa : \mathbb{R}^{\mathcal{U}(\mathcal{S})} \to \mathbb{R}^X : \phi \mapsto (X \to \mathbb{R} : x \mapsto d_1(\phi, \phi_x)).$$

The map κ was originally introduced in [6][1]. Note that it immediately follows from this definition that $\kappa(B(\mathcal{S}, \alpha)) \subseteq P(d_{\mathcal{S},\alpha})$ and that, by (TS3), $\kappa(\phi_x) =$

[1] In [6] this map is denoted by λ. Since our definition of κ is slightly different from the map λ presented in [6], we use κ as opposed to λ to prevent confusion. It can be easily checked that the results stated in [6] concerning λ also hold for κ.

h_x, for all $x \in X$. Moreover, using the fact that, for all $\phi \in B(\mathcal{S}, \alpha)$ and $x, y \in X$,

$$\kappa(\phi)(x) = 2 \sum_{A \in \mathcal{U}(\mathcal{S}),\ y \in A} |\phi(A) - \phi_x(A)|, \tag{2}$$

it is straight-forward to check that κ induces a non-expanding map from $B(\mathcal{S}, \alpha)$ to $P(d_{\mathcal{S},\alpha})$, i. e.

$$d_\infty(\kappa(\phi), \kappa(\psi)) \leq d_1(\phi, \psi)$$

holds for all $\phi, \psi \in B(\mathcal{S}, \alpha)$. In this section we will characterize those split systems $\mathcal{S}$ of X for which $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$ holds for any weighting α on $\mathcal{S}$.

We begin by proving two useful lemmas. Abusing notation, to any $\phi \in B(\mathcal{S}, \alpha)$ associate the graph $K(\phi)$ which has vertex set X and edge set consisting of those subsets $\{x, y\}$ of X with $d_1(\phi_x, \phi_y) = d_1(\phi_x, \phi) + d_1(\phi, \phi_y)$. It is straight-forward to check that $\{x, y\}$ is an edge of $K(\phi)$ if and only if $\{x, y\}$ is an edge of $K(\kappa(\phi))$.

Lemma 4.1 *Let $(\mathcal{S}, \alpha)$ be a weighted split system on X. Suppose C is a maximal cell in $B(\mathcal{S}, \alpha)$, ϕ is any generator of C, and $(\Gamma(C), d_1|_{\Gamma(C)})$ is antipodal. Then, for $x, y \in X$ distinct, the following statements are equivalent.*

(i) γ^x is the antipode of γ^y in C.

(ii) $\{x, y\} \subseteq X$ is an edge of $K(\kappa(\phi))$ or – equivalently – of $K(\phi)$.

(iii) $\kappa(\phi_x), \kappa(\gamma^x), \kappa(\phi), \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $P(d_{\mathcal{S},\alpha})$.

Proof: (i) $\Rightarrow$ (iii): Suppose γ^x is the antipode of γ^y in C. By Corollary 3.3(ii), $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$. Since κ is non-expanding, it immediately follows that $\kappa(\phi_x), \kappa(\gamma^x), \kappa(\phi), \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $P(d_{\mathcal{S},\alpha})$.

(iii) $\Rightarrow$ (ii): Suppose $\kappa(\phi_x), \kappa(\gamma^x), \kappa(\phi), \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $P(d_{\mathcal{S},\alpha})$. Then clearly

$$d_\infty(\kappa(\phi_x), \kappa(\phi)) + d_\infty(\kappa(\phi), \kappa(\phi_y)) = d_\infty(\kappa(\phi_x), \kappa(\phi_y)).$$

But, for any $z \in X$, $\kappa(\phi_z) = h_z$ and so $d_\infty(\kappa(\phi_z), \kappa(\phi)) = d_\infty(h_z, \kappa(\phi)) = \kappa(\phi)(z)$. (ii) now follows immediately.

(ii) $\Rightarrow$ (i): Suppose $\{x, y\}$ is an edge of $K(\kappa(\phi))$. Then $d_1(\phi_x, \phi_y) = d_1(\phi_x, \phi) + d_1(\phi, \phi_y)$. Since γ^x and γ^y are gates in C for x and y, respectively, it immediately follows that $\phi_x, \gamma_C^x, \phi, \gamma_C^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$. Hence, by Corollary 3.3(ii), γ_C^x is the antipode of γ_C^y in C. ∎

A split system $\mathcal{S} \subseteq \mathcal{S}(X)$ is called *antipodal* if for all $x \in X$ there exists some $y \in X$ such that $S(x) \neq S(y)$ holds for all $S \in \mathcal{S}$. Such split systems were studied in [8]. We now relate them to antipodal X-gated cells in the Buneman complex.

Lemma 4.2 *Suppose $C \subseteq B(\mathcal{S}, \alpha)$ is a cell with $\dim(C) > 0$ and ϕ is a generator of C. Then the following statements are equivalent.*

(i) C is antipodal X-gated.

(ii) $\mathcal{S}(\phi)$ is antipodal.

Proof: (i) $\Rightarrow$ (ii): This follows immediately from Proposition 3.2.
(ii) $\Rightarrow$ (i): Suppose $x \in X$. Since $\mathcal{S}(\phi)$ is antipodal by assumption, there is some $y \in X$ with $S(x) \neq S(y)$ holding for all $S \in \mathcal{S}(\phi)$. Note that if $y' \in X$ distinct from y with $S(x) \neq S(y')$ for all $S \in \mathcal{S}(\phi)$ then $S(y) = S(y')$ and so $\gamma^y = \gamma^{y'}$ follows by the definition of γ^y and $\gamma^{y'}$. Hence, the map which takes, for any $u \in X$, the gate γ^u to γ^v with $v \in \bigcap_{S \in \mathcal{S}(\phi)} \overline{S}(u)$ is a well-defined involution on $\Gamma(C)$. Moreover, for all $z \in X$ and all $A \in \mathcal{U}(\mathcal{S}(\phi))$,

$$|\phi_x(A) - \phi_y(A)| = |\phi_x(A) - \phi_z(A)| - |\phi_z(A) - \phi_y(A)|$$

and hence, by Lemma 3.1 (i),

$$d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \gamma^z) + d_1(\gamma^z, \gamma^y).$$

Thus $(\Gamma(C), d_1|_{\Gamma(C)})$ is antipodal and, therefore, C is antipodal X-gated. ∎

We now give the characterization promised above.

Theorem 4.3 *Suppose that $\mathcal{S}$ is a split system on X. Then the following statements are equivalent:*

(i) Every maximal incompatible split system in $\mathcal{S}$ is antipodal.

(ii) *For every weighting* $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$*, every maximal cell in* $B(\mathcal{S}, \alpha)$ *is antipodal* X*-gated.*

(ii') *For some weighting* $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$*, every maximal cell in* $B(\mathcal{S}, \alpha)$ *is antipodal* X*-gated.*

(iii) *For every weighting* $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$*, every cell in* $B(\mathcal{S}, \alpha)$ *with non-zero dimension is antipodal* X*-gated.*

(iii') *For some weighting* $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$*, every cell in* $B(\mathcal{S}, \alpha)$ *with non-zero dimension is antipodal* X*-gated.*

(iv) *For every weighting* $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$*,* $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$*.*

(iv') *For some weighting* $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$*,* $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$*.*

Proof: We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (ii') $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iii') $\Rightarrow$ (ii'), and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iv') $\Rightarrow$ (ii').

The implications (ii) $\Rightarrow$ (ii'), (iii) $\Rightarrow$ (iii'), (iv) $\Rightarrow$ (iv'), and (iii') $\Rightarrow$ (ii') clearly all hold.
(i) $\Rightarrow$ (ii): Suppose that $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, and C is a maximal cell in $B(\mathcal{S}, \alpha)$ with generator ϕ. By (B4), $\mathcal{S}(\phi)$ is maximal incompatible and so $\mathcal{S}(\phi)$ must be antipodal, by assumption. Thus, by Lemma 4.2, C is antipodal X-gated.
(ii) $\Rightarrow$ (iii): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, C is a cell in $B(\mathcal{S}, \alpha)$ with $\dim(C) > 0$, and D is any maximal cell containing C. Let ϕ and ψ be generators of C and D, respectively. By assumption, D is antipodal X-gated, and so for any $x \in X$ there exists some $y \in X$ with γ_D^y the antipode of γ_D^x in D. By Proposition 3.2, $S(x) \neq S(y)$ for all $S \in \mathcal{S}(\psi)$. By (B1), $\mathcal{S}(\phi) \subseteq \mathcal{S}(\psi)$ and so $\mathcal{S}(\phi)$ is antipodal. (iii) now follows by Lemma 4.2.
(ii') $\Rightarrow$ (i): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting so that every maximal cell in $B(\mathcal{S}, \alpha)$ is antipodal X-gated. Suppose $\mathcal{S}' \subseteq \mathcal{S}$ is a maximal incompatible split system. Then, by (B3), $\mathcal{S}' = \mathcal{S}(\phi)$ where $\phi \in B(\mathcal{S}, \alpha)$ is a generator of some maximal cell in $B(\mathcal{S}, \alpha)$. Since $[\phi]$ is antipodal X-gated by assumption, $\mathcal{S}'$ is antipodal by Lemma 4.2.
(ii) $\Rightarrow$ (iv): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, C is a maximal cell in $B(\mathcal{S}, \alpha)$, ϕ is a generator of C, and $x \in X$. Then there must exist some $y \in X$ distinct from x with γ^y the antipode of γ^x in C. By Lemma 4.1, $\{x, y\}$ is an edge of $K(\kappa(\phi))$. Since $\kappa(\phi) \in P(d_{\mathcal{S},\alpha})$, (TS2) implies $\kappa(\phi) \in T(d_{\mathcal{S},\alpha})$. By (TS1), it follows that $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$.

(iv') ⇒ (ii'): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting with $\kappa(B(\mathcal{S},\alpha)) \subseteq T(d_{\mathcal{S},\alpha})$. Let C be a maximal cell in $B(\mathcal{S},\alpha)$ with generator ϕ, and let $x \in X$. Since $\kappa(\phi) \in T(d_{\mathcal{S},\alpha})$, by (TS2) there is some $y \in X$ distinct from x with $\{x, y\}$ an edge of $K(\kappa(\phi))$. Hence, for all $S \in \mathcal{S}(\phi)$, we must have $S(x) \neq S(y)$ since, otherwise, if there were some $S \in \mathcal{S}(\phi)$ with $S(x) = S(y)$ then

$$0 = \alpha_S \delta_S(x,y) = \sum_{A \in S} |\phi_x(A) - \phi(A)| + |\phi_y(A) - \phi(A)| = 4\phi(S(x)),$$

which is impossible since $S \in \mathcal{S}(\phi)$. Thus $\mathcal{S}(\phi)$ is antipodal, and so, by Lemma 4.2, C is antipodal X-gated. ∎

We call a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ *Teutoburgan* if every maximal incompatible subset of splits in $\mathcal{S}$ is antipodal. In view of the last theorem, this definition is equivalent to the one presented in the introduction. Since every weakly compatible, yet incompatible split system is antipodal [8], it immediately follows that every weakly compatible split system is Teutoburgan. Note, however, that a Teutoburgan split system is not necessarily weakly compatible (e.g. take the split system of cardinality 3 on the set of vertices of a 3-cube induced by removing collections of parallel edges).

Remark 4.4 *If $(\mathcal{S},\alpha)$ is a weighted split system for which the map $\Phi : X \to B(\mathcal{S},\alpha)$ maps X surjectively onto the set of vertices of $B(\mathcal{S},\alpha)$, then it is straight-forward to check that $\mathcal{S}$ is Teutoburgan. Moreover, it can be shown that such a split system can be associated to any* median graph *(by taking X to be the vertex set of the graph, and $\mathcal{S}$ to be the split system induced by the 'parallel classes' of edges of the median graph). This provides a large additional class of Teutoburgan split systems.*

5 Maximal cells of the tight-span

In this section we shall show that if $\mathcal{S}$ is a Teutoburgan split system then, for any weighting α on $\mathcal{S}$, κ induces an injective map from the set of maximal cells of $B(\mathcal{S},\alpha)$ into the set of maximal cells of $T(d_{\mathcal{S},\alpha})$. This is essentially a consequence of the following result.

Theorem 5.1 *Let $(\mathcal{S},\alpha)$ be a weighted split system on X with $\mathcal{S}$ Teutoburgan. Suppose C is a maximal cell of $B(\mathcal{S},\alpha)$, and ϕ is any generator of C. Then the following statements hold.*

(i) For all $x \in X$, $\kappa(\gamma^x) \in [\kappa(\phi)]$.

(ii) $\kappa(\phi)$ is a generator of a maximal cell of $T(d_{\mathcal{S},\alpha})$.

(iii) Suppose $\psi \in B(\mathcal{S}, \alpha)$. Then $\psi \in C$ if and only if $\kappa(\psi) \in [\kappa(\phi)]$.

Proof: (i): Suppose $\{u, v\}$ is an edge of $K(\kappa(\phi))$ with $u \neq v$, which exists by (TS2). Since $\mathcal{S}$ is Teutoburgan, by Lemma 4.1 γ^u is the antipode of γ^v in C. By Proposition 3.2 and Corollary 3.3(ii), $\phi_u, \gamma^u, \gamma^z, \gamma^v, \phi_v$ is a geodesic in $B(\mathcal{S}, \alpha)$. Hence, since γ^u and γ^v are gates in C for u and v, respectively,

$$d_1(\phi_u, \phi_v) = d_1(\phi_u, \gamma^z) + d_1(\gamma^z, \phi_v) = \kappa(\gamma^z)(u) + \kappa(\gamma^z)(v).$$

Hence $\{u, v\}$ is an edge of $K(\kappa(\gamma^z))$. Thus, $K(\kappa(\phi)) \subseteq K(\kappa(\gamma^z))$, and so by (TS1) $\kappa(\gamma^z) \in [\kappa(\phi)]$.
(ii): By Theorem 4.3 $[\kappa(\phi)]$ is a cell of $T(d_{\mathcal{S},\alpha})$. Suppose that $[\kappa(\phi)]$ is not maximal. Then there exists some $f \in T(d_{\mathcal{S},\alpha})$ with $[\kappa(\phi)] \subsetneq [f]$. By (TS1), $K(f) \subsetneq K(\kappa(\phi))$ and so there exist $x_1, y_1 \in X$ with $\{x_1, y_1\}$ an edge of $K(\kappa(\phi))$ but not of $K(f)$. Note that $x_1 \neq y_1$. For, if not, then $\kappa(\phi) = h_{x_1}$ by (TS3), and, taking γ^z to be the antipode of γ^{x_1} in C, for $z \in X$ (which exists by Theorem 4.3), by (i) we obtain $\kappa(\gamma^z) = \kappa(\phi) = h_{x_1}$. So $\kappa(\gamma^z)(x_1) = 0$, which is impossible because, since γ^z is the antipode of γ^{x_1} in C,

$$d_1(\gamma^z, \phi_{x_1}) \geq \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\gamma^z(A) - \phi_{x_1}(A)| = \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\gamma^z(A) - \gamma^{x_1}(A)| > 0.$$

Now define

$$\begin{aligned} Z &= \{z \in X \mid \gamma_C^z = \gamma_C^{y_1}\}, \text{ and} \\ Y &= \{z \in X \mid \gamma_C^z \text{ is the antipode of } \gamma_C^{y_1} \text{ in } C\}. \end{aligned}$$

Clearly, $y_1 \in Z$ and, by Lemma 4.1, $x_1 \in Y$. Since $f \in T(d_{\mathcal{S},\alpha})$ and $\{x_1, y_1\}$ is not an edge of $K(f)$, by (TS2) there exist $x_2, y_2 \in X$ with $x_1 \neq y_2$ and $x_2 \neq y_1$ such that $\{x_1, y_2\}$ and $\{x_2, y_1\}$ are edges of $K(f)$. Since $K(f) \subseteq K(\kappa(\phi))$, Lemma 4.1 implies $x_2 \in Y$ and $y_2 \in Z$. Hence, by Corollary 3.3 (iii), $d_{\mathcal{S},\alpha}(x_1, y_1) + d_{\mathcal{S},\alpha}(x_2, y_2) = d_{\mathcal{S},\alpha}(x_1, y_2) + d_{\mathcal{S},\alpha}(x_2, y_1)$. But then

$$\begin{aligned} f(y_1) + f(x_2) + f(y_2) + f(x_1) &= d_{\mathcal{S},\alpha}(y_1, x_2) + d_{\mathcal{S},\alpha}(y_2, x_1) \\ &= d_{\mathcal{S},\alpha}(y_1, x_1) + d_{\mathcal{S},\alpha}(y_2, x_2) \\ &\leq f(y_1) + f(x_2) + f(y_2) + f(x_1), \end{aligned}$$

and so $d_{\mathcal{S},\alpha}(y_1, x_1) = f(y_1) + f(x_1)$ and $d_{\mathcal{S},\alpha}(y_2, x_2) = f(y_2) + f(x_2)$. Hence, $\{x_1, y_1\}$ is an edge of $K(f)$ which is a contradiction.
(iii): Suppose $\psi \in [\phi]$ and let $\{x, y\}$ be an edge of $K(\kappa(\phi))$. By Proposition 3.2, Corollary 3.3(ii), and Theorem 4.3 $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ and $\phi_x, \gamma^x, \psi, \gamma^y, \phi_y$ are geodesics in $B(\mathcal{S}, \alpha)$. Thus, since γ^x and γ^y are gates in C, for x and y respectively,

$$d_{\mathcal{S},\alpha}(x, y) = \kappa(\phi)(x) + \kappa(\phi)(y) = \kappa(\psi)(x) + \kappa(\psi)(y)$$

Hence, $\{x, y\}$ is an edge of $K(\kappa(\psi))$. Thus, by (TS1) $\kappa(\psi) \in [\kappa(\phi)]$.

Conversely, suppose $\kappa(\psi) \in [\kappa(\phi)]$. We can assume $\mathcal{S}(\phi) \neq \mathcal{S}$ since otherwise $supp(\psi) \subseteq \mathcal{U}(\mathcal{S}) = supp(\phi)$ and so $\psi \in [\phi]$. We first claim that if $S \in \mathcal{S} - \mathcal{S}(\phi)$, then there exist elements $x, y \in X$ with $S(x) = S(y)$ and γ^x the antipode of γ^y in $[\phi]$. Indeed, suppose $S \in \mathcal{S} - \mathcal{S}(\phi)$. By (B4), $\mathcal{S}(\phi)$ is a maximal incompatible split system in $\mathcal{S}$, and so there exists some $S' \in \mathcal{S}(\phi)$ with S' and S compatible. Hence there exists some $x \in X$ with $S(x) \cup S'(x) = X$. Since $[\phi]$ is antipodal X-gated by Theorem 4.3, there exists some $y \in X$ with γ^x is the antipode of γ^y in $[\phi]$. By Proposition 3.2, $y \notin S'(x)$ and so $y \in S(x)$. Hence, $S(x) = S(y)$, which completes the proof of the first claim.

We now claim that $\phi(A) = \psi(A)$ holds for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. Suppose $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. Put $S_0 = S_A$. Then, by the claim just above, there exist elements $x, y \in X$ with $S_0(x) = S_0(y)$ and γ^x the antipode of γ^y in $[\phi]$. Hence, by Proposition 3.2 and Corollary 3.3(i), $\phi(S_0(x)) = 0$, and, by Lemma 4.1, $\{x, y\}$ is an edge of $K(\kappa(\phi)) \subseteq K(\kappa(\psi))$. Thus, $d_1(\phi_x, \psi) + d_1(\psi, \phi_y) = d_{\mathcal{S},\alpha}(x, y)$. Since for all $S \in \mathcal{S}$

$$\alpha_S \delta_S(x, y) \leq \sum_{A \in S} |\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)|,$$

it follows that $0 = \alpha_{S_0} \delta_{S_0}(x, y) = \sum_{A \in S_0} |\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)|$. Thus, $\phi_x(A) = \psi(A)$, for all $A \in S_0$, and so $\psi(S_0(x)) = 0$. In particular, it follows that $\phi(A) = \psi(A)$ holds for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$ which concludes the proof of the claim. Using (B1), it is now straight-forward to conclude that $\psi \in [\phi]$. ∎

In view of the last theorem it follows that the map $\kappa' = \kappa'_{\mathcal{S},\alpha}$ defined by taking any maximal cell C in $B(\mathcal{S}, \alpha)$ to the cell $[\kappa(\phi)]$, where ϕ is any generator of C, is a well-defined map from the set of maximal cells of $B(\mathcal{S}, \alpha)$ to the set of maximal cells of $T(d_{\mathcal{S},\alpha})$. Moreover, we have

Corollary 5.2 *If $(\mathcal{S}, \alpha)$ is a weighted split system on X with $\mathcal{S}$ Teutoburgan, then the map κ' defined above is injective.*

Proof: Suppose that C and C' are maximal cells in $B(\mathcal{S}, \alpha)$ with $\kappa'(C) = \kappa'(C')$. Let ϕ and ϕ' be generators for C and C', respectively. Then $[\kappa(\phi)] = [\kappa(\phi')]$. Hence $\kappa(\phi) \in [\kappa(\phi')]$ and so $\phi \in [\phi']$ by Theorem 5.1 (iii). Thus $[\phi] \subseteq [\phi']$ by (TS1). Interchanging the roles of ϕ and ϕ' yields $[\phi'] \subseteq [\phi]$. Therefore $C = [\phi] = [\phi'] = C'$. Hence κ' is injective. ∎

6 Totally split-decomposable metrics

For $(\mathcal{S}, \alpha)$ a weighted split system on X, by the main result of [6] $\kappa(B(\mathcal{S}, \alpha)) = T(d_{\mathcal{S},\alpha})$ if and only if $\mathcal{S}$ is weakly compatible. We now use this fact to prove that in case $\mathcal{S}$ is a weakly compatible split system, the map κ' defined at the end of the last section is a bijection.

Theorem 6.1 *Let $(\mathcal{S}, \alpha)$ be a weighted split system on X. If $\mathcal{S}$ is weakly compatible, then the map κ' is a bijection between the set of maximal cells of $B(\mathcal{S}, \alpha)$ and the set of maximal cells of $T(d_{\mathcal{S},\alpha})$.*

Proof: Since any weakly compatible split system is Teutoburgan, by Corollary 5.2 it follows that the map κ' is injective. Hence it suffices to prove that κ' is surjective.

To this end, suppose that Z is a maximal cell in $T(d_{\mathcal{S},\alpha})$. Let h be any generator of Z. Since $\mathcal{S}$ is weakly compatible κ maps $B(\mathcal{S}, \alpha)$ onto $T(d_{\mathcal{S},\alpha})$ [6]. Hence, there must be some $\psi \in B(\mathcal{S}, \alpha)$ with $\kappa(\psi) = h$. Suppose C is a maximal cell in $B(\mathcal{S}, \alpha)$ which contains ψ, and let ϕ be a generator of C. Since $\mathcal{S}$ is Teutoburgan, $[\kappa(\phi)]$ is a maximal cell in $T(d_{\mathcal{S},\alpha})$ by Theorem 5.1(ii). So $h = \kappa(\psi) \in [\kappa(\phi)]$ by Theorem 5.1(iii), and thus, by (TS1), $Z = [h] \subseteq [\kappa(\phi)]$. But Z is maximal, and so $Z = [\kappa(\phi)]$. Thus κ' is surjective. ∎

We conclude this section by giving some new characterizations of weakly compatible split systems (see [6] for some further characterizations). Given a metric d on a finite set Y, define the *underlying graph* $UG(Y, d)$ to be the graph with vertex set Y and edge set consisting of those subsets $\{x, y\} \subseteq Y$ for which there is no $z \in Y$ distinct from x and y with $d(x, y) = d(x, z) +$

$d(z, y)$. In addition, define a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ to be *3-cube-free* if for all 3-subsets $\{S_1, S_2, S_3\} \subseteq \mathcal{S}$ there exists $A_k \in S_k$ for $k = 1, 2, 3$ with $A_1 \cap A_2 \cap A_3 = \emptyset$.

Theorem 6.2 *Suppose that $\mathcal{S} \subseteq \mathcal{S}(X)$ is a Teutoburgan split system. Then the following statements are equivalent.*

(i) $\mathcal{S}$ is weakly compatible.

(ii) $\mathcal{S}$ is 3-cube-free.

(iii) for every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S}, \alpha)$ with $\dim(C) = 3$, then $|\Gamma(C)| \leq 6$.

(iii') for some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S}, \alpha)$ with $\dim(C) = 3$, then $|\Gamma(C)| \leq 6$.

(iv) for every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S}, \alpha)$ with $\dim(C) \neq 0$, then $d_1|_{\Gamma(C)}$ is totally split-decomposable.

(iv') for some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S}, \alpha)$ with $\dim(C) \neq 0$, then $d_1|_{\Gamma(C)}$ is totally split-decomposable.

Proof: Clearly (iii) $\Rightarrow$ (iii') and (iv) $\Rightarrow$ (iv').
(i) $\Rightarrow$ (iv): Suppose $\mathcal{S}$ is weakly compatible, $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, C is a cell of $B(\mathcal{S}, \alpha)$ with $\dim(C) > 0$, and ϕ is a generator of C. Then, by Lemma 3.1(iii),

$$d_1(\gamma^x, \gamma^y) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S \delta_S(x, y)$$

for all $x, y \in X$. Since $\mathcal{S}(\phi) \subseteq \mathcal{S}$ and $\mathcal{S}$ is weakly compatible, $\mathcal{S}(\phi)$ is weakly compatible, and hence $d_1|_{\Gamma(C)}$ is totally split-decomposable.
(iv) $\Rightarrow$ (iii): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, and that there is some 3-dimensional cell C in $B(\mathcal{S}, \alpha)$ with $|\Gamma(C)| \geq 7$. By Theorem 4.3 C is antipodal X-gated, and, since C has eight vertices, $|\Gamma(C)| = 8$. Suppose ϕ is a generator of C and $x_0 \in X$. Then, by (B5), $|\mathcal{S}(\phi)| = 3$. Put $\mathcal{S}(\phi) = \{S_1, S_2, S_3\}$. Since each vertex of C is a gate, $\bigcap_{i=1}^{3} A_i \neq \emptyset$ for all $A_i \in S_i$, $i = 1, 2, 3$, and so we can choose some $x_i \in \overline{S_i}(x_0) \cap S_j(x_0) \cap S_k(x_0)$ with $\{i, j, k\} = \{1, 2, 3\}$. But then, by Lemma 3.1(iii),

$$d_1(\gamma^{x_0}, \gamma^{x_i}) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S \delta_S(x_0, x_i) = \alpha_{S_i}$$

holds for $i = 1,2,3$. It follows that γ^{x_0} is a vertex in $UG(\Gamma(C), d_1|_{\Gamma(C)})$ with degree 3. But, since we are assuming that $d_1|_{\Gamma(C)}$ is totally split-decomposable, it follows by [12, Theorem 1.2] that $UG(\Gamma(C), d_1|_{\Gamma(C)})$ is an 8-cycle. This is a contradiction.
(iv') $\Rightarrow$ (iii'): This can be proven using similar arguments to (iv) $\Rightarrow$ (iii).
(iii) $\Rightarrow$ (ii): This follows in a straight-forward manner from the definition of γ^z, $z \in X$.
(ii) $\Rightarrow$ (i): Suppose that $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting and that $\mathcal{S}$ is not weakly compatible. Then there exist distinct splits $S_1, S_2, S_3 \in \mathcal{S}$ and distinct elements $x_0, x_1, x_2, x_3 \in X$ so that (1) holds. Note that $\mathcal{S}' = \{S_1, S_2, S_3\}$ is incompatible. Hence, $B(\mathcal{S}', \alpha|_{\mathcal{S}'}) = H(\mathcal{S}', \alpha|_{\mathcal{S}'})$ by [5, Proposition 3.3], and so there must exist some $\phi' \in B(\mathcal{S}', \alpha|_{\mathcal{S}'})$ with $\mathcal{S}(\phi') = \mathcal{S}'$. By (B2) there exists some $\phi \in B(\mathcal{S}, \alpha)$ with $\phi|_{\mathcal{S}'} = \phi'$. Without loss of generality, we may assume that $\mathcal{S}(\phi) = \mathcal{S}(\phi') = \mathcal{S}'$. Let $C = [\phi]$. Since $\mathcal{S}$ is Teutoburgan, Theorem 4.3 implies that C is antipodal X-gated and, by (B5), $\dim(C) = |\mathcal{S}(\phi)| = 3$. Now by (1), for all $i = 1,2,3$ and all $k, l \in \{1,2,3\} - i$ distinct, $x_i \in S_i(x_0) \cap \overline{S_k}(x_0) \cap \overline{S_l}(x_0)$ and so $\gamma^{x_0}, \gamma^{x_1}, \gamma^{x_2}, \gamma^{x_3}$ are all distinct gates in C and, by Proposition 3.2, for all $i, j \in \{0,1,2,3\}$ the antipode of γ^{x_i} in C is not γ^{x_j}. Hence, $|\Gamma(C)| = 8$. But then, by the definition of γ^{x_j}, $j = 0,1,2,3$, $\bigcap_{i=1,2,3} A_i \neq \emptyset$ where $A_i \in S_i$, $i = 1,2,3$. It follows that $\mathcal{S}$ is not 3-cube-free. ∎

7 Proof of Theorem 1.1

Suppose that d is a totally split-decomposable. Let $(\mathcal{S}, \alpha)$ be the unique weighted split system on X with $\mathcal{S}$ weakly compatible and $d = d_{\mathcal{S},\alpha}$. Note that since $\mathcal{S}$ is weakly compatible it is Teutoburgan.

We first show that every cell in $T(d)$ is a zonotope. It clearly suffices to show that any maximal cell in $T(d)$ is a zonotope, since every face of a zonotope is again a zonotope [16]. So, suppose Z is a maximal cell of $T(d)$. We will show that Z is centrally symmetric. Note that since $\mathcal{S}$ is weakly compatible, by Theorem 6.1 there exists a unique maximal cell C in $B(\mathcal{S}, \alpha)$ with $\kappa'(C) = Z$. Suppose that ψ is a generator of C, and consider the map

$$\xi : \mathcal{U}(\mathcal{S}) \to \mathbb{R}^{\geq 0} : A \mapsto \begin{cases} \frac{\alpha_{S_A}}{4} & \text{if } A \in \mathcal{U}(\mathcal{S}(\psi)), \\ \psi(A) & \text{else,} \end{cases}$$

which is contained in $[\psi]$ by (B1). By Theorem 5.1(iii), $\kappa(\xi)$ is contained in $[\kappa(\psi)] = Z$. We claim that $\kappa(\xi)$ is a center for Z. We must show that $\kappa(\xi) + g \in [\psi]$ if and only if $\kappa(\xi) - g \in [\psi]$ holds for all $g \in T(d)$. Suppose $f = \kappa(\xi)+g \in [\psi]$. Since $\mathcal{S}$ is weakly compatible κ must be surjective [6], and so there exists some map ϕ in $B(\mathcal{S}, \alpha)$ with $\kappa(\phi) = f$. By Theorem 5.1(iii), $\phi \in [\psi]$ and so, by Lemma 3.1(i), for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$, $\phi(A) = \psi(A)$. Consider the map

$$\phi' : \mathcal{U}(\mathcal{S}) \to \mathbb{R}^{\geq 0} : A \mapsto \begin{cases} \frac{\alpha_{S_A}}{2} - \phi(A) \text{ if } A \in \mathcal{U}(\mathcal{S}(\psi)), \\ \psi(A) \text{ else,} \end{cases}$$

which is contained in $[\psi]$ by (B1). A straight-forward computation shows that, for all $A \in \mathcal{U}(\mathcal{S})$ and all $x \in X$,

$$2|\xi(A) - \phi_x(A)| - |\phi(A) - \phi_x(A)| = |\phi'(A) - \phi_x(A)|.$$

Now, summing over all $A \in \mathcal{U}(\mathcal{S})$ and using (2), it follows that $2\kappa(\xi) - \kappa(\phi) = \kappa(\phi')$. Since $\phi' \in [\psi]$ and so, by Theorem 5.1(iii), $\kappa(\phi') \in [\kappa(\psi)]$, we have

$$\kappa(\xi) - g = \kappa(\xi) - f + \kappa(\xi) = 2\kappa(\xi) - \kappa(\phi) = \kappa(\phi').$$

Hence $\kappa(\xi) - g \in [\psi]$ and so $\kappa(\xi)$ is a center for Z. Thus Z is a zonotope.

Now suppose that Z is a maximal cell of $T(d)$. Let C be the maximal cell in $B(\mathcal{S}, \alpha)$ with $\kappa'(C) = Z$, which exists by Theorem 6.1. We claim that if $x \in X$, then $\kappa(\gamma_C^x)$ is a gate for x in Z.

Suppose $f \in Z$ and let ϕ be a generator of C. Then, since κ is surjective, there must exist some $\psi \in B(\mathcal{S}, \alpha)$ with $\kappa(\psi) = f$. Since C is a maximal cell, $\psi \in C$ by Theorem 5.1(iii). Since $\mathcal{S}$ is Teutoburgan, by Theorem 4.3 and Corollary 3.3(ii) there must exist some $y \in X$ with $\phi_x, \gamma_C^x, \phi, \gamma_C^y, \phi_y$ a geodesic in $B(\mathcal{S}, \alpha)$. By Proposistion 3.2, the fact that κ is a non-expnding map, and, by Theorem 4.3, it follows that $\kappa(\phi_x), \kappa(\gamma^x), f, \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $T(d)$. But then by (TS4)

$$d_\infty(\kappa(\phi_x), \kappa(\gamma^x)) + d_\infty(\kappa(\gamma^x), f) = d_\infty(\kappa(\phi_x), f) = d_\infty(h_x, f).$$

Hence $\kappa(\gamma_C^x)$ is a gate for x in Z, as claimed.

Now put $d' := d_1|_{\Gamma(C)}$ and $d'' := d_\infty|_{G(Z)}$. Using the last claim it immediately follows that Z is X-gated in $T(d)$ and, since κ is a non-expanding map, that κ induces an isometry between $(\Gamma(C), d')$ and $(G(Z), d'')$.

We next claim that Z is polytope isomorphic to either a hypercube or a rhombic dodecahedron. Since Z is X-gated, it immediately follows by [12, Theorem 1.1] that d'' is antipodal. Hence, since $(\Gamma(C), d')$ and $(G(Z), d'')$ are isometric, d' is antipodal. Thus, by Theorem 6.2, d' is totally split-decomposable, and so d'' is also totally split-decomposable. But by [12, Theorem 1.2] it immediately follows that $T(d'')$ is polytope isomorphic to either a hypercube or a rhombic dodecahedron, which concludes the proof of the claim.

To conclude the proof of Theorem 1.1 it remains to show that every cell in $T(d)$ is X-gated. Let W be any cell of $T(d)$. Suppose that Z is any maximal cell in $T(d)$ containing W, and put $d'' = d_\infty|_{G(Z)}$. Since Z is X-gated, by Claim 4 in the proof of [12, Theorem 1.1] there is a bijective isometry $\chi : Z \to T(d'')$ that induces a polytope isomorphism between Z and $T(d'')$. Moreover, since d'' is antipodal and totally split-decomposable, by [12, Theorem 1.2] it follows that every cell in $T(d'')$ is $G(Z)$-gated.

Now let $x \in X$, and let $g^x \in Z$ be the gate for x in Z. Let p be the element of Z that is mapped by χ to the gate for $\chi(g^x)$ in $\chi(W)$. We claim that p is a gate for x in W. Let $f \in W$. Since χ is a bijective isometry

$$d_\infty(g^x, f) = d_\infty(g^x, p) + d_\infty(p, f),$$

and, since Z is X-gated,

$$d_\infty(x, f) = d_\infty(x, g^x) + d_\infty(g^x, f).$$

But, since $p \in Z$ and Z is X-gated,

$$d_\infty(x, p) = d_\infty(x, g^x) + d_\infty(g^x, p).$$

Using these last three equalities, it immediately follows that $d_\infty(x, f) = d_\infty(x, p) + d_\infty(p, f)$. Hence p is a gate for x in W, and so W is X-gated. This concludes the proof of the theorem. ∎

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