A decomposable graph and its subgraphs

Sung-Ho Kim [1]

Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, Daejeon, 305-701, South Korea.

Abstract:

We explore the properties of subgraphs (called Markovian subgraphs) of a decomposable graph under some condition. For a decomposable graph $\mathcal{G}$ and a collection γ of its Markovian subgraphs, we show that the set $\chi(\mathcal{G})$ of the intersections of all the neighboring cliques of $\mathcal{G}$ contains $\cup_{g\in\gamma}\chi(g)$. We also show that $\chi(\mathcal{G}) = \cup_{g\in\gamma}\chi(g)$ holds for a certain type of $\mathcal{G}$ which we call a maximal Markovian supergraph of γ.

Keywords: Edge-subgraph; Markovian subgraph; Markovian supergraph; Prime separator.

1 Introduction

Graphs are used effectively in representing model structures in a variety of research fields such as statistics, artificial intelligence, data mining, biological science, medicine, decision science, educational science, etc. Different forms of graphs are used according to the intrinsic inter-relationship among the random variables involved. Arrows are used when the relationship is causal, temporal, or asymmetric, and undirected edges are used when the relationship is associative or symmetric.

Among the graphs, triangulated graphs [1] are favored mostly when Markov random fields ([11, 7]) are considered with respect to undirected graphs. When a random field is Markov with respect to a triangulated graph, its corresponding probability model is expressed in a factorized form which facilitates computation over the probability distribution of the random field [7]. This computational feasibility, among others, makes such a Markov random field a most favored random field.

The triangulated graph is called a rigid circuit [4], a chordal graph [5], or a decomposable graph [9]. A survey on this type of graphs is given in [2]. One of the attractive properties (see Chapter 4 of [6]) of the triangulated graph is that its induced subgraphs and Markovian subgraphs (defined in section 2) are triangulated. While induced subgraphs are often used in literature (see Chapter 2 of [8]), Markovian subgraphs are introduced in this paper. We will explore the relationship between a triangulated graph and its Markovian subgraphs and find explicit expressions for the relationship. The relationship is useful for understanding the

[1] E-mail address: shkim@amath.kaist.ac.kr

relationship between a probability model P, which is Markov with respect to the triangulated graph, and submodels of P. Since the terminology "decomposable graph" is more contextual than any others as long as Markov random fields are concerned, we will call the triangulated graph a decomposable graph in the remainder of the paper.

This paper consists of 6 sections. Section 2 presents notation and graphical terminologies. Markovian subgraphs are defined here. We define decomposable graphs in Section 3 and introduce a class of separators. In Section 4, we present the notion of Markovian supergraph and the relationship between Markovian supergraph and Markovian subgraph. In Section 5, we compare Markovian supergraphs between a pair of collections of Markovian subgraphs of a given graph. Section 6 concludes the paper with summarizing remarks.

2 Notation and terminology

We will consider only undirected graphs in the paper. We denote a graph by $\mathcal{G} = (V, E)$, where V is the set of the nodes involved in $\mathcal{G}$ and E is a collection of ordered pairs, each pair representing that the nodes of the pair are connected by an edge. Since $\mathcal{G}$ is undirected, $(u, v) \in E$ is the same edge as (v, u). We say that a set of nodes of $\mathcal{G}$ forms a complete subgraph of $\mathcal{G}$ if every pair of nodes in the set are connected by an edge. A maximal complete subgraph is called a clique of $\mathcal{G}$, where the maximality is in the sense of set-inclusion. We denote by $\mathcal{C}(\mathcal{G})$ the set of cliques of $\mathcal{G}$.

If $(u, v) \in E$, we say that u is a neighbor node of v or vice versa and write it as $u \sim v$. A path of length n is a sequence of nodes $u = v_0, \cdots, v_n = v$ such that $(v_i, v_{i+1}) \in E$, $i = 0, 1, \cdots, n-1$ and $u \neq v$. If $u = v$, the path is called an n-cycle. If $u \neq v$ and u and v are connected by a path, we write $u \rightleftharpoons v$. Note that $\rightleftharpoons$ is an equivalence relation. We define the connectivity component of u as

$$[u] = \{v \in V;\ \ v \rightleftharpoons u\} \cup \{u\}.$$

So, we have

$$v \in [u] \iff u \rightleftharpoons v \iff u \in [v].$$

For $v \in V$, we define $ne(v) = \{u \in V;\ \ v \sim u \text{ in } \mathcal{G}\}$ and, for $A \subseteq V$, $bd(A) = \cup_{v \in A} ne(v) \setminus A$. If we have to specify the graph $\mathcal{G}$ in which $bd(A)$ is obtained, we will write $bd_{\mathcal{G}}(A)$. A path, $v_1, \cdots, v_n$, $v_1 \neq v_n$, is intersected by A if $A \cap \{v_1, \cdots, v_n\} \neq \emptyset$ and neither of the end nodes of the path is in A. We say that nodes u and v are separated by A if all the paths from u and v are intersected by A, and we call such a set A a *separator*. In the same context, we say that, for three disjoint sets A, B, C, A is separated from B by C if all the paths from A to B are intersected by C, and we write $\langle A|C|B\rangle_{\mathcal{G}}$. The notation $\langle\cdot|\cdot|\cdot\rangle_{\mathcal{G}}$ follows [10]. A non-empty set B is said to be intersected by A if B is partitioned into three sets B_1, B_2, and $B \cap A$ and B_1 and B_2 are separated by A in $\mathcal{G}$.

For $A \subset V$, an induced subgraph of $\mathcal{G}$ confined to A is defined as $\mathcal{G}_A^{ind} = (A,\ E \cap (A \times A))$. The complement of a set A is denoted by A^c. For $A \subset V$, we let $\mathcal{J}_A$ be the collection of the connectivity components in $\mathcal{G}_{A^c}^{ind}$ and $\beta(\mathcal{J}_A) = \{bd(B);\ B \in \mathcal{J}_A\}$. Then we define a graph $\mathcal{G}_A = (A, E_A)$ where

$$E_A = [E \cup \{B \times B;\ B \in \beta(\mathcal{J}_A)\}] \cap A \times A. \tag{1}$$

We will call $\mathcal{G}_A$ the Markovian subgraph of $\mathcal{G}$ confined to A and write $\mathcal{G}_A \subseteq^M \mathcal{G}$. $\mathcal{J}_A$ and $\beta(\mathcal{J}_A)$ are defined with respect to a given graph $\mathcal{G}$. Note that E_A is not necessarily a subset of E, while $E_A^{ind} \subseteq E$. When the graph is to be specified, we will write them as $\mathcal{J}_A^{\mathcal{G}}$ and $\beta_{\mathcal{G}}(\mathcal{J}_A)$.

If $\mathcal{G} = (V, E)$, $\mathcal{G}' = (V, E')$, and $E' \subseteq E$, then we say that $\mathcal{G}'$ is an edge-subgraph of $\mathcal{G}$ and write $\mathcal{G}' \subseteq^e \mathcal{G}$. For us, a subgraph of $\mathcal{G}$ is either a Markovian subgraph, an induced subgraph, or an edge-subgraph of $\mathcal{G}$. If $\mathcal{G}'$ is a subgraph of $\mathcal{G}$, we call $\mathcal{G}$ a supergraph of $\mathcal{G}'$. The cardinality of a set A will be denoted by $|A|$. For two collections A, B of sets, if, for every $a \in A$, there exists a set b in B such that $a \subseteq b$, we will write $A \preceq B$.

3 Separators as a characterizer of decomposable graphs

In this section, we will present separators as a tool for characterizing decomposable graphs. Although decomposable graphs are well known in literature, we will define them here for completeness.

Definition 3.1. A triple (A, B, C) of disjoint, nonempty subsets of V is said to form a decomposition of $\mathcal{G}$ if $V = A \cup B \cup C$ and the two conditions below both hold:
(i) A and B are separated by C;
(ii) $\mathcal{G}_C^{ind}$ is complete.

By recursively applying the notion of graph decomposition, we can define a decomposable graph.

Definition 3.2. *A graph* $\mathcal{G}$ is said to be decomposable if it is complete, or if there exists a decomposition (A, B, C) into decomposable subgraphs $\mathcal{G}_{A \cup C}^{ind}$ and $\mathcal{G}_{B \cup C}^{ind}$.

According to this definition, we can find a sequence of cliques $C_1, \cdots, C_k$ of a decomposable graph $\mathcal{G}$ which satisfies the following condition [see Proposition 2.17 of [8]]: with $C_{(j)} = \cup_{i=1}^{j} C_i$ and $S_j = C_j \cap C_{(j-1)}$,

$$\text{for all } i > 1, \text{ there is a } j < i \text{ such that } S_i \subseteq C_j. \tag{2}$$

By this condition for a sequence of cliques, we can see that S_j is expressed as an intersection of neighboring cliques of $\mathcal{G}$. If we denote the collection of these S_j's by $\chi(\mathcal{G})$, we have, for a decomposable graph $\mathcal{G}$, that

$$\chi(\mathcal{G}) = \{a \cap b;\ \ a, b \in \mathcal{C}(\mathcal{G}),\ a \neq b\}. \tag{3}$$

The cliques are elementary graphical components and the S_j is obtained as intersection of neighboring cliques. So, we will call the S_j's prime separators (PSs) of the decomposable graph $\mathcal{G}$. The PSs in a decomposable graph may be extended to separators of prime graphs in any undirected graph, where the prime graphs are defined in [3] as the maximal subgraphs without a complete separator.

4 Markovian subgraphs

Let $\mathcal{G}$ be decomposable and consider its Markovian subgraphs, $\mathcal{G}_1, \cdots, \mathcal{G}_m$. The m Markovian subgraphs may be regarded as the graphs of the Markov random fields of $V_1, \cdots, V_m$. In this context, we may refer to a Markovian subgraph as a *marginal graph.*

Definition 4.1. *Suppose there are m marginal graphs, $\mathcal{G}_1, \cdots, \mathcal{G}_m$. Then we say that graph $\mathcal{H}$ of a set of variables V is a* Markovian supergraph *of $\mathcal{G}_1, \cdots, \mathcal{G}_m$, if the following conditions hold:*
(i) $\cup_{i=1}^m V_i = V$.
(ii) $\mathcal{H}_{V_i} = \mathcal{G}_i$, for $i = 1, \cdots, m$. That is, $\mathcal{G}_i$ are Markovian subgraphs of $\mathcal{H}$.

We will call $\mathcal{H}$ a maximal *Markovian supergraph (MaxG) of $\mathcal{G}_1, \cdots, \mathcal{G}_m$ if adding any edge to $\mathcal{H}$ invalidates condition (ii) for at least one $i = 1, \cdots, m$. Since $\mathcal{H}$ depends on $\mathcal{G}_1, \cdots, \mathcal{G}_m$, we denote the collection of the MaxGs formally by $\Omega(\mathcal{G}_1, \cdots, \mathcal{G}_m)$.*

According to this definition, the graph $\mathcal{G}$ is a Markovian supergraph of each $\mathcal{G}_i$, $i = 1, \cdots, m$. There may be many Markovian supergraphs that are obtained from a collection of marginal graphs. For the graphs, $\mathcal{G}, \mathcal{G}_1, \cdots, \mathcal{G}_m$, in the definition, we say that $\mathcal{G}_1, \cdots, \mathcal{G}_m$ are *combined into* $\mathcal{G}$.

In the lemma below, $\mathcal{C}_\mathcal{G}(A)$ is the collection of the cliques which include nodes of A in graph $\mathcal{G}$. The proof is intuitive.

Lemma 4.2. *Let $\mathcal{G}' = (V', E')$ be a Markovian subgraph of $\mathcal{G}$ and suppose that, for three disjoint subsets A, B, C of V', $\langle A|B|C\rangle_{\mathcal{G}'}$. Then*

(i) $\langle A|B|C\rangle_\mathcal{G}$;

(ii) *For $W \in \mathcal{C}_\mathcal{G}(A)$ and $W' \in \mathcal{C}_\mathcal{G}(C)$, $\langle W|B|W'\rangle_\mathcal{G}$.*

The following theorem is similar to Corollary 2.8 in [8], but it is different in that an induced subgraph is considered in the corollary while a Markovian subgraph is considered here.

Theorem 4.3. *Every Markovian subgraph of a decomposable graph is decomposable.*

Proof: Suppose that a Markovian subgraph $\mathcal{G}_A$ of a decomposable graph $\mathcal{G}$ is not decomposable. Then there must exist a chordless cycle, say C, of length ≥ 4 in $\mathcal{G}_A$.

Denote the nodes on the cycle by $\nu_1, \cdots, \nu_l$ and assume that they form a cycle in that order where ν_1 is a neighbor of ν_l.

We need to show that C itself forms a cycle in $\mathcal{G}$ or is contained in a chordless cycle of length $> l$ in $\mathcal{G}$. By Lemma 4.2, there is no edge in $\mathcal{G}$ between any pair of non-neighboring nodes on the cycle. If C itself forms a cycle in $\mathcal{G}$, our argument is done. Otherwise, we will show that the nodes $\nu_1, \cdots, \nu_l$ are on a cycle which is larger than C. Without loss of generality, we may consider the case where there is no edge between ν_1 and ν_2. If there is no path in $\mathcal{G}$ between the two nodes other than the path which passes through $\nu_3, \cdots, \nu_l$, then, since C forms a chordless cycle in $\mathcal{G}_A$, there must exist a path between v_1 and v_2 other than the path which passes through $v_3, \cdots, v_l$. Thus the nodes $\nu_1, \cdots, \nu_l$ must lie in $\mathcal{G}$ on a chordless cycle of length $> l$. This completes the proof. □

This theorem and expression (3) imply that, as for a decomposable graph $\mathcal{G}$, the PSs are always given in the form of a complete subgraph in $\mathcal{G}$ and in its Markovian subgraphs.

Lemma 4.2 states that a separator of a Markovian subgraph of $\mathcal{G}$ is also a separator of $\mathcal{G}$. We will next see that if the MaxG is decomposable provided that all the marginal graphs, $\mathcal{G}_1, \cdots, \mathcal{G}_m$, are decomposable.

Theorem 4.4. *Let $\mathcal{G}_1, \cdots, \mathcal{G}_m$ be decomposable. Then every graph in $\Omega(\mathcal{G}_1, \cdots, \mathcal{G}_m)$ is also decomposable.*

Proof: Suppose that there is a MaxG, say $\mathcal{H}$, which contains an n-cycle ($n \geq 4$) and let A be the set of the nodes on the cycle. Since $\mathcal{H}$ is maximal, we can not add any edge to it. This implies that no more than three nodes of A are included in any of V_i's, since any four or more nodes of A that are contained in a V_i form a cycle in $\mathcal{G}_i$, which is impossible due to the decomposability of the $\mathcal{G}_i$'s. Hence, the cycle in $\mathcal{H}$ may become a clique by edge-additions on the cycle, contradicting that $\mathcal{H}$ is maximal. Therefore, $\mathcal{H}$ must be decomposable. □

Theorem 4.4 does not hold for a Markovian supergraph. For example, in Figure 1, graph $\mathcal{G}$ is not decomposable. However, the Markovian subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are

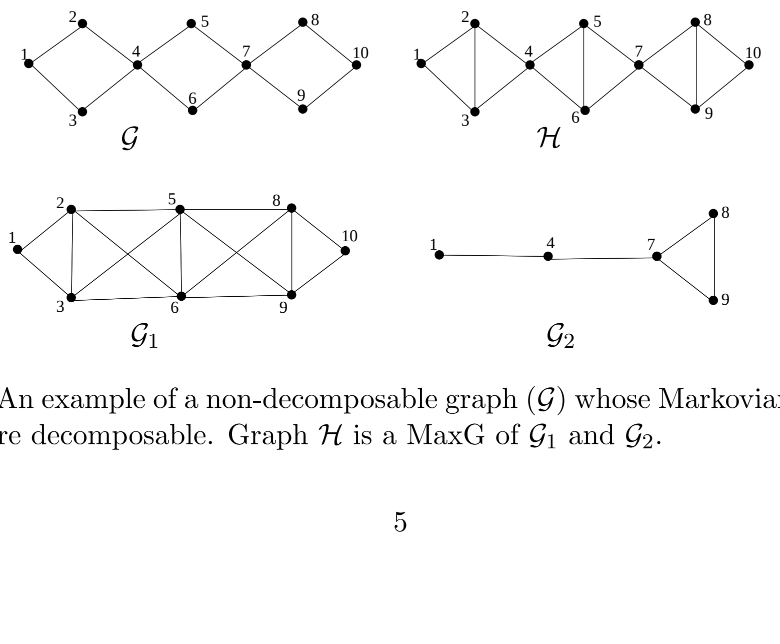

Figure 1: An example of a non-decomposable graph ($\mathcal{G}$) whose Markovian subgraphs ($\mathcal{G}_1$, $\mathcal{G}_2$) are decomposable. Graph $\mathcal{H}$ is a MaxG of $\mathcal{G}_1$ and $\mathcal{G}_2$.

both decomposable. And $\chi(\mathcal{G}) = \{\{4\}, \{7\}\}$, $\chi(\mathcal{G}_1) = \{\{2,3\}, \{5,6\}, \{8,9\}\}$, and $\chi(\mathcal{G}_2) = \{\{4\}, \{7\}\}$. Note that, for $\mathcal{H}$ in the figure, $\chi(\mathcal{H}) = \chi(\mathcal{G}_1) \cup \chi(\mathcal{G}_2)$, which holds true in general as is shown in Theorem 4.7 below. The theorem characterizes a MaxG in a most unique way. Before stating the theorem, we will see if a set of nodes can be a PS in a marginal graph while it is not in another marginal graph.

Theorem 4.5. *Let $\mathcal{G}$ be a decomposable graph and $\mathcal{G}_1$ and $\mathcal{G}_2$ Markovian subgraphs of $\mathcal{G}$. Suppose that a set $C \in \chi(\mathcal{G}_1)$ and that $C \subseteq V_2$. Then C is not intersected in $\mathcal{G}_2$ by any other subset of V_2.*

Proof: Suppose that there are two nodes u and v in C that are separated in $\mathcal{G}_2$ by a set S. Then, by Lemma 4.2, we have $\langle u|S|v\rangle_{\mathcal{G}}$. Since $C \in \chi(\mathcal{G}_1)$ and $\mathcal{G}_1$ is decomposable, C is an intersection of some neighboring cliques of $\mathcal{G}_1$ by equation (3). So, S can not be a subset of V_1 but a proper subset of S can be. This means that there are at least one pair of nodes, v_1 and v_2, in $\mathcal{G}_1$ such that all the paths between the two nodes are intersected by C in $\mathcal{G}_1$, with v_1 appearing in one of the neighboring cliques and v_2 in another.

Since v_1 and v_2 are in neighboring cliques, each node in C is on a path from v_1 to v_2 in $\mathcal{G}_1$. From $\langle u|S|v\rangle_{\mathcal{G}}$ follows that there is an l-cycle ($l \geq 4$) that passes through the nodes u, v, v_1, and v_2 in $\mathcal{G}$. This contradicts to the assumption that $\mathcal{G}$ is decomposable. Therefore, there can not be such a separator S in $\mathcal{G}_2$. □

This theorem states that, if $\mathcal{G}$ is decomposable, a PS in a Markovian subgraph of $\mathcal{G}$ is either a PS or a complete subgraph in any other Markovian subgraph of $\mathcal{G}$. If the set of the nodes of the PS is contained in only one clique of a Markovian subgraph, the set is embedded in the clique. For a subset V' of V, if we put $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_2 = \mathcal{G}_{V'}$ in Theorem 4.5, we have the following corollary.

Corollary 4.6. Let $\mathcal{G}$ be a decomposable graph and suppose that a set $C \in \chi(\mathcal{G})$ and that $C \subseteq V' \subset V$. Then C is not intersected in a Markovian subgraph $\mathcal{G}_{V'}$ of $\mathcal{G}$ by any other subset of V'.

Recall that if $\mathcal{G}_i,\ i = 1, 2, \cdots, m$ are Markovian subgraphs of $\mathcal{G}$, then $\mathcal{G}$ is a Markovian supergraph. For a given set $\mathcal{S}$ of Markovian subgraphs, there may be many MaxGs, and they are related with $\mathcal{S}$ through PSs as in the theorem below.

Theorem 4.7. Let there be Markovian subgraphs $\mathcal{G}_i$, $i = 1, 2, \cdots, m$, of a decomposable graph $\mathcal{G}$. Then

(i)
$$\cup_{i=1}^{m} \chi(\mathcal{G}_i) \subseteq \chi(\mathcal{G});$$

(ii) for any MaxG $\mathcal{H}$,
$$\cup_{i=1}^{m} \chi(\mathcal{G}_i) = \chi(\mathcal{H}).$$

Proof: See Appendix.

For a given set of marginal graphs, we can readily obtain the set of PSs under the decomposability assumption. By (3), we can find $\chi(\mathcal{G})$ for any decomposable graph

$\mathcal{G}$ simply by taking all the intersections of the cliques of the graph. An apparent feature of a MaxG in contrast to a Markovian supergraph is stated in Theorem 4.7.

For a set γ of Markovian subgraphs of a graph $\mathcal{G}$, there can be more than one MaxG of γ. But there is only one such MaxG that contains $\mathcal{G}$ as its edge-subgraph.

Theorem 4.8. Suppose there are m Markovian subgraphs $\mathcal{G}_1, \cdots, \mathcal{G}_m$ of a decomposable graph $\mathcal{G}$. Then there exists a unique MaxG $\mathcal{H}^*$ of the m node-subgraphs such that $\mathcal{G} \subseteq^e \mathcal{H}^*$.

Proof: By Theorem 4.7 (i), we have

$$\cup_{i=1}^m \chi(\mathcal{G}_i) \subseteq \chi(\mathcal{G}).$$

If $\cup_{i=1}^m \chi(\mathcal{G}_i) = \chi(\mathcal{G})$, then since $\mathcal{G}$ is decomposable, $\mathcal{G}$ itself is a MaxG. Otherwise, let $\chi' = \chi(\mathcal{G}) - \cup_{i=1}^m \chi(\mathcal{G}_i) = \{A_1, \cdots, A_g\}$. Since $A_1 \notin \cup_{i=1}^m \chi(\mathcal{G}_i)$, we may add edges so that $\cup_{C \in \mathcal{C}_\mathcal{G}(A_1)} C$ becomes a clique, and the resulting graph $\mathcal{G}^{(1)}$ becomes a Markovian supergraph of $\mathcal{G}_1, \cdots, \mathcal{G}_m$ with $\chi(\mathcal{G}^{(1)}) - \cup_{i=1}^m \chi(\mathcal{G}_i) = \{A_2, \cdots, A_g\}$.

We repeat the same clique-merging process for the remaining A_i's in χ'. Since each clique-merging makes the corresponding PS disappear into the merged, new clique while maintaining the resulting graph as a Markovian supergraph of $\mathcal{G}_1, \cdots, \mathcal{G}_m$, the clique-merging creates a Markovian supergraph of $\mathcal{G}_1, \cdots, \mathcal{G}_m$ as an edge-supergraph of the preceding graph. Therefore, we obtain a MaxG, say $\mathcal{H}^*$, of $\mathcal{G}_1, \cdots, \mathcal{G}_m$ at the end of the sequence of the clique-merging processes for all the PSs in χ'. $\mathcal{H}^*$ is the desired MaxG as an edge-supergraph of $\mathcal{G}$.

Since the clique-merging begins with $\mathcal{G}$ and, for each PS in $\mathcal{G}$, the set of the cliques which meet at the PS only is uniquely defined, the uniqueness of $\mathcal{H}^*$ follows. □

The relationship among Markovian subgraphs is transitive as shown in

Theorem 4.9. *For three graphs, $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}$ with $\mathcal{G}_1 \subseteq^M \mathcal{G}_2 \subseteq^M \mathcal{G}$, it holds that $\mathcal{G}_1 \subseteq^M \mathcal{G}$.*

Proof: For $u, v \in bd_\mathcal{G}(V_2 \setminus V_1) \cap V_1 \times V_1$ with $u \not\sim v$ in $\mathcal{G}_1$, we have

$$\langle u|(V_1 \setminus \{u,v\})|v\rangle_{\mathcal{G}_2} \quad (4)$$

by the condition of the theorem. Expression (4) means that there is no path between u and v in $\mathcal{G}_2$ bypassing $V_1 \setminus \{u,v\}$. Since $\mathcal{G}_2 \subseteq^M \mathcal{G}$, expression (4) implies that $\langle u|(V_1 \setminus \{u,v\})|v\rangle_\mathcal{G}$.

Now consider $u, v \in bd_\mathcal{G}(V_2 \setminus V_1) \cap V_1 \times V_1$ such that $(u,v) \in E_1$ but $(u,v) \notin E_2$. This means that there is a path between u and v in $\mathcal{G}_2$ bypassing $V_1 \setminus \{u,v\}$. Either there is at least one path between u and v in $\mathcal{G}_{V_2}^{ind}$ bypassing $V_1 \setminus \{u,v\}$, or there is no such path in $\mathcal{G}_{V_2}^{ind}$ at all. In the former situation, it must be that $u \sim v$ in $\mathcal{G}_1$ as a Markovian subgraph of $\mathcal{G}$. In the latter situation, at least one path is newly created in $\mathcal{G}_{V_2}^{ind}$ when $\mathcal{G}_{V_2}^{ind}$ becomes a Markovian subgraph of $\mathcal{G}$. This new path contains an edge, (v_1, v_2) say, in $\{B \times B;\ B \in \beta_\mathcal{G}(\mathcal{J}_{V_2})\} \cap V_2 \times V_2$ where $\mathcal{J}_{V_2}$ is the connectivity components in $\mathcal{G}^{ind}_{V_2^c}$. This also implies that there is at least one path between v_1 and v_2 in $\mathcal{G}$ bypassing $V_2 \setminus \{v_1, v_2\}$. In a nutshell, the statement that $(u, v) \in E_1$ but $(u, v) \notin E$ implies that there is at least one path between u and v in $\mathcal{G}$ bypassing $V_1 \setminus \{u, v\}$. This completes the proof. □

5 Markovian supergraphs from marginal graphs

Given a collection γ of marginal graphs, a Markovian supergraph of γ may not exist unless the marginal graphs are Markovian subgraphs of a graph. We will consider in this section collections of Markovian subgraphs of a graph $\mathcal{G}$ and investigate the relationship of a Markovian supergraph of a collection with those of another collection.

Let $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ be Markovian subgraphs of $\mathcal{G}_1$ with $V_{11} \cup V_{12} = V_1$, and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be Markovian subgraphs of $\mathcal{G}$ with $V_1 \cup V_2 = V$. For $H \in \Omega(\mathcal{G}_1, \mathcal{G}_2)$, we have, by Theorem 4.9, that $\mathcal{G}_{11} \subseteq^M H$ and $\mathcal{G}_{12} \subseteq^M H$, since $\mathcal{G}_1 \subseteq^M H$. Thus, H is a Markovian supergraph of $\mathcal{G}_{11}, \mathcal{G}_{12}$, and $\mathcal{G}_2$, but may not be a MaxG of them since $\chi(\mathcal{G}_{11}) \cup \chi(\mathcal{G}_{12}) \subseteq \chi(\mathcal{G}_1)$ by Theorem 4.7 (i). We can generalize this as follows. We denote by $V(\mathcal{G})$ the set of nodes of $\mathcal{G}$.

Theorem 5.1. Consider two collections, γ_1 and γ_2, of Markovian subgraphs of $\mathcal{G}$ with $\cup_{g\in\gamma_1} V(g) = \cup_{g\in\gamma_2} V(g) = V(\mathcal{G})$. For every $g \in \gamma_2$, there exists a graph $h \in \gamma_1$ such that $g \subseteq^M h$. Then, every $H \in \Omega(\gamma_1)$ is a Markovian supergraph of γ_2.

Proof: For $H \in \Omega(\gamma_1)$, every $h \in \gamma_1$ is a Markovian subgraph of H. By the condition of the theorem, for each $g \in \gamma_2$, we have $g \subseteq^M h'$ for some $h' \in \gamma_1$. Thus, by Theorem 4.9, $g \subseteq^M H$. Since $\cup_{g\in\gamma_2} V(g) = V(\mathcal{G})$, H is a Markovian supergraph of γ_2. □

From this theorem and Theorem 4.7 we can deduce that, for $H \in \Omega(\gamma_1)$,

$$\cup_{g\in\gamma_2} \chi(g) \subseteq \chi(H).$$

This implies that H cannot be a proper supergraph of any H' in $\Omega(\gamma_2)$. Since γ_1 and γ_2 are both from the same graph $\mathcal{G}$, H is an edge-subgraph of some $H' \in \Omega(\gamma_2)$ when $\cup_{g\in\gamma_2} \chi(g) \subset \chi(H)$. However, it is noteworthy that every pair H and H', $H \in \Omega(\gamma_1)$ and $H' \in \Omega(\gamma_2)$, are not necessarily comparable as we will see below.

Example 5.2. Consider the graph $\mathcal{G}$ in Figure 2 and let $V_1 = \{3, 4, 5, 6, 7, 8\}$ and $V_2 = \{1, 2, 3, 5, 7, 9\}$. The Markovian subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are also in Figure 2. Note that

$$\chi(\mathcal{G}_1) \cup \chi(\mathcal{G}_2) = \chi(\mathcal{G}) = \{\{3\}, \{2, 3\}, \{5\}, \{6\}, \{7\}\}$$

and that $\mathcal{G} \in \Omega(\mathcal{G}_1,\ \mathcal{G}_2)$.

Let $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ be two Markovian subgraphs of $\mathcal{G}_1$ as in Figure 2. Then $\{\mathcal{F}_1\} = \Omega(\mathcal{G}_{11}, \mathcal{G}_{12})$. $\chi(\mathcal{F}_1) = \{\{5\}, \{6\}\}$ and $\chi(\mathcal{G}_2) = \{\{2, 3\}, \{5\}, \{7\}\}$. Let $\gamma_1 = \{\mathcal{F}_1, \mathcal{G}_2\}$. Then, for every $H \in \Omega(\gamma_1)$,

$$\chi(H) = \{\{2, 3\}, \{5\}, \{6\}, \{7\}\} \tag{5}$$

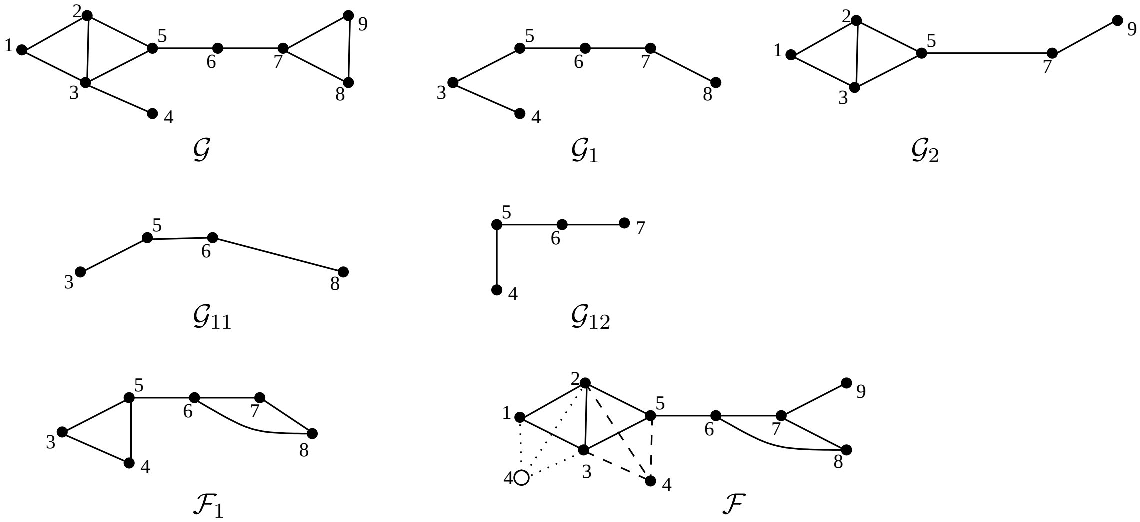

Figure 2: Markovian subgraphs and supergraphs. In graph $\mathcal{F}$, $bd(4) = \{1, 2, 3\}$ or $\{2, 3, 5\}$. We denote the $\mathcal{F}$ by $\mathcal{F}_\circ$ when $bd(4) = \{1, 2, 3\}$ and by $\mathcal{F}_\bullet$ when $bd(4) = \{2, 3, 5\}$.

by Theorem 4.7. In $\mathcal{G}_2$, we have $\langle \{2, 3\}|5|7\rangle$; and in $\mathcal{F}_1$, $\langle 5|6|7\rangle$. Thus, the four PSs in (5) are to be arranged in a path, $\{2, 3\}$ — $\{5\}$ — $\{6\}$ — $\{7\}$. The remaining nodes can be added to this path as the two graphs in γ_1 suggest in such a way that equation (5) may hold. Note that $\langle 4|5|6\rangle$ in $\mathcal{F}_1$. This means that node 4 must form a clique either with $\{1, 2, 3\}$ or with $\{2, 3, 5\}$ because $\{2, 3\}$ is a PS. This is depicted in $\mathcal{F}$ of Figure 2 representing two possible cliques which include node 4. The two different MaxGs are denoted by $\mathcal{F}_\circ$ and $\mathcal{F}_\bullet$ which are explained in the caption of Figure 2. □

It is worthwhile to note that $\mathcal{G}$ is not an edge-subgraph of either of the two MaxGs in $\Omega(\gamma_1)$ while $\{\mathcal{G}\} = \Omega(\mathcal{G}_1, \mathcal{G}_2)$. This phenomenon seems to contradict Theorem 4.8 which says that there always exists a MaxG which is an edge-supergraph of $\mathcal{G}$. But recall that $\mathcal{F}_1$ is a MaxG of $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$. If we let $\gamma_2 = \{\mathcal{G}_{11}, \mathcal{G}_{12}, \mathcal{G}_2\}$, Example 5.2 shows that $\Omega(\gamma_2)$ is not the same as $\Omega(\gamma_1)$.

The $\mathcal{F}$ in Figure 2 is obtained first by combining $\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ into $\mathcal{F}_1$ and then by combining the graphs in γ_1. This is a sequential procedure. It is apparent from Example 5.2 that a sequential combination of graphs does not necessarily lead us to a MaxG which is an edge-supergraph of $\mathcal{G}$. For a collection γ of marginal graphs, it is desirable that $\Omega(\gamma)$ is obtained by considering the graphs in γ simultaneously.

6 Concluding remarks

In this paper, we have explored the relationship between a decomposable graph and its Markovian subgraph which is summarized in Theorem 4.7. Let there be a collection γ of Markovian subgraphs of a decomposable graph $\mathcal{G}$. Theorem 4.8 states that there always exists a MaxG of γ which contains $\mathcal{G}$ as an edge-subgraph.

According to Theorem 5.1, we may consider a sequence of collections, $\gamma_1, \cdots, \gamma_r$, of Markovian subgraphs of $\mathcal{G}$, where γ_i and γ_j, $i < j$, are ordered such that for every $g \in \gamma_j$, there exists $h \in \gamma_i$ satisfying $g \subseteq^M h$. Every $H \in \Omega(\gamma_i)$ is a Markovian supergraph of $g \in \gamma_j$, but, as shown in Example 5.2, an $H \in \Omega(\gamma_j)$ may not be a Markovian supergraph of a graph in γ_i. This implies that if we are interested in Markovian supergraphs of $V(\mathcal{G})$, the collection γ_1 is the best to use among the collections, $\gamma_1, \cdots, \gamma_r$.

Appendix: Proof of theorem 4.7

We will first prove result (i). For a subset of nodes V_j, the followings hold:

(i') If V_j does not contain a subset which is a PS of $\mathcal{G}$, then $\chi(\mathcal{G}_j) = \emptyset$.

(ii') Otherwise, i.e., if there are PSs, $C_1, \cdots, C_r$, of $\mathcal{G}$ as subsets of V_j,

- (ii'-a) if there are no nodes in V_j that are separated by any of $C_1, \cdots, C_r$ in $\mathcal{G}$, then $\chi(\mathcal{G}_j) = \emptyset$.
- (ii'-b) if there is at least one of the PSs, say C_s, such that there are a pair of nodes, say u and v, in V_j such that $\langle u|C_s|v\rangle_{\mathcal{G}}$, then $\chi(\mathcal{G}_j) \neq \emptyset$.

We note that, since $\mathcal{G}$ is decomposable, the condition that V_j contains a separator of $\mathcal{G}$ implies that V_j contains a PS of $\mathcal{G}$. As for (i'), every pair of nodes, say u and v, in V_j have at least one path between them that bypasses $V_j \setminus \{u, v\}$ in the graph $\mathcal{G}$ since V_j does not contain any PS of $\mathcal{G}$. Thus, (i') follows.

On the other hand, suppose that there are PSs, $C_1, \cdots, C_r$, of $\mathcal{G}$ as a subset of V_j. The result (ii'-a) is obvious, since for each of the PSs, $C_1, \cdots, C_r$, the rest of the nodes in V_j are on one side of the PS in $\mathcal{G}$.

As for (ii'-b), let there be two nodes, u and v, in V_j such that $\langle u|C_s|v\rangle_{\mathcal{G}}$. Since $\mathcal{G}$ is decomposable, C_s is an intersection of neighboring cliques in $\mathcal{G}$, and the nodes u and v must appear in some (not necessarily neighboring) cliques that are separated by C_s. Thus, the two nodes are separated by C_s in $\mathcal{G}_j$ with C_s as a PS in $\mathcal{G}_j$. Any proper subset of C_s can not separate u from v in $\mathcal{G}$ and in any of its Markovian subgraphs.

From the results (i') and (ii') follows that

(iii') if $C \in \chi(\mathcal{G})$ and $C \subseteq V_j$, then either $C \in \chi(\mathcal{G}_j)$ or C is contained in only one clique of $\mathcal{G}_j$.

(iv') that $\chi(\mathcal{G}_j) = \emptyset$ does not necessarily implies that $\chi(\mathcal{G}) = \emptyset$.

To check if $\chi(\mathcal{G}_j) \not\subseteq \chi(\mathcal{G})$ for any $j \in \{1, 2, \cdots, m\}$, suppose that $C \in \chi(\mathcal{G}_j)$ and $C \notin \chi(\mathcal{G})$. This implies, by Lemma 4.2, that C is a separator but not a PS in $\mathcal{G}$. Thus, there is a proper subset C' of C in $\chi(\mathcal{G})$. By (iii'), $C' \in \chi(\mathcal{G}_j)$ or is contained in only one clique of $\mathcal{G}_j$. However, neither is possible, since $C' \subset C \in \chi(\mathcal{G}_j)$ and C is an intersection of cliques of $\mathcal{G}_j$. Therefore,

$$\chi(\mathcal{G}_j) \subseteq \chi(\mathcal{G}) \quad \text{for all } j.$$

This proves result (i) of the theorem.

We will now prove result (ii). If $\cup_{i=1}^m \chi(\mathcal{G}_i) = \emptyset$, then, since all the $\mathcal{G}_i$'s are decomposable by Theorem 4.3, they are complete graphs themselves. So, by definition, the MaxG must be a complete graph of V. Thus, the equality of the theorem holds.

Next, suppose that $\cup_{i=1}^m \chi(\mathcal{G}_i) \neq \emptyset$. Then there must exist a marginal model structure, say $\mathcal{G}_j$, such that $\chi(\mathcal{G}_j) \neq \emptyset$. Let $A \in \chi(\mathcal{G}_j)$. Then, by Theorem 4.5, A is either a PS or embedded in a clique if $A \subseteq V_i$ for $i \neq j$. Since a PS is an intersection of cliques by equation (3), the PS itself is a complete subgraph. Thus, by the definition of MaxG and by Lemma 4.2, $A \in \chi(\mathcal{H})$. This implies that $\cup_{i=1}^m \chi(\mathcal{G}_i) \subseteq \chi(\mathcal{H})$.

To show that the set inclusion in the last expression comes down to equality, we will suppose that there is a set B in $\chi(\mathcal{H}) \setminus (\cup_{i=1}^m \chi(\mathcal{G}_i))$ and show that this leads to a contradiction to the condition that $\mathcal{H}$ is a MaxG. $\mathcal{H}$ is decomposable by Theorem 4.4. So, B is the intersection of the cliques in $\mathcal{C}_{\mathcal{H}}(B)$. By supposition, $B \notin \chi(\mathcal{G}_i)$ for all $i = 1, \cdots, m$. This means either (a) that $B \subseteq V_j$ for some j and $B \subseteq C$ for only one clique C of $\mathcal{G}_j$ by Corollary 4.6 or (b) that $B \not\subseteq V_j$ for all $j = 1, \cdots, m$. In both of the situations, B need not be a PS in $\mathcal{H}$, since $\mathcal{G}_i$ are decomposable and so $B \cap V_i$ are complete in $\mathcal{G}_i$ in both of the situations. In other words, edges may be added to $\mathcal{H}$ so that $\mathcal{C}_{\mathcal{H}}(B)$ becomes a clique, which contradicts that $\mathcal{H}$ is a MaxG. This completes the proof. □

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