

A PRECONDITIONER FOR THE FETI-DP FORMULATION WITH MORTAR METHODS IN THREE DIMENSIONS

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ABSTRACT. In this paper, we extend a FETI-DP method with mortar discretizations developed in [7] to three dimensional elliptic problems. We use the mortar matching condition as the continuity constraints for the FETI-DP formulation. In addition, the redundant continuity constraints are needed to achieve the same condition number bound as two dimensional problems. We propose a Neumann-Dirichlet preconditioner for the FETI-DP operator and show that the condition number of the preconditioned FETI-DP operator is bounded by $C \max_{i=1, \dots, N} \{(1 + \log(H_i/h_i))^2\}$, where H_i and h_i are sizes of domain and mesh for each subdomain, respectively, and the constant C is independent of H_i and h_i and may depend on coefficients of the elliptic problems. Numerical results are included.

1. INTRODUCTION

FETI-DP methods were introduced by Farhat *et al.* [4] and applied to solve elliptic problems with conforming discretizations both in two and three dimensions [5]. In three dimensions, subdomains intersect with neighboring subdomains on faces, edges, or at corners, while they intersect on edges or at corners in two dimensions; the continuity of solution is imposed on faces and edges with dual variables and at corners with primal variables in the dual-primal FETI (FETI-DP) methods. However, numerical results in [4, 5] show that we need redundant continuity constraints for three dimensional problems to attain the same efficiency as two dimensional problems. For these constraints, additional Lagrange multipliers are introduced and they are treated as primal variables in the FETI-DP formulation. FETI-DP methods with various redundant constraints have been studied and their condition number bound was analyzed by Klawonn *et al.* [9, 10] for elliptic problems with heterogeneous coefficients. Further, numerical results were provided in [8].

Recently FETI-DP methods have been applied to mortar finite elements methods [2, 3, 7, 14]. In [2, 3], the condition number bound of FETI-DP operator was analyzed for various types of preconditioners but it depends on ratios of mesh sizes between neighboring subdomains. In [7], a Neumann-Dirichlet preconditioner was proposed and analyzed for elliptic problems with heterogeneous coefficients. In this case, the condition number bound does not depend on the mesh sizes and the

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coefficients. Moreover, numerical results show that the Neumann-Dirichlet preconditioner works much more efficiently than other FETI-DP preconditioners for elliptic problems with highly discontinuous coefficients. For three dimensional problems, FETI methods with mortar discretizations were developed and their numerical results were provided in [13]. To our best knowledge, there is no condition number bound analysis for three dimensional problems with mortar discretizations.

The primary contribution of our work is the extension of the FETI-DP method in [7] to three dimensional problems. In the FETI-DP formulation, we need redundant continuity constraints to get the same condition number bound as two dimensional problems. The redundant constraints are that averages of the solution across subdomain interfaces are the same, which is so called face constraints in [10]. With the similar idea to the previous work in [7], we propose a Neumann-Dirichlet preconditioner for the FETI-DP operator, which is obtained from the augmented FETI-DP formulation with mortar constraints and face constraints, and show that the condition number bound is $C \max_{i=1, \dots, N} \left\{ (1 + \log(H_i/h_i))^2 \right\}$ for elliptic problems whose coefficients do not change rapidly across subdomain interfaces. Here, H_i and h_i are sizes of domain and mesh for each subdomain, respectively, and the constant C is independent of H_i and h_i and may depend on the coefficients of elliptic problems. Further, with an assumption on mesh sizes according to the magnitude of coefficients, we get the same condition number bound for elliptic problems with discontinuous constant coefficients. In this case, the constant C does not depend on the coefficients.

This paper is organized as follows. In Section 2, we introduce finite element spaces and norms and in Section 3, we derive the FETI-DP operator with the mortar matching constraints and redundant constraints, and propose a Neumann-Dirichlet preconditioner. Section 4 is devoted to the condition number bound analysis of the preconditioned FETI-DP operator. Numerical results are provided in Section 5.

2. FINITE ELEMENT SPACES AND NORMS

2.1. A model problem and Sobolev spaces. Let Ω be a bounded polyhedral domain in \mathbb{R}^3 and $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the norm $\|\cdot\|_{0,\Omega}$:

$$\|v\|_{0,\Omega}^2 := \int_{\Omega} v^2 dx.$$

The space $H^1(\Omega)$ is the set of functions, which are square integrable up to the first weak derivatives, and the norm is given by

$$\|v\|_{1,\Omega} := \left(\int_{\Omega} \nabla v \cdot \nabla v dx + \frac{1}{d_{\Omega}^2} \int_{\Omega} v^2 dx \right)^{1/2},$$

where d_{Ω} denotes the diameter of Ω .

We consider a FETI-DP method on nonmatching grids for the following elliptic problem:

For $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$ such that

$$(2.1) \quad \begin{aligned} -\nabla \cdot (A(x)\nabla u(x)) + \beta(x)u(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \Gamma_D, \\ \mathbf{n} \cdot (A(x)\nabla u(x)) &= 0 \quad \text{on } \Gamma_N. \end{aligned}$$

Here, $A(x) = (\alpha_{ij}(x))$ for $i, j = 1, \dots, 3$ and \mathbf{n} is the outward unit vector normal to Γ_N . We assume that $\alpha_{ij}(x), \beta(x) \in L^\infty(\Omega)$, $A(x)$ is uniformly elliptic, $\beta(x) \geq 0$ for all $x \in \Omega$ and $|\Gamma_D| \neq 0$, where $|\Gamma_D|$ denotes the measure of Γ_D .

Let Ω be partitioned into nonoverlapping polyhedral subdomains $\{\Omega_i\}_{i=1}^N$. We assume that the partition is geometrically conforming, which means that the subdomains intersect with neighboring subdomains on a whole face, a whole edge or at a vertex. The subdomain Ω_i is equipped with a quasi uniform triangulation Ω_i^h , which consists of tetrahedrons. The quasi-uniformity means that there exist constants γ and σ such that $\gamma h_i \leq d_\tau \leq \sigma \rho_\tau$ for all $\tau \in \Omega_i^h$, where ρ_τ is the diameter of the sphere inscribed in τ , d_τ is the diameter of τ and $h_i = \max_{\tau \in \Omega_i^h} d_\tau$. These triangulations need not to be aligned across subdomain interfaces.

For each subdomain Ω_i , we introduce a finite element space

$$X_i := \{v \in H_D^1(\Omega_i) : v|_\tau \in P_1(\tau), \tau \in \Omega_i^h\},$$

where $H_D^1(\Omega_i) := \{v \in H^1(\Omega_i) : v = 0 \text{ on } \Gamma_D \cap \partial\Omega_i\}$ and $P_1(\tau)$ is a set of polynomials of degree ≤ 1 in τ . For $(u_i, v_i) \in X_i \times X_i$, we define a bilinear form

$$a_i(u_i, v_i) := \int_{\Omega_i} A(x)\nabla u_i \cdot \nabla v_i dx + \int_{\Omega_i} \beta(x)u_i v_i dx.$$

To get the FETI-DP formulation, we need a finite element space in Ω as follows:

$$X := \left\{ v \in \prod_{i=1}^N X_i : v \text{ is continuous at subdomain vertices} \right\}.$$

By restricting the space X_i on the boundaries of each subdomain, we define

$$W_i := X_i|_{\partial\Omega_i} \quad \forall i = 1, \dots, N.$$

Then we let

$$W := \left\{ w \in \prod_{i=1}^N W_i : w \text{ is continuous at subdomain vertices} \right\}.$$

In this paper, we will use the same notation for finite element functions and the corresponding vectors of nodal values. For example, w_i is used to denote a finite element function or the vector of nodal values of that function. The same applies to the notations for function spaces such as W_i, X, W , etc.

We define S^i as the Schur complement matrix obtained from the bilinear form $a_i(\cdot, \cdot)$ over the finite elements X_i (see p. 50 in [12]). Using this operator, a semi-norm is defined for $w_i \in W_i$:

$$|w_i|_{S^i}^2 := \langle S^i w_i, w_i \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the l^2 -inner product of vectors. Since $w \in W$ is continuous at subdomain vertices, we define a norm for w by summing up semi-norms

$$(2.2) \quad \|w\|_W^2 := \sum_{i=1}^N |w_i|_{S^i}^2, \quad w_i = w|_{\partial\Omega_i}.$$

Now, we introduce Sobolev spaces defined on the boundaries of subdomains. The space $H^{1/2}(\partial\Omega_i)$ is the trace space of $H^1(\Omega_i)$ equipped with the norm

$$\|w_i\|_{1/2, \partial\Omega_i}^2 := |w_i|_{1/2, \partial\Omega_i}^2 + \frac{1}{d_{\Omega_i}} \|w_i\|_{0, \partial\Omega_i}^2,$$

where

$$|w_i|_{1/2, \partial\Omega_i}^2 := \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{|w_i(x) - w_i(y)|^2}{|x - y|^3} ds(x) ds(y).$$

For any $\Gamma_{ij} \in \partial\Omega_i$, $H_{00}^{1/2}(\Gamma_{ij})$ is the set of functions in $L^2(\Gamma_{ij})$ whose zero extension into $\partial\Omega_i$ is contained in $H^{1/2}(\partial\Omega_i)$. For $v \in H_{00}^{1/2}(\Gamma_{ij})$, let

$$|v|_{H_{00}^{1/2}(\Gamma_{ij})}^2 := |v|_{1/2, \Gamma_{ij}}^2 + \int_{\Gamma_{ij}} \frac{v^2(x)}{\text{dist}(x, \partial\Gamma_{ij})} ds$$

and the norm is given by

$$\|v\|_{H_{00}^{1/2}(\Gamma_{ij})} := \left(|v|_{H_{00}^{1/2}(\Gamma_{ij})}^2 + \frac{1}{d_{\Omega_i}} \|v\|_{0, \Gamma_{ij}}^2 \right)^{1/2}.$$

From Section 4.1 in [18], we have the following relation for $v \in H_{00}^{1/2}(\Gamma_{ij})$:

$$(2.3) \quad C_1 \|\tilde{v}\|_{1/2, \partial\Omega_i} \leq \|v\|_{H_{00}^{1/2}(\Gamma_{ij})} \leq C_2 \|\tilde{v}\|_{1/2, \partial\Omega_i},$$

where the constants C_1 and C_2 are independent of d_{Ω_i} and \tilde{v} denotes the zero extension of v into $\partial\Omega_i$.

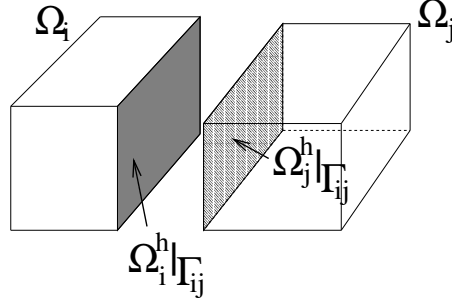
2.2. Mortar matching conditions. We note that the space X is not contained in $H^1(\Omega)$. In order to approximate the solution of the problem (2.1) in the nonconforming finite element space X , we impose the mortar matching condition on X , for which jumps of a function in X across a common face are orthogonal to a Lagrange multiplier space.

Let $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$ be the common face of subdomains Ω_i and Ω_j . On Γ_{ij} , we distinguish $\Omega_i^h|_{\Gamma_{ij}}$ and $\Omega_j^h|_{\Gamma_{ij}}$ as in Figure 1 and choose one as a mortar side and the other as a nonmortar side. In each subdomain Ω_i , we define sets

$$\begin{aligned} m_i &:= \{j : |\Gamma_{ij}| \neq 0, \Omega_j^h|_{\Gamma_{ij}} \text{ is a mortar side of } \Gamma_{ij}\}, \\ s_i &:= \{j : |\Gamma_{ij}| \neq 0, \Omega_j^h|_{\Gamma_{ij}} \text{ is a nonmortar side of } \Gamma_{ij}\}. \end{aligned}$$

For $j \in m_i$, $\Omega_i^h|_{\Gamma_{ij}}$ is the nonmortar side of Γ_{ij} and from the finite elements on the nonmortar side, we get

$$W_{ij} := \{v|_{\Gamma_{ij}} : v \in X_i\}.$$

FIGURE 1. Mortar and nonmortar sides of Γ_{ij}

Furthermore, we define

$$W_{ij}^0 := \{v \in W_{ij} : v = 0 \text{ on } \partial\Gamma_{ij}\}$$

and

$$W^0 = \prod_{i=1}^N \prod_{j \in m_i} W_{ij}^0.$$

For $w_{ij} \in W_{ij}^0$, we define $\tilde{w}_{ij} \in W_i$ by the zero extension of w_{ij} into $\partial\Omega_i$. Let $\tilde{w}_i = \sum_{j \in m_i} \tilde{w}_{ij}$ and $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_N)$. Since \tilde{w} is continuous at subdomain vertices, \tilde{w} is contained in the space X , and a norm for $w \in W^0$ is given by

$$(2.4) \quad \|w\|_{W^0} := \|\tilde{w}\|_X.$$

Let us assume that a suitable Lagrange multiplier space M_{ij} is chosen on each interface Γ_{ij} equipped with a triangulation from the nonmortar side. Then we take the Lagrange multiplier space

$$M := \prod_{i=1}^N \prod_{j \in m_i} M_{ij}$$

and impose the following mortar matching condition on X , i.e., $v \in X$ satisfies

$$(2.5) \quad \int_{\Gamma_{ij}} (v_i - v_j) \lambda_{ij} ds = 0 \quad \forall \lambda_{ij} \in M_{ij}, \quad i = 1, \dots, N, \quad j \in m_i.$$

In our FETI-DP formulation, we use (2.5) as continuity constraints and define a bilinear form $b(\cdot, \cdot) : W \times M \rightarrow \mathbb{R}$ as

$$b(w, \mu) := \sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} (w_i - w_j) \mu_{ij} ds \quad \forall (w, \mu) \in W \times M.$$

For $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$, we denote $\partial\Omega_i \cap \partial\Omega_j$ as Γ_{ij} if $\Omega_i^h|_{\Gamma_{ij}}$ is a nonmortar side and as Γ_{ji} , otherwise. We assume that $\Omega_i^h|_{\Gamma_{ij}}$ is the nonmortar side and $\Omega_j^h|_{\Gamma_{ij}}$ is the mortar side of Γ_{ij} . Denote basis for M_{ij} by $\{\xi_k^{ij}\}_{k=1}^{N_{ij}}$ and, let $\{\phi_k^{ij}\}_{k=1}^{N_i}$ and

$\{\phi_k^{ji}\}_{k=1}^{N_j}$ be basis functions for $W_i|_{\Gamma_{ij}}$ and $W_j|_{\Gamma_{ij}}$, respectively. From these basis functions, we obtain matrices B_i^{ij} and B_j^{ij} with entries

$$\begin{aligned} (B_i^{ij})_{lk} &= \int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ij} ds, \quad \text{for } l = 1, \dots, N_{ij}, \quad k = 1, \dots, N_i, \\ (B_j^{ij})_{lk} &= - \int_{\Gamma_{ij}} \xi_l^{ij} \phi_k^{ji} ds, \quad \text{for } l = 1, \dots, N_{ij}, \quad k = 1, \dots, N_j. \end{aligned}$$

Then we rewrite (2.5) as

$$(2.6) \quad B_i^{ij} w_i^{ij} + B_j^{ij} w_j^{ij} = 0,$$

where $w_i^{ij} = v_i|_{\Gamma_{ij}}$ and $w_j^{ij} = v_j|_{\Gamma_{ij}}$.

Now define $E_{ij} : M_{ij} \rightarrow M$, an extension operator from M_{ij} to M by zero and $R_{ij}^l : W_l \rightarrow W_l|_{\Gamma_{ij}}$ for $l = i, j$, a restriction operator. Let

$$\begin{aligned} B_i &= \sum_{j \in m_i} E_{ij} B_i^{ij} R_{ij}^i + \sum_{j \in s_i} E_{ji} B_i^{ji} R_{ji}^i, \\ B &= (B_1 \quad \dots \quad B_N), \\ w &= \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}, \end{aligned}$$

where $w_i = v_i|_{\partial\Omega_i}$. Then the mortar matching condition (2.5) becomes

$$(2.7) \quad Bw = 0.$$

To guarantee the optimal order approximation of the mortar finite elements, we need the following abstract conditions on the space M_{ij} ;

(A.1) The basis $\{\xi_k^{ij}\}_{k=1}^{N_{ij}}$ are locally supported, that is, the number of elements in $\Omega_i^h|_{\Gamma_{ij}}$, which have nonempty intersections with the simply connected support of ξ_k^{ij} , is bounded independently of mesh sizes and Γ_{ij} .

(A.2) W_{ij}^0 and M_{ij} have the same dimension.

(A.3) There is a constant C such that

$$\|\phi\|_{0,\Gamma_{ij}} \leq C \sup_{\psi \in M_{ij}} \frac{\int_{\Gamma_{ij}} \phi \psi ds}{\|\psi\|_{0,\Gamma_{ij}}} \quad \forall \phi \in W_{ij}^0.$$

(A.4) For $\mu \in H^{k-1/2}(\Gamma_{ij})$, there exists $\mu_h \in M_{ij}$ such that

$$\|\mu - \mu_h\|_{0,\Gamma_{ij}}^2 \leq Ch_i^{2k-1} |\mu|_{k-1/2,\Gamma_{ij}}^2,$$

where k is the order of finite elements in X_i .

The condition (A.4) implies that $1 \in M_{ij}$. In the following, we assume that the Lagrange multiplier space M_{ij} satisfies the above conditions; the standard Lagrange multiplier space in [1] and the Lagrange multipliers with dual basis in [6] are those examples.

3. FETI-DP FORMULATION

3.1. FETI-DP operator. In this section, we formulate the FETI-DP operator for the problem (2.1) with the mortar matching condition as constraints. For 3D elliptic problems, it was shown from the numerical results in [4, 5] that using the primal variables at corners is not enough to get the same condition number bound as 2D problems. Hence, redundant continuity constraints are added to the coarse problem to accelerate the convergence of the FETI-DP method.

For the 3D elliptic problems with conforming discretizations, Klawonn *et al.* [9] developed FETI-DP methods with various redundant constraints. They introduced additional continuity constraints on edges or on faces to achieve the same condition number bound as 2D elliptic problems. The continuity constraints on edges are that the averages of functions across a common edge are the same. The same is applied to faces also. In [10], they extended the results to a case with face constraints only. Since the constraints on edges are not redundant to the mortar matching condition, we will only impose the face constraints as the redundant constraints

$$\int_{\Gamma_{ij}} v_i ds = \int_{\Gamma_{ij}} v_j ds \quad \forall i = 1, \dots, N, j \in m_i.$$

From $1 \in M_{ij}$, the above constraints are redundant to the mortar constraints (2.5) and they are written into the following algebraic equations:

$$(3.1) \quad R^t B w = 0,$$

where the matrix R has 0 or 1 as its entries and $R^t \lambda = 0$ means that sum of $\lambda|_{\Gamma_{ij}}$ is zero on each interface Γ_{ij} .

For $w_i \in W_i$ we write

$$w_i = \begin{pmatrix} w_r^i \\ w_c^i \end{pmatrix},$$

where r and c stand for the nodal values on faces or edges, and at vertices, respectively. These notational conventions hold throughout this paper.

Let W_r be a space which consists of vectors

$$w_r = \begin{pmatrix} w_r^1 \\ \vdots \\ w_r^N \end{pmatrix}.$$

Define W_c as a set of vectors which have d.o.f. corresponding to the union of subdomain vertices, that is, global corner points. Since $w = (w_1, \dots, w_N) \in W$ is continuous at subdomain vertices, there exists $w_c \in W_c$ such that $L_c^i w_c = w_c^i \forall i = 1, \dots, N$, where the matrix L_c^i consists of 0 and 1 and restricts the value of w_c on the vertices of subdomain Ω_i . For $w = (w_1, \dots, w_N) \in W$, we write

$$w_i = \begin{pmatrix} w_r^i \\ L_c^i w_c \end{pmatrix} \quad \forall i, \text{ for some } w_c \in W_c.$$

Recall that S^i is the Schur complement matrix obtained from the bilinear form $a_i(\cdot, \cdot)$ and let g^i be the Schur complement forcing vector obtained from $\int_{\Omega_i} f v_i dx$.

The matrix S^i and vector g^i are ordered in the following way:

$$S^i = \begin{pmatrix} S_{rr}^i & S_{rc}^i \\ S_{cr}^i & S_{cc}^i \end{pmatrix}, \quad g^i = \begin{pmatrix} g_r^i \\ g_c^i \end{pmatrix}.$$

Let $B_{i,r}$ and $B_{i,c}$ be matrices that consist of the columns of B_i corresponding to the nodal points on faces or edges, and at vertices, respectively. Let U be a Lagrange multiplier space corresponding to the redundant constraints (3.1). Then, we have the following mixed formulation of the problem (2.1) with the constraints (2.7) and (3.1):

Find $(w_r, w_c, \mu, \lambda) \in W_r \times W_c \times U \times M$ satisfying

$$(3.2) \quad \begin{aligned} S_{rr}w_r + S_{rc}w_c + B_r^t R\mu + B_r^t \lambda &= g_r, \\ S_{cr}w_r + S_{cc}w_c + B_c^t R\mu + B_c^t \lambda &= g_c, \\ R^t B_r w_r + R^t B_c w_c &= 0, \\ B_r w_r + B_c w_c &= 0, \end{aligned}$$

where

$$\begin{aligned} S_{rr} &= \text{diag}_{i=1, \dots, N} (S_{rr}^i), \\ S_{rc} &= \begin{pmatrix} S_{rc}^1 L_c^1 \\ \vdots \\ S_{rc}^N L_c^N \end{pmatrix}, \\ S_{cr} &= S_{rc}^t, \\ S_{cc} &= \sum_{i=1}^N (L_c^i)^t S_{cc}^i L_c^i, \\ B_r &= (B_{1,r}, \dots, B_{N,r}), B_c = \sum_{i=1}^N B_{i,c} L_c^i, \\ g_r &= \begin{pmatrix} g_r^1 \\ \vdots \\ g_r^N \end{pmatrix}, g_c = \sum_{i=1}^N (L_c^i)^t g_c^i, w_r = \begin{pmatrix} w_r^1 \\ \vdots \\ w_r^N \end{pmatrix}. \end{aligned}$$

In the above equations, we regard $\tilde{w}_c = \begin{pmatrix} w_c \\ \mu \end{pmatrix}$ as primal variables in the FETI-DP formulation and follow the augmented FETI-DP formulation introduced in [5]. Let

$$\begin{aligned} K_{rr} &= S_{rr}, \\ K_{rc} &= (S_{rc} \quad B_r^t R), \quad K_{cr} = K_{rc}^t, \\ K_{cc} &= \begin{pmatrix} S_{cc} & B_c^t R \\ R^t B_c & 0 \end{pmatrix}, \\ \tilde{B}_c &= (B_c \quad 0), \quad \tilde{g}_c = \begin{pmatrix} g_c \\ 0 \end{pmatrix}. \end{aligned}$$

Then we have

$$(3.3) \quad \begin{pmatrix} K_{rr} & K_{rc} & B_r^t \\ K_{cr} & K_{cc} & \tilde{B}_c^t \\ B_r & \tilde{B}_c & 0 \end{pmatrix} \begin{pmatrix} w_r \\ \tilde{w}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} g_r \\ \tilde{g}_c \\ 0 \end{pmatrix}.$$

Since K_{rr} is invertible, after eliminating w_r in (3.3), we obtain

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{lc} & F_{ll} \end{pmatrix} \begin{pmatrix} \tilde{w}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} -d_c \\ d_l \end{pmatrix},$$

where

$$\begin{aligned} F_{cc} &= K_{cc} - K_{cr}K_{rr}^{-1}K_{rc}, \\ F_{lc} &= B_rK_{rr}^{-1}K_{rc} - \tilde{B}_c, \quad F_{cl} = F_{lc}^t, \\ F_{ll} &= B_rK_{rr}^{-1}B_r^t \\ d_l &= B_rS_{rr}^{-1}g_r, \quad d_c = \tilde{g}_c - K_{cr}K_{rr}^{-1}g_r. \end{aligned}$$

From the fact that B_r^tR has a full column rank, we can show that F_{cc} is invertible. Hence, eliminating \tilde{w}_c in the above equation, the FETI-DP equation of (3.2) follows:

$$(3.4) \quad F_{DP}\lambda = d_l - F_{lc}F_{cc}^{-1}d_c,$$

with $F_{DP} = F_{ll} + F_{lc}F_{cc}^{-1}F_{cl}$. we call F_{DP} the FETI-DP operator. Since, we added the redundant mortar matching constraints to the FETI-DP formulation, the solution of FETI-DP equation is not uniquely determined in M . Let us define a subspace

$$M_R := \{\lambda \in M : R^t\lambda = 0\}.$$

In Section 4, we will show that F_{DP} is symmetric and positive definite (s.p.d.) on M_R . Hence, the solution $\lambda \in M_R$ is uniquely determined.

3.2. Preconditioner. Since F_{DP} is s.p.d. on M_R , we will solve (3.4) by the preconditioned conjugate gradient method using a suitable preconditioner. We derive a preconditioner from the similar idea to [7], in which a Neumann-Dirichlet preconditioner is derived from a dual norm on the Lagrange multiplier space by using a duality pairing between the Lagrange multiplier space and finite elements on nonmortar sides. In the following, the idea is provided in more detail.

Let us define the following subspaces equipped with norms induced from W and W^0 :

$$\begin{aligned} W_R &:= \{w \in W : R^tBw = 0\}, \\ W_R^0 &:= \{w \in W^0 : R^tB\tilde{w} = 0\}, \end{aligned}$$

where \tilde{w} is the zero extension of $w \in W^0$ into the space W . A duality pairing between the spaces M_R and W_R^0 is defined as

$$\langle \lambda, w \rangle_m = \sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} \lambda_{ij} w_{ij} ds.$$

Then, a dual norm on $\lambda \in M_R$ is given by

$$(3.5) \quad \|\lambda\|_{M_R} := \max_{w \in W_R^0} \frac{\langle \lambda, w \rangle_m}{\|w\|_{W^0}}.$$

Similarly to the 2D problems in [7], we will find an operator \widehat{F}_{DP} which gives

$$(3.6) \quad \langle \widehat{F}_{DP} \lambda, \lambda \rangle = \|\lambda\|_{M_R}^2$$

and propose \widehat{F}_{DP}^{-1} as a preconditioner for the operator F_{DP} .

In order to obtain a matrix form of the operator \widehat{F}_{DP}^{-1} , we need some projections, restrictions and extensions. We let $P_{W_R^0} : W^0 \rightarrow W_R^0$ and $P_{M_R} : M \rightarrow M_R$ be l^2 -orthogonal projections. From the definitions W_R^0 and M_R , we observe that these projections are composed of diagonal blocks of projections

$$P_{W_R^0} = \text{diag}_{i=1}^N \text{diag}_{j \in m_i} (P_{W_R^0}^{ij}),$$

$$P_{M_R} = \text{diag}_{i=1}^N \text{diag}_{j \in m_i} (P_{M_R}^{ij}),$$

where $P_{W_R^0}^{ij} : W^0|_{\Gamma_{ij}} \rightarrow W_R^0|_{\Gamma_{ij}}$ and $P_{M_R}^{ij} : M|_{\Gamma_{ij}} \rightarrow M_R|_{\Gamma_{ij}}$ are l^2 -orthogonal projections restricted on Γ_{ij} . We introduce the following restriction and extension

$$R_{ij} : W^0 \rightarrow W_{ij}^0,$$

$$E_{ij}^i : W_{ij}^0 \rightarrow W_i,$$

and recall the matrices B_i^{ij} and B_j^{ij} in (2.6). We obtain the matrices $B_{i,r}^{ij}$ from B_i^{ij} after deleting columns corresponding to the d.o.f. on the boundary of Γ_{ij} . Let

$$\widehat{S} = \sum_{i=1}^N \left(\sum_{j \in m_i} E_{ij}^i R_{ij} \right)^t S^i \left(\sum_{j \in m_i} E_{ij}^i R_{ij} \right),$$

$$\widehat{B} = \text{diag}_{i=1}^N \text{diag}_{j \in m_i} (B_{i,r}^{ij}).$$

By using the above matrices, the norm on the space W_R^0 and the duality pairing between M_R and W_R^0 are written into

$$\|w\|_{W^0}^2 = \langle \widehat{S}_p w, w \rangle \quad \text{for } w \in W_R^0,$$

$$\langle \lambda, w \rangle_m = \lambda^t \widehat{B}_p w \quad \text{for } \lambda \in M_R, w \in W_R^0,$$

where

$$\widehat{S}_p = P_{W_R^0}^t \widehat{S} P_{W_R^0},$$

$$\widehat{B}_p = P_{M_R}^t \widehat{B} P_{W_R^0}.$$

It can be shown that \widehat{S}_p and \widehat{B}_p are invertible on W_R^0 and \widehat{B}_p^t is invertible on M_R . Hence, the maximum in (3.5) occurs when $\widehat{S}_p w = \widehat{B}_p^t \lambda$ and this gives

$$\langle \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t \lambda, \lambda \rangle = \|\lambda\|_{M_R}^2 \quad \text{for } \lambda \in M_R.$$

As a result, we have $\widehat{F}_{DP} = \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t$. From the observation that \widehat{B}_p is composed of invertible block matrices $\widehat{B}_p^{ij} = (P_{M_R}^{ij})^t B_{i,r}^{ij} P_{W_R}^{ij}$, we get

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^N \left(\sum_{j \in m_i} E_{ij}^i (\widehat{B}_p^{ij})^{-1} R_{ij} \right)^t S^i \left(\sum_{j \in m_i} E_{ij}^i (\widehat{B}_p^{ij})^{-1} R_{ij} \right).$$

Hence, the computation of $\widehat{F}_{DP}^{-1} \lambda$ can be done parallelly in each subdomain.

4. CONDITION NUMBER ESTIMATION FOR THE PRECONDITIONED FETI-DP OPERATOR

The following well-known result is given when $a_i(u, v) = \int_{\Omega_i} \nabla u \cdot \nabla v \, dx$ (see Theorem 4.1.3 in [12]). With slight modification, we can obtain the similar result for a general case.

Lemma 4.1. *For $w_i \in W_i$, we have*

$$C_1 \|w_i\|_{1/2, \partial\Omega_i}^2 \leq \langle S^i w_i, w_i \rangle \leq C_2 \|w_i\|_{1/2, \partial\Omega_i}^2,$$

where C_1 and C_2 are constants depending on $A(x)$ and $\beta(x)$, but not depending on H_i and h_i .

In the following, we obtain a formula that is useful to analyze the condition number bound and the result is the same as Lemma 4.3 of Mandel and Tezaur [11].

Lemma 4.2. *For $\lambda \in M_R$, we have*

$$\max_{w \in W_R \setminus \{0\}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_W^2} = \langle F_{DP} \lambda, \lambda \rangle.$$

Proof. The problem (3.3) is equivalent to solving the following saddle-point problem

$$\max_{\lambda \in B(W_R)} \min_{w \in W_R} \left(\frac{1}{2} w^t S w + w^t g + \lambda^t B w \right),$$

where g is a vector obtained from the vectors g_r and g_c in (3.2). It can be shown easily that $B(W_R) = M_R$. Let $P_{W_R} : W \rightarrow W_R$ be an l^2 -orthogonal projection. Then, Euler-Lagrange equations from the above problem are

$$(4.1) \quad \begin{aligned} S_p w + B_p^t \lambda &= P_{W_R}^t g, \\ B_p w &= 0, \end{aligned}$$

where

$$\begin{aligned} S_p &= P_{W_R}^t S P_{W_R}, \\ B_p &= P_{M_R}^t B P_{W_R}. \end{aligned}$$

Since S_p is s.p.d. on W_R , we eliminate w in (4.1) and get

$$B_p S_p^{-1} B_p^t \lambda = d,$$

where $d = B_p S_p^{-1} P_{W_R}^t g$. Notice that this equation is obtained from the problem (3.3) by eliminating unknowns other than λ . Hence, we have

$$(4.2) \quad F_{DP} = B_p S_p^{-1} B_p^t.$$

Using the identity

$$\|w\|_W^2 = \langle Sw, w \rangle$$

and the projections P_{W_R} and P_{M_R} , we can see that

$$(4.3) \quad \max_{w \in W_R \setminus \{0\}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_W^2} = \langle B_p S_p^{-1} B_p^t \lambda, \lambda \rangle \quad \text{for } \lambda \in M_R.$$

From (4.2) and (4.3), we prove the lemma. \square

Remark 4.1. For $\lambda \in M_R$, $B_p^t \lambda = 0$ gives $\lambda = 0$ and S_p is s.p.d. on W_R . Hence, we can see that F_{DP} is s.p.d. on M_R from (4.2).

Now, we estimate the lower bound of the operator F_{DP} , which gives a lower bound for the smallest eigenvalue of the preconditioned FETI-DP operator.

Lemma 4.3. For any $\lambda \in M_R$, we have

$$\max_{w \in W_R \setminus \{0\}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_W^2} \geq \|\lambda\|_{M_R}^2.$$

Proof. Let $\tilde{w} \in W$ be the zero extension of $w \in W_R^0$. Then, we can see that $\tilde{w} \in W_R$. From the definitions of $\|\lambda\|_{M_R}$, $\|w\|_{W^0}$ and $\langle \lambda, w \rangle_m$, we get

$$\begin{aligned} \|\lambda\|_{M_R}^2 &= \max_{w \in W_R^0 \setminus \{0\}} \frac{\langle \lambda, w \rangle_m^2}{\|w\|_{W^0}^2} \\ &= \max_{w \in W_R^0 \setminus \{0\}} \frac{\langle B\tilde{w}, \lambda \rangle^2}{\|\tilde{w}\|_W^2} \\ &\leq \max_{w \in W_R \setminus \{0\}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_W^2}. \end{aligned}$$

This completes the proof. \square

To estimate the upper bound of the operator F_{DP} , we define an interpolation $I_0^i w_i \in W_i$ corresponding to a face $F \subset \partial\Omega_i$ by

$$(I_0^i w_i)(x) = \begin{cases} w_i(x), & x \in \partial F \cap \partial\Omega_i^h, \\ C_F(w_i), & x \in F \cap \partial\Omega_i^h, \\ 0, & \text{on the remaining nodes,} \end{cases}$$

where $\partial\Omega_i^h$ is the set of nodes on the boundary of Ω_i and $C_F(w_i)$ is the average of w_i on the face $F \subset \partial\Omega_i$, that is,

$$C_F(w_i) = \frac{\int_F w_i ds}{\int_F ds}.$$

Note that faces and edges are open sets which do not include their boundaries. In the following, C is a generic constant which does not depend on mesh sizes or the number of subdomains but may depend on $A(x)$ and $\beta(x)$. Recall the definition of

norms $\|\cdot\|_{H_{00}^{1/2}(F)}$ and $\|\cdot\|_{1/2,\partial\Omega_i}$ in Section 2.1. From the Hölder inequality, we obtain

$$(4.4) \quad |C_F(w_i)| \leq CH_i^{-1/2} \|w_i\|_{1/2,\partial\Omega_i}.$$

For a set $A \subset \partial\Omega_i$, let $I_A^h w_i \in W_i$ denote a nodal value interpolation of w_i on the set $A \cap \partial\Omega_i^h$, which means that $I_A^h w_i(x)$ has the same value with $w_i(x)$ on nodes $x \in A \cap \partial\Omega_i^h$ and has zero value on the remaining nodes.

The interpolation $I_0^i w_i$ has the following approximation properties.

Lemma 4.4. *For $w_i \in W_i$, we have*

$$(4.5) \quad \|w_i - I_0^i w_i\|_{H_{00}^{1/2}(F)} \leq C \left(1 + \log \frac{H_i}{h_i}\right) |w_i|_{1/2,\partial\Omega_i},$$

$$(4.6) \quad \|I_0^i w_i - C_F(w_i)\|_{0,F} \leq Ch_i^{1/2} \left(1 + \log \frac{H_i}{h_i}\right)^{1/2} |w_i|_{1/2,\partial\Omega_i}.$$

Proof. First, we consider

$$\begin{aligned} \|w_i - I_0^i w_i\|_{H_{00}^{1/2}(F)} &= \|I_F^h w_i - I_F^h C_F(w_i)\|_{H_{00}^{1/2}(F)} \\ &\leq \|I_F^h w_i\|_{H_{00}^{1/2}(F)} + |C_F(w_i)| \|I_F^h 1\|_{H_{00}^{1/2}(F)}. \end{aligned}$$

The identity in the above equations follows from the definitions of interpolations $I_0^i w_i$ and $I_F^h w_i$: on the nodes $x \in F \cap \partial\Omega_i^h$, $(w_i - I_0^i w_i)(x) = w_i(x) - C_F(w_i)$ and $w_i - I_0^i w_i = 0$ on the nodes in $\partial F \cap \partial\Omega_i^h$.

By applying the Lemma 4.10 and Lemma 4.11 in [18] to $I_F^h w_i$ and $I_F^h 1$, and (4.4), we get

$$\|w_i - I_0^i w_i\|_{H_{00}^{1/2}(F)} \leq C \left(1 + \log \frac{H_i}{h_i}\right) \|w_i\|_{1/2,\partial\Omega_i}.$$

We replace the norm $\|\cdot\|_{1/2,\partial\Omega_i}$ by the semi-norm $|\cdot|_{1/2,\partial\Omega_i}$ from the shift invariance of the expression $w_i - I_0^i w_i$.

Now, we consider the second estimate. From the definition of $I_0^i w_i$ and the quasi-uniform assumption on the triangulation, we get

$$\begin{aligned} \|I_0^i w_i - C_F(w_i)\|_{0,F} &= \|I_{\partial F}^h (w_i - C_F(w_i))\|_{0,F} \\ &\leq Ch_i^{1/2} \|I_{\partial F}^h (w_i - C_F(w_i))\|_{0,\partial F} \\ &\leq Ch_i^{1/2} \sum_{E \subset \partial F} \|I_E^h (w_i - C_F(w_i))\|_{0,E} \\ &\leq Ch_i^{1/2} \left(\sum_{E \subset \partial F} \|w_i\|_{0,E} + \sum_{E \subset \partial F} \|C_F(w_i)\|_{0,E} \right), \end{aligned}$$

where E is a closed edge on ∂F . From Lemma 4.9 in [18], we have

$$(4.7) \quad \|w_i\|_{0,E} \leq C \left(1 + \log \frac{H_i}{h_i}\right)^{1/2} \|w_i\|_{1/2,\partial\Omega_i},$$

and

$$\|C_F(w_i)\|_{0,E} \leq |E|^{1/2} |C_F(w_i)|.$$

The bound (4.4) and $|E| \leq CH_i$ yield

$$(4.8) \quad \|C_F(w_i)\|_{0,E} \leq C \|w_i\|_{1/2, \partial\Omega_i}.$$

From (4.7), (4.8) and the shift invariance of the term $I_0^i w_i - C_F(w_i)$, we complete the proof of (4.6). \square

Definition 4.1. We define a projection $\pi_{ij} : H_{00}^{1/2}(\Gamma_{ij}) \rightarrow W_{ij}^0$ for $v \in H_{00}^{1/2}(\Gamma_{ij})$ by

$$\int_{\Gamma_{ij}} (v - \pi_{ij}v) \lambda_{ij} ds = 0 \quad \forall \lambda_{ij} \in M_{ij}.$$

For the space M_{ij} satisfying the assumptions (A.1)-(A.4), we can show that the projection π_{ij} is continuous on the space $H_{00}^{1/2}(\Gamma_{ij})$ (see [6] or [17]);

$$(4.9) \quad \|\pi_{ij}v\|_{H_{00}^{1/2}(\Gamma_{ij})} \leq C \|v\|_{H_{00}^{1/2}(\Gamma_{ij})} \quad \forall v \in H_{00}^{1/2}(\Gamma_{ij}),$$

where C is a constant not depending on H_i and h_i under quasi-uniform meshes in each subdomain. Moreover, the projection is continuous on the space $L^2(\Gamma_{ij})$.

From Lemma 4.4 and (4.9), we have the following result similarly to the 2D case in [7].

Lemma 4.5. For $w \in W_R$, we have

$$\begin{aligned} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} &\leq C \left(\left(1 + \log \frac{H_i}{h_i}\right) |w_i|_{1/2, \partial\Omega_i} \right. \\ &\quad \left. + \left(1 + \frac{h_j}{h_i}\right)^{1/2} \left(1 + \log \frac{H_j}{h_j}\right) |w_j|_{1/2, \partial\Omega_j} \right). \end{aligned}$$

Proof. By the inverse inequality and the continuity of π_{ij} , we get

$$\begin{aligned} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} &\leq \|\pi_{ij}(w_i - I_0^i w_i)\|_{H_{00}^{1/2}(\Gamma_{ij})} + \|\pi_{ij}(w_j - I_0^j w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} \\ &\quad + \|\pi_{ij}(I_0^i w_i - I_0^j w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})} \\ &\leq C \left(\|w_i - I_0^i w_i\|_{H_{00}^{1/2}(\Gamma_{ij})} + \|w_j - I_0^j w_j\|_{H_{00}^{1/2}(\Gamma_{ij})} \right. \\ &\quad \left. + h_i^{-1/2} \|I_0^i w_i - I_0^j w_j\|_{0, \Gamma_{ij}} \right), \end{aligned}$$

where the interpolations $I_0^i w_i$ and $I_0^j w_j$ correspond to the common face $F(= \Gamma_{ij})$. Since $w \in W_R$, $C_F(w_i)$ and $C_F(w_j)$ are the same. Then, we have

$$\|I_0^i w_i - I_0^j w_j\|_{0, \Gamma_{ij}} \leq \|I_0^i w_i - C_F(w_i)\|_{0, \Gamma_{ij}} + \|I_0^j w_j - C_F(w_j)\|_{0, \Gamma_{ij}}.$$

From the above equation and the approximation properties of $I_0^i w_i$ in Lemma 4.4, we complete the proof. \square

Now, we estimate the upper bound of the operator F_{DP} , which provides an upper bound for the largest eigenvalue of the preconditioned FETI-DP operator. Let us define

$$r_i = \max_{j \in s_i} \left\{ 1 + \frac{h_i}{h_j} \right\} \quad \text{for } i = 1, \dots, N.$$

Lemma 4.6. *For $\lambda \in M_R$, we have*

$$\max_{w \in W_R \setminus \{0\}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_W^2} \leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 r_i \right\} \|\lambda\|_{M_R}^2,$$

where C is a constant depending on coefficients $A(x)$ and $\beta(x)$, but not depending on mesh parameters h_i and H_i .

Proof. From the definitions of B and π_{ij} , we have

$$\langle Bw, \lambda \rangle^2 = \left(\sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} \pi_{ij}(w_i - w_j) \lambda_{ij} ds \right)^2.$$

We consider $z \in W^0$ such that $z|_{\Gamma_{ij}} = \pi_{ij}(w_i - w_j)$. Since $w \in W_R$, we can see that $z \in W_R^0$. Then the above equation is the duality pairing between $\lambda \in M_R$ and $z \in W_R^0$. Hence, using the definition of dual norm on λ , we get

$$(4.10) \quad \langle Bw, \lambda \rangle^2 \leq \|\lambda\|_{M_R}^2 \|z\|_{W^0}^2.$$

Let $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_N) \in W$ be the zero extension of z . From (2.4), (2.2), Lemma 4.1, (2.3), and Lemma 4.5, it follows that

$$\begin{aligned} \|z\|_{W^0}^2 &= \sum_{i=1}^N \langle S^i \tilde{z}_i, \tilde{z}_i \rangle \\ &\leq C \sum_{i=1}^N \|\tilde{z}_i\|_{1/2, \partial\Omega_i}^2 \\ &\leq C \sum_{i=1}^N \sum_{j \in m_i} \|\pi_{ij}(w_i - w_j)\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\ (4.11) \quad &\leq C \sum_{i=1}^N \sum_{j \in m_i} \left(\left(1 + \log \frac{H_i}{h_i} \right)^2 |w_i|_{1/2, \partial\Omega_i}^2 \right. \\ &\quad \left. + \left(1 + \frac{h_j}{h_i} \right) \left(1 + \log \frac{H_j}{h_j} \right)^2 |w_j|_{1/2, \partial\Omega_j}^2 \right) \\ &\leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 r_i \right\} \|w\|_W^2. \end{aligned}$$

Combining (4.10) and (4.11), we complete the proof. \square

Remark 4.2. *When the coefficients $A(x)$ and $\beta(x)$ do not change rapidly across subdomain interfaces, it is appropriate to use triangulations which have similar mesh sizes between neighboring subdomains. Hence, in this case, we may assume that r_i is bounded independently of the mesh sizes.*

Now, we consider the following elliptic problem with discontinuous constant coefficients:

$$(4.12) \quad \begin{aligned} -\nabla \cdot (\alpha(x)\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with $\alpha(x)|_{\Omega_i} = \rho_i (> 0)$ for all $i = 1, \dots, N$. Then, we have the similar estimates to Lemma 4.1:

$$C_1 \rho_i |w_i|_{1/2, \partial\Omega_i} \leq \langle S^i w_i, w_i \rangle \leq C_2 \rho_i \|w_i\|_{1/2, \partial\Omega_i},$$

where C_1 and C_2 are constants not depending on ρ_i , h_i and H_i . Using the above bound, we follow the proofs of Lemma 4.6 and obtain

$$(4.13) \quad \begin{aligned} \|z\|_{W^0}^2 &\leq C \sum_{i=1}^N \sum_{j \in m_i} \left(\left(1 + \log \frac{H_i}{h_i}\right)^2 |w_i|_{S_i}^2 \right. \\ &\quad \left. + \left(1 + \frac{h_j}{h_i}\right) \frac{\rho_i}{\rho_j} \left(1 + \log \frac{H_j}{h_j}\right)^2 |w_j|_{S_j}^2 \right), \end{aligned}$$

where C is a constant independent of ρ_i , h_i and H_i . For the same elliptic problem in $2D$, Wohlmuth [15] observed that the optimal ratio $\frac{h_j}{h_i}$ tends to become $\left(\frac{\rho_j}{\rho_i}\right)^{1/4}$ as an adaptivity strategy is applied successively. At this point, we need a reasonable assumption on the ratio of meshes for $3D$ problems.

Assumption on meshes: For each Γ_{ij} , we assume that

$$(4.14) \quad \frac{h_j}{h_i} \leq C \left(\frac{\rho_j}{\rho_i}\right)^\gamma, \quad \text{with } 0 \leq \gamma \leq 1,$$

where the constant C does not depend on mesh parameters h_i, H_i , and coefficients ρ_i .

If we choose Ω_i with smaller ρ_i as a nonmortar side on Γ_{ij} then from the above assumption on meshes, (4.13) is written into

$$\begin{aligned} \|z\|_{W^0}^2 &\leq C \sum_{i=1}^N \sum_{j \in m_i} \left(\left(1 + \log \frac{H_i}{h_i}\right)^2 |w_i|_{S_i}^2 \right. \\ &\quad \left. + \max \left\{ \frac{\rho_i}{\rho_j}, \left(\frac{\rho_i}{\rho_j}\right)^{1-\gamma} \right\} \left(1 + \log \frac{H_j}{h_j}\right)^2 |w_j|_{S_j}^2 \right), \end{aligned}$$

where C is a constant independent of ρ_i , h_i and H_i . Since the nonmortar side has smaller ρ_i 's, in the above equation $\max \left\{ \frac{\rho_i}{\rho_j}, \left(\frac{\rho_i}{\rho_j} \right)^{1-\gamma} \right\} \leq 1$. Therefore, we obtain the following result.

Lemma 4.7. *For the elliptic problem (4.12) with the assumption (4.14) on meshes, we have*

$$\max_{w \in W_R \setminus \{0\}} \frac{\langle Bw, \lambda \rangle^2}{\|w\|_W^2} \leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} \|\lambda\|_{M_R}^2,$$

where C is a constant independent of mesh parameters and coefficients.

Remark 4.3. *The result is the same as 2D case in [7]. However, we need an additional assumption on the ratio of meshes for 3D problems.*

Now, we restrict ourselves to the elliptic problem (2.1) with coefficients $A(x)$ and $\beta(x)$ that do not change rapidly across subdomain interfaces or the elliptic problem (4.12) with discontinuous constant coefficients ρ_i 's. From Lemma 4.2, Lemma 4.3, Lemma 4.6, Remark 4.2 and Lemma 4.7, we have the following result.

Theorem 4.1. *For $\lambda \in M_R$, we have*

$$\|\lambda\|_{M_R}^2 \leq \langle F_{DP} \lambda, \lambda \rangle \leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\} \|\lambda\|_{M_R}^2,$$

where the constant C does not depend on mesh parameters H_i and h_i but may depend on coefficients $A(x)$ and $\beta(x)$ for the elliptic problem (2.1). For the elliptic problem (4.12) with discontinuous coefficients ρ_i 's, the constant C is independent of mesh parameters and the coefficients.

From (3.6) and the above theorem, we obtain the condition number bound.

Corollary 4.1. *For the elliptic problems (2.1) or (4.12), we have*

$$\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \leq C \max_{i=1, \dots, N} \left\{ \left(1 + \log \frac{H_i}{h_i} \right)^2 \right\},$$

where the constant C is the same as one in the above theorem.

5. NUMERICAL RESULTS

In this section, we provide numerical tests for the FETI-DP formulation developed in this paper. We consider the following model problem:

$$\begin{aligned} -\nabla \cdot (\alpha(x, y, z) \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = (0, 1)^3$ is a unit cube, $u(x, y, z) = \sin(\pi x)y(1-y)\sin(\pi z)$ is the exact solution, and $\alpha(x, y, z) = 1$.

We divide the domain Ω into $N \times N \times N$ cubic subdomains with side length $H = 1/N$. Each subdomain is discretized by conforming trilinear finite elements

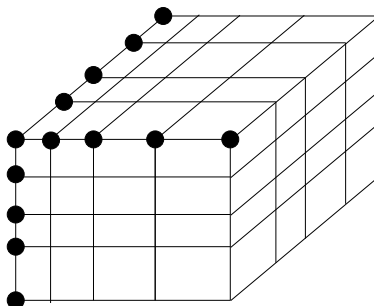


FIGURE 2. Hexahedral elements generated by the quasi-uniform nodes (black dots) along each axis when $n = 5$

and these elements are nonmatching across subdomain interfaces. To make non-matching grids, we generate hexahedral elements in each subdomain as follows. In a subdomain Ω_i , we choose n random quasi-uniform nodes along each axis including its end points. From these nodes, we generate nonuniform structured grids, which consists of hexahedrons with mesh parameter h_i (see Figure 2). Since the finite elements are obtained from the quasi-uniform nodes, the mesh parameter h_i is comparable to $H/(n - 1)$. The corresponding Lagrange multiplier is given by the tensor product of two dimensional multipliers considered in [16]. Even though the theory provided in the previous section was developed for tetrahedral finite elements, it extends to the approximation described above without difficulty.

To see the scalability of the preconditioner, we perform two types of experiments. First, we keep the number of subdomain fixed and increase the number of nodes n along each axis. In the second test, we have the number of subdomain increasing with a fixed subdomain problem size. We solve the FETI-DP equation using conjugate gradient method with and without preconditioners. The conjugate gradient iteration continues until the relative residual norm is reduced below 10^{-6} .

In Table 1, the number of CG iterations and condition numbers are shown when the number of nodes n increases with the fixed number of subdomains $N^3 = 4^3$. From the result, we observe the \log^2 -growth of the condition number for the proposed preconditioner. It also shows that the preconditioner effectively reduces CG iterations. Table 2 shows the numerical results when we fix $n = 5$ and increase the number of subdomains. As proved in theory, the condition number becomes stable as the number of subdomains is increasing.

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$n - 1$	Without preconditioner		With preconditioner	
	Iter	Cond	Iter	Cond
4	26	1.75e+1	17	6.90
8	67	2.17e+2	23	1.29e+1
12	75	3.15e+2	25	1.56e+1
16	83	3.89e+2	26	1.74e+1
24	92	5.26e+2	29	2.04e+1
32	99	6.39e+2	30	2.24e+1

TABLE 1. The number of CG iterations (Iter) and corresponding condition numbers (Cond) for the FETI-DP operator with or without preconditioner when subdomain problem size n^3 increases with the fixed number of subdomains N^3 ($N = 4$)

N^3	Without preconditioner		With preconditioner	
	Iter	Cond	Iter	Cond
2^3	21	1.63e+1	15	5.72
3^3	24	2.14e+1	17	7.13
4^3	26	1.75e+1	17	6.90
6^3	26	2.09e+1	18	7.91
8^3	27	2.22e+1	18	8.41

TABLE 2. The number of CG iterations (Iter) and corresponding condition numbers (Cond) for the FETI-DP operator with or without preconditioner when number of subdomains N^3 increases with the fixed subdomain problem size n^3 ($n = 5$)

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