Oblique projections, pseudo-inverses and the fibre principle of shift-invariant spaces *

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Abstract

We find a relationship between the existence of the oblique projection between two, not necessarily regular, finitely generated shift-invariant spaces of $L^2(\mathbb{R}^d)$ and the pseudo-inverses of the mixed Gramians of the fibre spaces of the shift-invariant spaces, thereby improving

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a result of Aldroubi. Our result is another illustration of the so-called fibre principle of the shift-invariant spaces. Applications to the theory of biorthogonal multiresolution analyses are discussed.

1 Introduction and main results

The notions of oblique projection and the angle between subspaces are closely related with the theory of biorthogonal (or non-orthogonal) multiresolution analyses [1, 2, 22]. The purpose of this article is to clarify further this relationship. In particular, we do this by using the fibre principle of the shift-invariant subspaces of $L^2(\mathbb{R}^d)$.

Let us first introduce the definitions of the various concepts which interest us. Let U and V be closed subspaces of a separable Hilbert space $\mathcal{H}$ over the complex field $\mathbb{C}$. We say that the *oblique projection* $P_{U\perp V}$ of $\mathcal{H}$ on U along $V^\perp$ is well-defined if $\mathcal{H} = U \dot{+} V^\perp$, that is, if $\mathcal{H} = U + V^\perp$ and $U \cap V^\perp = \{0\}$ [1]. In this case, for any $f \in \mathcal{H}$ there exist unique $u \in U$ and $v^\perp \in V^\perp$ such that $f = u + v^\perp$. We define $P_{U\perp V} f := u$. This concept is closely related with that of the angle between U and V which is defined as follows [1]: The angle $R(U, V)$ is defined to be

$$R(U, V) := \inf_{u \in U \setminus \{0\}} \frac{||P_V u||}{||u||},$$

where P_V denotes the orthogonal projection of $\mathcal{H}$ onto V. We define $R(U, V)$ to be 1 if either U or V is trivial. See [1, 23] for the geometric meaning of this concept and its applications to signal processing, and see [2, 8, 16, 17] for its applications to the theory of wavelets. Note that, in general, it does not hold that $R(U,V) = R(V,U)$ [8]. The following proposition, which is Theorem 2.3 of [22], shows the connexion between two concepts.

Proposition 1 ([22]) *Let U and V be closed subspaces of $\mathcal{H}$. The following conditions are equivalent:*

(1) $\mathcal{H} = U \dotplus V^{\perp}$;

(2) $\mathcal{H} = U^{\perp} \dotplus V$;

(3) *There exist Riesz bases $\{u_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ for U and V, respectively, such that $\{u_i\}_{i\in I}$ is biorthogonal to $\{v_i\}_{i\in I}$;*

(4) $R(U,V) > 0$ *and* $R(V,U) > 0$.

The definition of a Riesz basis is found in the next section.

We now introduce the concept of multiresolution analysis which is the main tool in the construction of wavelets [10, 18]. We refer to [10] for the applications of multiresolution analyses to the theory of wavelets. Let $D : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denote the unitary dyadic dilation operator defined via $Df(x) := 2^{d/2}f(2x)$. For $y \in \mathbb{R}^d$, $T_y : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ denotes the unitary translation operator such that $T_y f(x) := f(x-y)$.

Definition 2 *A sequence of closed subspaces $\{V_k\}_{k\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ is said to be a multiresolution analysis if*

(1) $V_k \subset V_{k+1}, \quad k \in \mathbb{Z}$;

(2) $\overline{\cup_{k\in\mathbb{Z}} V_k} = L^2(\mathbb{R}^d)$, $\cap_{k\in\mathbb{Z}} V_k = \{0\}$;

(3) $D(V_k) = V_{k+1}, \quad k \in \mathbb{Z}$;

(4) *There exists a finite set of scaling functions* $\Phi \subset V_0$ *such that* $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ *is a Riesz basis for* V_0.

Suppose that $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$ are multiresolution analyses. If their respective sets Φ and $\tilde{\Phi}$ of scaling functions have the same number of elements and if $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ and $\{T_k\tilde{\varphi} : k \in \mathbb{Z}^d, \tilde{\varphi} \in \tilde{\Phi}\}$ are biorthogonal, then we say that they are *biorthogonal multiresolution analyses*. This definition is a direct generalisation of the 1-dimensional definition of Wang [24]. Proposition 1 now implies that this condition is equivalent to the other conditions of Proposition 1. This in turn implies that if we construct a pair of biorthogonal multiresolution analyses, then the central spaces of the two multiresolution analyses must satisfy the conditions of Proposition 1. This result is not only theoretically important, but also often useful in the actual construction of biorthogonal multiresolution analyses [2].

In some applications, more general multiresolution analyses are considered. In particular, if Condition (4) of Definition 2 is replaced by

(4′) There exists a finite set of scaling functions $\Phi \subset V_0$ such that $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a frame for V_0;

then we say that $\{V_j\}_{j\in\mathbb{Z}}$ is a *frame multiresolution analysis*. The definition and the basic properties of a frame are found in the next section. For the time being we only mention that a Riesz basis is a frame. Hence a frame multiresolution analysis is a more general concept than a multiresolution analysis. Benedetto and Li successfully applied the theory of frame multiresolution analyses in the analysis of narrow band signals [3]. Suppose we try to define the concept of biorthogonal frame multiresolution analyses. The situation is as follows: We have two frame multiresolution analyses $\{V_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{V}_j\}_{j\in\mathbb{Z}}$. Let Φ and $\tilde{\Phi}$ be the sets of scaling functions of the respective frame multiresolution analyses. It does not have to be that the cardinality of Φ and that of $\tilde{\Phi}$ coincide. Therefore, we cannot use the variant of Condition (3) of Proposition 1 as the definition of the biorthogonality of two frame multiresolution analyses. We may, however, use Condition (4) of Proposition 1 as the definition of the biorthogonality. Hence, we say that they are *biorthogonal* if $R(V_0, \tilde{V_0}) > 0$. In [17] Kim et al. developed the 1-dimensional version of the theory of biorthogonal frame multiresolution analyses in which the numbers of scaling functions of the two frame multiresolution analyses concerned are all one. The full analysis of biorthogonal frame multiresolution analyses is outside the scope of this article. We are merely interested in finding conditions easier to check than those of Proposition 1. We do this by using the theory of shift-invariant spaces and the fibre principle which will be explained briefly in a moment. All of the results on the theory of shift-invariant spaces we use are contained in [5, 7, 13, 15, 19].

A subspace $S \subset L^2(\mathbb{R}^d)$ is said to be a *shift-invariant subspace* of $L^2(\mathbb{R}^d)$ if it is closed and is invariant under each (multi-)integer translation operator $T_k, k \in \mathbb{Z}^d$. In particular, the central space V_0 of a multiresolution analysis or a frame multiresolution analysis is a shift-invariant subspace. For $f \in L^2(\mathbb{R}^d), x \in \mathbb{T}^d$, we let

$$\hat{f}_{||x} := (\hat{f}(x+k))_{k\in\mathbb{Z}^d},$$

which belongs to $\ell^2(\mathbb{Z}^d)$ almost every $x \in \mathbb{T}^d$. Here $\wedge$ denotes the Fourier transform defined by

$$\hat{f}(x) := \int_{\mathbb{R}^d} f(t)^{-2\pi i x\cdot t}\, dt$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and extended to be a unitary operator on $L^2(\mathbb{R}^d)$ by the Plancherel theorem. For a shift-invariant subspace S and $x \in \mathbb{T}^d$ we let

$$\hat{S}_{||x} := \overline{\text{span}}\{\hat{f}_{||x} : f \in S\}.$$

It is known that $\hat{S}_{||x}$, called the *fibre* of S at x, is a closed subspace of $\ell^2(\mathbb{Z}^d)$ for almost every $x \in \mathbb{T}^d$. The *spectrum* $\sigma(S)$ of S is defined to be

$$\sigma(S) := \{x \in \mathbb{T}^d : \hat{S}_{||x} \neq \{0\}\}.$$

If there exists $n \in \mathbb{N}$ such that $\dim \hat{S}_{||x} = n$ for almost every $x \in \mathbb{T}$, we say that S is *regular*. If Φ is a subset of $L^2(\mathbb{R}^d)$, then we let

$$\mathcal{S}(\Phi) := \overline{\text{span}}\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\},$$

which is clearly a shift-invariant subspace. We then say that $\mathcal{S}(\Phi)$ is a *shift-invariant space generated by* Φ. In case Φ is finite, we say that it is *finitely generated*. It is known that a shift-invariant subspace of $L^2(\mathbb{R}^d)$ has a generator set whose cardinality is at most countable. Moreover, it is shown in [5] that S is regular if and only if there exists a finite subset Φ of S such that $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a Riesz basis for S. The so-called *fibre principle* is roughly stated as follows: A property holds for a shift-invariant space S if and only if it holds for each fibre space of S in a uniform way. It is best understood by looking at examples. Hence we introduce some examples of the fibre principle which will be used later in proving our main results.

The following is Proposition 2.10 of [8].

Proposition 3 ([8]) *If U and V are shift-invariant subspaces of $L^2(\mathbb{R}^d)$, then*

$$R(U,V) = \text{ess-inf}_{x\in\mathbb{T}^d} R(\hat{U}_{||x}, \hat{V}_{||x}).$$

The following is Theorem 2.3 of [7].

Proposition 4 ([7]) *Suppose that $\Phi \subset L^2(\mathbb{R}^d)$ is countable. Then $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a frame/Riesz basis for $\mathcal{S}(\Phi)$ with frame/Riesz bounds A and B if and only if, for almost every $x \in \mathbb{T}^d$, $\{\hat{\varphi}_{||x} : \varphi \in \Phi\}$ is a frame/Riesz basis for $(\mathcal{S}(\Phi))^{\wedge}_{||x}$ with frame/Riesz bounds A and B.*

The readers are now convinced that if one is to analyse a shift-invariant subspace, then it probably is best to analyse the fibre spaces separately and then to patch up the fibre-wise analyses together to produce a result on the original shift-invariant space.

There is an elegant theory, called the Gramian/dual Gramian analysis, that somehow formalises this method [20, 21]. The following is an example.

Suppose that $S := \mathcal{S}(\Phi)$ is finitely generated. The matrix $G_\Phi(x) := (\langle \hat{\varphi}_{||x}, \hat{\psi}_{||x} \rangle)_{\psi,\varphi\in\Phi}$ is called the *Gramian* of Φ at $x \in \mathbb{T}^d$. Let $\lambda(x), \lambda^+(x)$ and $\Lambda(x)$ denote the smallest eigenvalue, the smallest non-zero eigenvalue and the largest eigenvalue of $G(x)$. The proof of the following proposition is found in [19].

Proposition 5 ([19]) *$\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a Riesz basis for $\mathcal{S}(\Phi)$ with Riesz bounds A and B if and only if*

$$A \le \lambda(x) \le \Lambda(x) \le B$$

for almost every $x \in \mathbb{T}^d$. It is a frame for S with frame bounds A and B if and only if

$$A \le \lambda^+(x) \le \Lambda(x) \le B$$

for almost every $x \in \sigma(\mathcal{S}(\Phi))$.

The following theorem, which is also an illustration of the fibre principle, is the main result of this article. Here $G^\dagger$ denotes the pseudo-inverse of an operator/matrix G with closed range [11], the theory of which will be reviewed in the next section.

Theorem 6 *Let $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_m\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$, and let $U := \mathcal{S}(\Phi), V := \mathcal{S}(\Psi)$. Suppose that $\{T_k\varphi_j : k \in \mathbb{Z}^d, 1 \le j \le m\}$ and $\{T_k\psi_i : k \in \mathbb{Z}^d, 1 \le i \le n\}$ are frames for U and V, respectively. Let*

$$G(x) := G_{\Phi,\Psi}(x) := (\langle \hat{\varphi}_{j||x}, \hat{\psi}_{i||x} \rangle)_{1\le i\le n, 1\le j\le m},\ x \in \mathbb{T}^d.$$

Then the following assertions are equivalent:

(1) $L^2(\mathbb{R}^d) = U \dot{+} V^\perp$*;*

(2) $\operatorname{rank} G(x) = \dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ *for almost every* $x \in \mathbb{T}^d$*; and there exists a positive constant* C *such that* $||G(x)^\dagger|| \le C$ *for almost every* $x \in \mathbb{T}^d$.

The matrix $G(x)$ is called the *mixed Gramian* of Φ and Ψ at x. The frame condition is not restrictive since it is shown in [5, 7, 19] that any shift-invariant subspace S has a generator Φ such that $\{T_k\varphi : k \in \mathbb{Z}^d, \varphi \in \Phi\}$ is a frame for S. It is, however, generally impossible to find a generator for S which generates a Riesz basis unless $\sigma(S) = \mathbb{T}^d$ [5, 7, 19]. The following corollary is a special case of one of the main results of Aldroubi [1]. He considers the subspace of a general Hilbert space $\mathcal{H}$ of the form

$$\{\sum_{i=1}^{r}\sum_{j\in\mathbb{Z}} c_i(j)O^j\varphi_i : c_i \in \ell^2(\mathbb{Z}), 1 \le i \le r\},$$

where $\varphi_i \in \mathcal{H}, 1 \leq i \leq r$ and O is a unitary operator on $\mathcal{H}$. If we let $\mathcal{H} := L^2(\mathbb{R})$ and $O := T_1$, then Theorem 3.1 of [1] reduces to the $L^2(\mathbb{R})$ version as in the following corollary.

Corollary 7 *Let* $\Phi := \{\varphi_1, \varphi_2, \cdots, \varphi_n\}, \Psi := \{\psi_1, \psi_2, \cdots, \psi_n\} \subset L^2(\mathbb{R}^d)$, *and let* $U := \mathcal{S}(\Phi), V := \mathcal{S}(\Psi)$. *Suppose that* $\{T_k\varphi_j : k \in \mathbb{Z}^d, 1 \leq j \leq n\}$ *and* $\{T_k\psi_i : k \in \mathbb{Z}^d, 1 \leq i \leq n\}$ *are Riesz bases for* U *and* V*, respectively. Let*

$$G(x) := G_{\Phi,\Psi}(x) := (\langle \hat{\varphi}_{j||x}, \hat{\psi}_{i||x} \rangle)_{1 \leq i,j \leq n}, x \in \mathbb{T}^d,$$

be the mixed Gramian of Φ *and* Ψ*. Then the following assertions are equivalent:*

(1) $L^2(\mathbb{R}^d) = U \dot{+} V^\perp$;

(2) $G(x)$ *is invertible for almost every* $x \in \mathbb{T}^d$*; and there exists a positive real number* C *such that* $||G(x)^{-1}|| \leq C$ *for almost every* $x \in \mathbb{T}^d$.

Theorem 6 is inspired by Corollary 7, which is the shift-invariant subspace interpretation of the result of Aldroubi [1]. The method of proof, however, is rather different, and uses the full power of the fibre principle.

Moreover, we have the following result. For an operator/matrix X, X^* denotes its adjoint operator/matrix.

Theorem 8 *Suppose that the hypotheses of Theorem 6 hold. If any one of the conditions of Theorem 6 is satisfied, then*

$$\begin{aligned} R(U,V) &= \text{ess-inf}_{x\in\sigma(U)} ||T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger T^*_{\hat{V}_{||x}}||^{-1} \\ &= \text{ess-inf}_{x\in\sigma(U)} ||G_\Phi(x)^{1/2} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2}||^{-1} \\ &= R(V,U), \end{aligned}$$

where $T_{\hat{U}_{||x}}$ and $T_{\hat{V}_{||x}}$ are pre-frame operators defined in the next section and $G_\Phi(x)$ and $G_\Psi(x)$ are the Gramians of Φ and Ψ at x, respectively.

The following obvious corollary is also a special case of Theorem 3.2 of [1].

Corollary 9 *Suppose that the hypotheses of Corollary 7 hold. If any one of the conditions of Corollary 7 is satisfied, then*

$$R(U,V) = R(V,U) = \text{ess-inf}_{x\in\mathbb{T}^d} ||G_\Phi(x)^{1/2} G_{\Phi,\Psi}(x)^{-1} G_\Psi(x)^{1/2}||^{-1},$$

where $G_\Phi(x)$ and $G_\Psi(x)$ are the Gramians of Φ and Ψ at x, respectively.

Compare the above theorems with Lemma 3.2 in [16] and Lemma 5.11 of [17].

The rest of the paper is organised as follows: We present preliminary results on the theory of frames and that of pseudo-inverses in Section 2. The proofs of the main results are found in Section 3

2 Preliminary discussions

In this section we introduce the relevant definitions and preliminary results which will be used later.

We first present basic facts about pseudo-inverses. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces over $\mathbb{C}$, and $X : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded linear operator with closed range. For each $b \in \mathcal{H}_2$, $\{a \in \mathcal{H}_1 : Xa = P_{\operatorname{ran} X} b\}$ is a closed convex subset of $\mathcal{H}_1$, where $P_{\operatorname{ran} X}$ denotes the orthogonal projection of $\mathcal{H}_2$ onto $\operatorname{ran} X$. Hence it contains a unique element a of minimal norm. We let

$a := X^\dagger b$. It is known that the map: $X^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ is a bounded linear operator, called the *pseudo-inverse* of X [11].

We introduce two results which will be used later.

Proposition 10 ([11]) *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces over $\mathbb{C}$. Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with closed range. Then the following assertions hold:*

(1) $\operatorname{ran} X^\dagger = \operatorname{ran} X^*$*;*

(2) $XX^\dagger = P_{\operatorname{ran} X}$*;*

(2) $X^\dagger X = P_{\operatorname{ran} X^\dagger}$*.*

Proof. (1) is a part of Theorem 2.1.2 of [11]; and (2) and (3) are parts of Theorem 2.2.2 of [11]. □

The following is Theorem 3.1 of [6].

Proposition 11 ([6]) *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be separable Hilbert spaces over $\mathbb{C}$ and $X : \mathcal{H}_2 \to \mathcal{H}_3, Y : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operators with closed range. Then $(XY)^\dagger = Y^\dagger X^\dagger$ if and only if*

(i) $\operatorname{ran} XY$ *is closed;*

(ii) $\operatorname{ran} X^*$ *is invariant under YY^*;*

(iii) $\operatorname{ran} X^* \cap \ker Y^*$ *is invariant under X^*X.*

We now introduce the theory of frames briefly. Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$, and let $\{f_i : i \in I\}$ be a sequence in $\mathcal{H}$, where I is a countable index set. We say that $\{f_i : i \in I\}$ is a *frame* for $\mathcal{H}$ if there exist positive constants A and B such that

$$A||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B||f||^2$$

for each $f \in \mathcal{H}$. A and B are called the lower and upper frame bounds, respectively. Suppose that $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$ with frame bounds A and B. Define $T : \ell^2(I) \to \mathcal{H}$ via $T\alpha := \sum_{i\in I} \alpha_i f_i$, where $\alpha := (\alpha_i)_{i\in I}$. It is known that T, usually called the *pre-frame operator*, is an onto bounded linear operator [9, 14]. Moreover, $||T|| \leq B^{1/2}$. A direct calculation shows that $T^*f = (\langle f, f_i\rangle)_{i\in I}$. Let $S := TT^*$. Then S, called the *frame operator*, is a strictly positive (and hence self-adjoint) bounded linear operator with a bounded inverse [12]. More precisely, we have $Sf = \sum_{i\in I}\langle f, f_i\rangle f_i$ and $A \leq S \leq B$. We say that $\{f_i : i \in I\}$ is a *Riesz basis* for $\mathcal{H}$ with Riesz bounds A and B if it is complete and there exist positive constants A and B such that for any $(c_i)_{i\in I} \in \ell^2(I)$

$$A||f||^2 \leq ||\sum_{i\in I} c_i f_i||^2 \leq B||f||^2.$$

We refer to [12] for the basic properties of Riesz bases and frames of a separable Hilbert space. In particular, it is shown there that a Riesz basis is a frame. Note also that if I is a finite set, then a Riesz basis is just another ordinary basis treated in Linear Algebra.

Now, let $\{u_j\}_{j=1}^m$ be a frame for a closed subspace U of $\mathcal{H}$ with frame bounds A_U and B_U and let $\{v_i\}_{i=1}^n$ be a frame for a closed subspace V of $\mathcal{H}$ with frame bounds A_V and B_V. Suppose that $r = \dim U = \dim V$. Then obviously, $r \leq \min\{m, n\}$. Let $T_U : \mathbb{C}^m \to U$ and $T_V : \mathbb{C}^n \to V$

be the pre-frame operators of $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$, respectively. Also let $S_U : U \to U$ and $S_V : V \to V$ be the frame operators of $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$, respectively. Finally, let P_V be the orthogonal projection of $\mathcal{H}$ onto V, and $P := P_{V|U} : U \to V$ the restriction of P_V to U.

Lemma 12 *Suppose $G : \mathbb{C}^m \to \mathbb{C}^n$ is the mixed Gramian of the frames $\{u_j\}_{j=1}^m$ and $\{v_i\}_{i=1}^n$ such that $G_{i,j} := \langle u_j, v_i\rangle, 1 \le i \le n, 1 \le j \le m$. Then*

$$G = T_V^* P T_U.$$

Proof. Let $\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_m)^t \in \mathbb{C}^m$.

$$\begin{aligned}
T_V^* P T_U \alpha &= T_V^* P(\sum_{j=1}^m \alpha_j u_j) = T_V^*(\sum_{j=1}^m \alpha_j P u_j) \\
&= (\langle \sum_{j=1}^m \alpha_j P u_j, v_i\rangle)_{i=1}^n = (\sum_{j=1}^m \alpha_j \langle P u_j, v_i\rangle)_{i=1}^n \\
&= (\sum_{j=1}^m \alpha_j \langle P_V u_j, v_i\rangle)_{i=1}^n = (\sum_{j=1}^m \alpha_j \langle u_j, P_V v_i\rangle)_{i=1}^n \\
&= (\sum_{j=1}^m \alpha_j \langle u_j, v_i\rangle)_{i=1}^n = G\alpha.
\end{aligned}$$

□

We now calculate the pseudo-inverse of G. We need the following fact which is Theorem 1.6 of [4] (see also [9, Lemma 2.4]).

Proposition 13 ([4]) *If T is a bounded linear operator with closed range, then*

$$(T^\dagger)^* = (T^*)^\dagger.$$

The following is Theorem 3.1 of [9].

Proposition 14 ([9]) *If T is the pre-frame operator of a frame $\{f_i\}_{i\in I}$ with frame bounds A and B, then, for each $f \in \mathcal{H}$,*

$$T^\dagger f = (\langle f, S^{-1} f_i\rangle)_{i\in I},$$

where S denotes the frame operator. In particular, $||T^\dagger|| \leq A^{-1/2}$.

Lemma 15 *If $P : U \to V$ is invertible, then*

$$G^\dagger = T_U^\dagger P^{-1} (T_V^*)^\dagger = T_U^\dagger P^{-1} (T_V^\dagger)^*.$$

Proof. Let $X := T_V^*, Y := PT_U$. Since $\operatorname{ran} X$ and $\operatorname{ran} Y$ are finite dimensional, they are closed. Moreover, $\operatorname{ran} XY$ is also finite dimensional, hence closed. Since $\operatorname{ran} X^* = \operatorname{ran} T_V = V$, it is invariant under YY^*. $\ker Y^* = \ker(T_U^* P^*)$. Therefore $\operatorname{ran} X^* \cap \ker Y^* = \ker(T_U^* P^*)$. Since T_U is onto, T_U^* is one-to-one. Moreover, P^* is invertible since P is invertible. Hence $\ker(T_U^* P^*) = \{0\}$ which is clearly invariant under X^*X. We have, by Lemma 12 and Propositions 11 and 13

$$G^\dagger = (PT_U)^\dagger (T_V^*)^\dagger = (PT_U)^\dagger (T_V^\dagger)^*.$$

We now apply Proposition 11 once more. Let $X := P$ and $Y := T_U$. X, Y and XY have closed range since they are finite dimensional operators. Since $\operatorname{ran} X^* = \operatorname{ran} P^* = U$, it is invariant under YY^*. $\ker Y^* = \ker T_U^* = \{0\}$, since T_U is onto. Hence it is invariant under X^*X. This shows that $(PT_U)^\dagger = T_U^\dagger P^\dagger$. Since P is invertible, $P^\dagger = P^{-1}$. □

Lemma 16 *Suppose that U and V are shift-invariant subspaces of $L^2(\mathbb{R}^d)$. If $L^2(\mathbb{R}^d) = U \dotplus V^\perp$, then $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ for almost every $x \in \mathbb{T}^d$. In particular, $\sigma(U) = \sigma(V)$.*

Proof. Note that, by [5], $V^\perp$ is a shift-invariant space since V is. Note also that $(L^2(\mathbb{R}^d))^\wedge_{||x} = \ell^2(\mathbb{Z}^d)$ for almost every $x \in \mathbb{T}^d$. Now we have $\ell^2(\mathbb{Z}^d) = \hat{U}_{||x} \dot{+} (V^\perp)^\wedge_{||x}$ almost everywhere by an argument similar to the proof of Lemma 3.7 of [17]. This implies that the oblique projection Π_x of $\ell^2(\mathbb{Z}^d)$ on $\hat{U}_{||x}$ along $(V^\perp)^\wedge_{||x}$ is well-defined almost everywhere. Hence, $\ell^2(\mathbb{Z}^d)/\ker \Pi_x = \ell^2(\mathbb{Z}^d)/(V^\perp)^\wedge_{||x}$ is isomorphic to $\operatorname{ran} \Pi_x = \hat{U}_{||x}$. Now $\ell^2(\mathbb{Z}^d)/(V^\perp)^\wedge_{||x}$ is obviously isomorphic to $((V^\perp)^\wedge_{||x})^\perp$. The point-wise projection property of a shift-invariant space ([5, Result 3.7] or [7, Lemma 1.4]) implies that $((V^\perp)^\wedge_{||x})^\perp = \hat{V}_{||x}$. Hence $\hat{U}_{||x}$ is isomorphic to $\hat{V}_{||x}$ almost everywhere. In particular, they are of the same dimension almost everywhere. □

3 Proofs of the main results

Proof of Theorem 6: (1) ⇒ (2): Combining Propositions 1 and 3 we see that there exists a positive constant c such that $c \leq R(\hat{U}_{||x}, \hat{V}_{||x})$ for almost every $x \in \mathbb{T}^d$. Fix such $x \in \mathbb{T}^d$. Let $P_{\hat{V}_{||x}}$ denote the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto $\hat{V}_{||x}$ and P_x its restriction to $\hat{U}_{||x}$. Then, for any $u \in \hat{U}_{||x}$, we have $c||u|| \leq ||P_x u||$. This shows that P_x is one-to-one. It is onto since $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ by Lemma 16. It is now easy to see that $||P_x^{-1}|| \leq c^{-1}$. Propositions 14 and 4 imply that the norms of $T^\dagger_{\hat{U}_{||x}}$ and $T^\dagger_{\hat{V}_{||x}}$ are bounded uniformly. Hence the norm of $G_{\Phi,\Psi}(x)^\dagger$ is bounded uniformly by Lemma 15. Recall that

$$G_{\Phi,\Psi}(x) = T^*_{\hat{V}_{||x}} P_x T_{\hat{U}_{||x}}.$$

Since $T_{\hat{U}_{||x}}$ is onto, $\operatorname{ran} T_{\hat{U}_{||x}} = \hat{U}_{||x} = \operatorname{dom} P_x$. Since P_x is also onto, $\operatorname{ran} P_x T_{\hat{U}_{||x}} = \operatorname{ran} P_x = \hat{V}_{||x} = \operatorname{dom} T^*_{\hat{V}_{||x}}$. Hence $\operatorname{rank} G_{\Phi,\Psi}(x) = \operatorname{rank} T^*_{\hat{V}_{||x}}$. Now $\operatorname{rank} T^*_{\hat{V}_{||x}} =$

$\operatorname{rank} T_{\hat{V}_{||x}} = \dim \hat{V}_{||x}$ since $T_{\hat{V}_{||x}}$ is onto and since $\operatorname{ran} T_{\hat{V}_{||x}} = \hat{V}_{||x}$. Now, $\dim \hat{U}_{||x} = \dim \hat{V}_{||x}$ by Lemma 16.

(2) $\Rightarrow$ (1): Recall that, for almost every $x \in \mathbb{T}^d$,

$$G_{\Phi,\Psi}(x) = T^*_{\hat{V}_{||x}} P_x T_{\hat{U}_{||x}}.$$

Fix such x. We first show that the rank condition implies that P_x is invertible. Let $r := \operatorname{rank} G_{\Phi,\Psi}(x) = \dim \hat{V}_{||x} = \dim \hat{U}_{||x}$. Since $T_{\hat{U}_{||x}}$ is onto, $\operatorname{ran} T_{\hat{U}_{||x}} = \hat{U}_{||x} = \operatorname{dom} P_x$. Therefore $\operatorname{rank} T^*_{\hat{V}_{||x}} P_x = \operatorname{rank} G_{\Phi,\Psi}(x) = r$. Since $\operatorname{rank} P_x$ is at most r and since the rank of the product of two operators is less than or equal to the minimum of the ranks of two operators, $\operatorname{rank} P_x = r$. Hence P_x is onto. It is one-to-one, since its domain and co-domain are of the same dimension.

Hence, by Lemma 15,

$$G_{\Phi,\Psi}(x)^\dagger = T^\dagger_{\hat{U}_{||x}} P_x^{-1} (T^*_{\hat{V}_{||x}})^\dagger.$$

Proposition 10 implies that

$$(T^*_{\hat{V}_{||x}})^\dagger (T^*_{\hat{V}_{||x}}) = P_{\operatorname{ran}(T^*_{\hat{V}_{||x}})^\dagger} = P_{\operatorname{ran} T_{\hat{V}_{||x}}} = P_{\hat{V}_{||x}}.$$

Similarly, we have $T_{\hat{U}_{||x}} T^\dagger_{\hat{U}_{||x}} = P_{\hat{U}_{||x}}$. Therefore,

$$\begin{aligned} T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger (T^*_{\hat{V}_{||x}}) &= T_{\hat{U}_{||x}} T^\dagger_{\hat{U}_{||x}} P_x^{-1} (T^*_{\hat{V}_{||x}})^\dagger (T^*_{\hat{V}_{||x}}) \\ &= P_{\hat{U}_{||x}} P_x^{-1} P_{\hat{V}_{||x}} \\ &= P_x^{-1} P_{\hat{V}_{||x}} = P_x^{-1}. \end{aligned} \tag{1}$$

This implies that the norm of P_x^{-1} is bounded uniformly, say, by c, by Proposition 4, since, as in the discussion following Proposition 11, $||T_{\hat{U}_{||x}}||$ is less than or equal to the square root of an upper frame bound of $\{\hat{\varphi}_{j||x} : 1 \leq j \leq m\}$ and, likewise, $||T_{\hat{V}_{||x}}|| = ||T^*_{\hat{V}_{||x}}||$ is less than or equal to the square root of an upper frame bound of $\{\hat{\psi}_{i||x} : 1 \leq i \leq n\}$. It is easy to see that this implies that $c^{-1} \leq R(\hat{U}_{||x}, \hat{V}_{||x})$. We now invoke Proposition 3. □

Proof of Corollary 7: We recall that a Riesz basis is a frame. Note that Proposition 5 implies that the Gramians $G_\Phi(x)$ and $G_\Psi(x)$ is invertible almost everywhere. Moreover, Proposition 4 implies that $\dim \hat{U}_{||x} = \dim \hat{V}_{||x} = n$ almost everywhere, since a finite Riesz basis is a basis in the sense of Linear Algebra.

(1) ⇒ (2): Condition (2) of Theorem 6 implies that $\operatorname{rank} G(x) = n$. Therefore $G(x)$ is invertible almost everywhere, and $G(x)^{-1} = G(x)^\dagger$. The proof is complete by Condition (2) of Theorem 6.

(2) ⇒ (1): Trivial. □

Proof of Theorem 8: We have $\sigma(U) = \sigma(V)$ by Lemma 16. Suppose that $x \notin \sigma(U)$. Then $\hat{U}_{||x}$ and $\hat{V}_{||x}$ are trivial subspace of $\ell^2(\mathbb{Z}^d)$. Therefore $R(\hat{U}_{||x}, \hat{V}_{||x}) = 1$. Now fix $x \in \sigma(U)$. Let $P_{\hat{V}_{||x}}$ be the orthogonal projection of $\ell^2(\mathbb{Z}^d)$ onto $\hat{V}_{||x}$ and P_x its restriction to $\hat{U}_{||x}$. The proof of Theorem 6 implies that P_x is invertible almost everywhere. Now

$$\begin{aligned}
R(\hat{U}_{||x}, \hat{V}_{||x}) &= \inf_{u \in \hat{U}_{||x} \setminus \{0\}} \frac{||P_{\hat{V}_{||x}} u||}{||u||} \\
&= \inf_{u \in \hat{U}_{||x} \setminus \{0\}} \frac{||P_x u||}{||u||} \\
&= \inf_{u \in \hat{U}_{||x} \setminus \{0\}} \frac{||P_x u||}{||P_x^{-1} P_x u||} \\
&= \left(\sup_{u \in \hat{U}_{||x} \setminus \{0\}} \frac{||P_x^{-1} P_x u||}{||P_x u||} \right)^{-1} \\
&= \left(\sup_{v \in \hat{V}_{||x} \setminus \{0\}} \frac{||P_x^{-1} v||}{||v||} \right)^{-1} \\
&= ||P_x^{-1}||^{-1}.
\end{aligned}$$

This proves the first equality by Equation (1) and by Proposition 3.

Now recall that $G_\Phi(x)$ and $G_\Psi(x)$ is non-negative definite by definition. Note that $G_\Phi(x) = T^*_{\hat{U}_{||x}} T_{\hat{U}_{||x}}$ and $G_\Psi(x) = T^*_{\hat{V}_{||x}} T_{\hat{V}_{||x}}$ by Lemma 12.

$$\begin{aligned}
||P_x^{-1}|| &= ||P_x^{-1} (P_x^{-1})^*||^{1/2} \\
&= ||T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger T^*_{\hat{V}_{||x}} T_{\hat{V}_{||x}} (G_{\Phi,\Psi}(x)^\dagger)^* T^*_{\hat{U}_{||x}}||^{1/2} \text{ (by Equation (1))} \\
&= ||T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x) (G_{\Phi,\Psi}(x)^\dagger)^* T^*_{\hat{U}_{||x}}||^{1/2} \\
&= ||T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2} G_\Psi(x)^{1/2} (G_{\Phi,\Psi}(x)^\dagger)^* T^*_{\hat{U}_{||x}}||^{1/2} \\
&= ||T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2}|| \\
&= ||G_\Psi(x)^{1/2} (G_{\Phi,\Psi}(x)^\dagger)^* T^*_{\hat{U}_{||x}} T_{\hat{U}_{||x}} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2}||^{1/2} \\
&= ||G_\Psi(x)^{1/2} (G_{\Phi,\Psi}(x)^\dagger)^* G_\Phi(x) G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2}||^{1/2} \\
&= ||G_\Psi(x)^{1/2} (G_{\Phi,\Psi}(x)^\dagger)^* G_\Phi(x)^{1/2} G_\Phi(x)^{1/2} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2}||^{1/2} \\
&= ||G_\Phi(x)^{1/2} G_{\Phi,\Psi}(x)^\dagger G_\Psi(x)^{1/2}||,
\end{aligned}$$

where we have used $||X|| = ||XX^*||^{1/2} = ||X^*X||^{1/2}$ several times. This proves the second equality. The third is Corollary 2.9 of [8]. □

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