MONOTONICITY AND COMPLEX CONVEXITY IN BANACH LATTICES

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Abstract. The goal of this article is to study the relations among monotonicity properties of real Banach lattices and the corresponding convexity properties in the complex Banach lattices. We introduce the moduli of monotonicity of Banach lattices. We show that a Banach lattice E is uniformly monotone if and only if its complexification $E^{\mathbb{C}}$ is uniformly complex convex. We also prove that a uniformly monotone Banach lattice has finite cotype. In particular, we show that a Banach lattice is of cotype q for some $2 \leq q < \infty$ if and only if there is an equivalent lattice norm under which it is uniformly monotone and its complexification is q-uniformly PL-convex. We also show that a real Köthe function space E is strictly (resp. uniformly) monotone and a complex Banach space X is strictly (resp. uniformly) complex convex if and only if Köthe-Bochner function space $E(X)$ is strictly (resp. uniformly) complex convex.

1. Introduction and Preliminaries

The moduli of complex convexity of complex quasi-Banach spaces have been introduced by Davis, Garling and Tomczak-Jaegermann in [5]. In that paper the relation between complex convexity and cotype in complex Banach lattices have been also examined. Recently, Hudzik and Narloch [10] have observed that a real Köthe function space is strictly (resp. uniformly) monotone if and only if its complexification is strictly (resp. uniformly) complex convex. This observation was a motivation of our paper, where we investigate a number of monotonicity properties in real Banach lattices and we study their relations to convex properties in complex Banach lattices.

In particular, we introduce the moduli of monotonicity of Banach lattices and study the relations between monotonicity and complex convexity in real Banach lattices and its complexification. Together with the relations between cotype and monotonicity in Banach lattices, we can naturally define the monotone versions of some geometric properties of Banach spaces studied in [6]. We shall also discuss the lifting properties of complex convexity to Köthe-Bochner function spaces.

For the definitions and characterizations of strict and uniform monotonicity of various function spaces we refer to [2, 9]. The lifting properties of complex geometric properties from a continuously quasi-normed space X to $L^p(\mu, X)$, for $0 < p < \infty$, were discussed in [8].

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For the rest of the paper we reserve the symbol E for a real Banach lattice (called also just a Banach lattice), which in particular may be a Köthe function space [12]. The positive cone of E will be denoted by $E^+ := \{x \in E : x \geq 0\}$. For each pair of $x, y \in E$, we will use the standard notations:

(1) $x \vee y := \sup\{x, y\}$, $\quad x \wedge y := \inf\{x, y\}$;
(2) $x^+ := x \vee 0$, $\quad x^- := (-x) \vee 0$;
(3) $|x| := x \vee (-x)$.

Let $1 \leq p < \infty$. We say that E is *strictly p-monotone* if for every x, y in E^+ with $y \neq 0$ we have

$$\|x\| < \|(x^p + y^p)^{1/p}\|.$$

A Banach lattice E is said to be *uniformly p-monotone* if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E^+$ with $\|x\| = 1$, $\|y\| \geq \epsilon$ then

$$\|(x^p + y^p)^{1/p}\| \geq 1 + \delta$$

holds. For $p = 1$, strict 1-monotonicity and uniform 1-monotonicity are well known as *strict* and *uniform monotonicity*, respectively [9].

Notice that the Krivine functional calculus guarantees the existence of $(x^p + y^p)^{1/p}$ in E^+ for $x, y \in E^+$ [5, 12]. Notice also that if E is strictly (resp. uniformly) q-monotone then it is strictly (resp. uniformly) p-monotone for $1 \leq p < q < \infty$. For more details, see [12]. The *complexification* $E^{\mathbb{C}}$ of a real Banach lattice consists of $x + iy$ for $x, y \in E$ with the norm $\|x + iy\|_{E^{\mathbb{C}}} = \|(|x|^2 + |y|^2)^{1/2}\|_E$. Then $E^{\mathbb{C}}$ is a complex Banach space. For more details of complexification of Banach lattices, see [12, 13]. We call $E^{\mathbb{C}}$ a *complex Banach lattice* if it is a complexification of some Banach lattice E. It was observed in [5] that the Krivine functional calculus can be also applied to complex Banach lattices.

The following moduli of complex convexity of complex Banach space X were introduced in [5]: for $0 < p < \infty$ and $\epsilon \geq 0$, we define

$$H_p^X(\epsilon) = \inf\left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^p \, d\theta \right)^{1/p} - 1 : \|x\| = 1, \|y\| = \epsilon \right\},$$

and

$$H_\infty^X(\epsilon) = \inf\left\{ \sup\{\|x + e^{i\theta} y\| : 0 \leq \theta \leq 2\pi\} - 1 : \|x\| = 1, \|y\| = \epsilon \right\}.$$

Let X be a complex Banach space. A point x of a unit sphere S_X of X is called a *complex extreme point* of a unit ball B_X if $\|x + \zeta y\| \leq 1$ for every complex ζ with $|\zeta| \leq 1$ implies $y = 0$. We say that a complex Banach space X is *strictly complex convex* if every point of S_X is a complex extreme point of B_X. A complex Banach space X is *uniformly complex convex* if $H_\infty^X(\epsilon) > 0$ for all $\epsilon > 0$ and it is said to be *uniformly PL-convex* if $H_p^X(\epsilon) > 0$ for all $\epsilon > 0$ for some $0 < p < \infty$.

Let f and g be non-negative, non-decreasing functions on $[0, 1]$. We shall write $g \preceq f$ if there is $K \geq 1$ such that $g(\epsilon/K) \leq K f(\epsilon)$ for all $0 < \epsilon < 1/K$, and we write $f \sim g$ if $f \preceq g$ and $g \preceq f$ (f and g are then said to be equivalent at zero). It is well known that for $0 < p < \infty$, the moduli H_p^X are all equivalent at zero [5], and that there exists an absolute constant $A > 0$ such that for every complex Banach space X and $\epsilon > 0$, we have [6],

$$A(H_\infty^X(\epsilon))^2 \leq H_1^X(\epsilon) \leq H_\infty^X(\epsilon).$$

This implies among others that a complex Banach space is uniformly complex convex if and only if it is uniformly PL-convex.

We shall use the following theorem for characterization of complex extreme points.

Theorem 1.1. [8] *Let X be a complex Banach space and let $x \in S_X$. Then the following conditions are equivalent:*

(1) *x is a complex extreme point of B_X;*
(2) *there exists $0 < p < \infty$ such that for all non-zero $y \in X$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta > 1;$$

(3) *for each $0 < p < \infty$ and for each non-zero $y \in X$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta > 1.$$

For an increasing function g on $[0,1]$ with $g(0) = 0$, we shall say that a Banach space X is *g-uniformly PL-convex* if $H_1^X \succeq g$ holds. If $g(\epsilon) = \epsilon^r$ (where $2 \le r < \infty$) we say that X is *r-uniformly PL-convex* (g-uniformly H_∞-convexity and r-uniform H_∞-convexity are defined similarly). These notions are defined and used in [5].

Now let's sketch briefly the content of the paper. In section two we define moduli of monotonicity of Banach lattices and we study their basic properties. We also investigate some properties of strictly (uniformly) p-monotone Banach lattices, where $1 \le p < \infty$.

In the third section we investigate the relation between moduli of monotonicity of Banach lattices and moduli of PL-convexity. In particular we show that a Banach lattice is uniformly monotone if and only if its complexification is uniformly PL-convex.

In the fourth section, the relations between uniform monotonicity, uniform PL-convexity and cotype are studied. In particular, it is shown that a uniformly monotone Banach lattice has finite cotype, and conversely if a Banach lattice has finite cotype then it admits an equivalent lattice norm under which it is uniformly monotone and its complexification is uniformly PL-convex. It is also proved that there exists a uniformly monotone renorming of power type in E if E is of finite cotype (for the uniformly complex convex renorming of power type, see [5]).

In the last section, it is shown that a real Köthe function space E is strictly (resp. uniformly) monotone and a complex Banach space X is strictly (resp. uniformly) complex convex if and only if the Köthe-Bochner function space $E(X)$ is strictly (resp. uniformly) complex convex. The strict complex convexity of a generalized direct sum of complex Banach spaces is also discussed.

2. Moduli of p-monotonicity

The *modulus of p-monotonicity* M_p^E, $0 < p < \infty$, of a Banach lattice E is defined as follows: for each $\epsilon \ge 0$,

$$M_p^E(\epsilon) = \inf\left\{\left\|(|x|^p + |y|^p)^{1/p}\right\| - 1 : x, y \in E \text{ and } \|x\| = 1, \|y\| \ge \epsilon\right\}.$$

It is clear that $\epsilon \mapsto M_p^E(\epsilon)$ is increasing and $p \mapsto M_p^E(\epsilon)$ is decreasing. Notice also that E is uniformly p-monotone if and only if $M_p^E(\epsilon) > 0$ for all $\epsilon > 0$. We start with the the following elementary observation.

Proposition 2.1. *For each $\epsilon > 0$,*

$$M_p^E(\epsilon) = \inf\left\{ \left\| (|x|^p + |y|^p)^{1/p} \right\| - 1 : x, y \in E \text{ and } \|x\| = 1, \|y\| = \epsilon \right\}.$$

Proof. Letting for each $\epsilon > 0$,

$$N_p^E(\epsilon) = \inf\left\{ \left\| (|x|^p + |y|^p)^{1/p} \right\| - 1 : x, y \in E \text{ and } \|x\| = 1, \|y\| = \epsilon \right\},$$

we have that $N_p^E \geq M_p^E$. On the other hand, for each $x, y \in E$ with $\|x\| = 1$ and $\|y\| \geq \epsilon$, take $y_1 = \frac{\epsilon}{\|y\|} y$. Clearly $|y_1| \leq |y|$, $\|y_1\| = \epsilon$ and

$$N_p^E(\epsilon) \leq \left\| (|x|^p + |y_1|^p)^{1/p} \right\| - 1 \leq \left\| (|x|^p + |y|^p)^{1/p} \right\| - 1.$$

This gives $N_p^E \leq M_p^E$ and completes the proof. □

Recall that a Banach lattice E is said to be *order continuous* if every order bounded increasing sequence converges in the norm topology of E [1, 12, 13].

Proposition 2.2. *Let $1 \leq p < \infty$. Every uniformly p-monotone Banach lattice E is order continuous.*

Proof. We have only to show that c_0 is not lattice embeddable in E (see Theorem 14.12 in [1]). Suppose that E is uniformly p-monotone and for a contrary assume that there is a lattice isomorphism $T : c_0 \to E$ such that there is a positive constant K with

$$K\|x\| \leq \|Tx\| \leq \|T\| \, \|x\|$$

for all $x \in c_0$. Then choose a sequence (x_n) in S_{c_0} with $\|Tx_n\| \geq \frac{1}{2} \|T\|$ such that $\lim_{n\to\infty} \|Tx_n\| = \|T\|$. Further we choose a sequence (y_n) in B_{c_0} with $\|y_n\|_{c_0} \geq 1/2$ so that $\||x_n| + |y_n|\|_{c_0} = 1$ for all $n \in \mathbb{N}$. Thus for every $n \in \mathbb{N}$,

$$\begin{aligned} \|Tx_n\| \left(1 + M_p^E(\|Ty_n\| / \|Tx_n\|\right) &\leq \left\| (|Tx_n|^p + |Ty_n|^p)^{1/p} \right\| \\ &\leq \|T\| \left\| (|x_n|^p + |y_n|^p)^{1/p} \right\|_{c_0} \\ &\leq \|T\| \, \||x_n| + |y_n|\|_{c_0} \leq \|T\|. \end{aligned}$$

By taking the limit we obtain that

$$\lim_{n\to\infty} M_p^E(\|Ty_n\| / \|Tx_n\|) = 0$$

. Since $1/2 \leq \|y_n\|_{c_0} \leq K^{-1} \|Ty_n\|$ we have

$$M_p^E\left(K/2 \, \|T\|\right) \leq M_p^E\left(\|Ty_n\| / \|Tx_n\|\right)$$

for all $n \in \mathbb{N}$. This implies that $M_p^E(K/2 \, \|T\|) = 0$, which is a contradiction to the fact that E is uniformly p-monotone. □

Lemma 2.3. *Let E be an order continuous Banach lattice or a Köthe function space. Let x, y be nonzero positive elements in E. Then there are $\delta = \delta(\|x\|, \|y\|) > 0$ and nonzero $z \in E^+$ such that $z \leq y$, $\|z\| \geq \|y\|/2$ and*

$$(x^p + y^p)^{1/p} \geq x + \delta z$$

. In particular, we can take $\delta(\|x\|, \|y\|) = \frac{(2^p\|x\|^p + \|y\|^p)^{1/p} - 2\|x\|}{\|y\|}$.

Proof. Suppose that E is an order continuous Banach lattice. Let G be a subspace of E generated by x and y. Then there is an order continuous band $F \supseteq G$ in E with a weak unit [12]. Now we may assume that F is a Köthe function space on a probability space Ω (see Theorem 1.b.14 in [12]). Letting now

$$A = \left\{ t \in \Omega : x(t) < \frac{2\|x\|}{\|y\|} y(t) \right\},$$

we clearly get

$$\|x\| \geq \|x\chi_{\Omega\setminus A}\| \geq \frac{2\,\|x\|}{\|y\|} \|y\chi_{\Omega\setminus A}\|.$$

Taking $z = y\chi_A$, $z \leq y$ and

$$\|z\| \geq \|y\| - \|y\chi_{\Omega\setminus A}\| \geq \frac{\|y\|}{2}.$$

On the other hand, notice that for each $\epsilon > 0$ there is $\delta_1 = \delta_1(\epsilon) > 0$ such that for each $a \geq \epsilon$,

$$(1 + a^p)^{1/p} \geq 1 + \delta_1 a.$$

In fact, it is easy to check that we can take

$$\delta_1(\epsilon) = \frac{(1 + \epsilon^p)^{1/p} - 1}{\epsilon}.$$

Hence if we take $\delta = \delta_1(\|y\|/\|2x\|)$ then

$$\begin{aligned}(x^p + y^p)^{1/p} &= (x^p\chi_A + y^p\chi_A)^{1/p} + (x^p\chi_{\Omega\setminus A} + y^p\chi_{\Omega\setminus A})^{1/p} \\ &\geq x\chi_A + \delta y\chi_A + x\chi_{\Omega\setminus A} \\ &= x + \delta z,\end{aligned}$$

and we obtain the desired result. □

Combining Proposition 2.1 and Lemma 2.3, we immediately obtain the following result.

Proposition 2.4. *Let E be a Köthe function space or order continuous Banach lattice. Then for each $1 \leq p < \infty$, E is strictly p-monotone if and only if E is strictly monotone.*

It is clear that Propositions 2.1, 2.2 and Lemma 2.3 imply the following.

Proposition 2.5. *Let E be a Banach lattice. For each $1 \leq p < \infty$, E is uniformly p-monotone if and only if E is uniformly monotone. In particular we obtain the following inequalities: for each $1 \leq p < \infty$ and for each $\epsilon > 0$,*

$$M_1^E(\epsilon^p) \preceq M_p^E(\epsilon) \leq M_1^E(\epsilon).$$

Observe that a Banach lattice E is uniformly monotone with $M_p^E \succeq \epsilon^r$ for some $1 \leq p < \infty$ and for some $r > 1$ if and only if there is an $\lambda > 0$ such that

$$\|(|x|^p + |y|^p)^{1/p}\| \geq (\|x\|^r + \lambda\|y\|^r)^{1/r}$$

holds for all x and y in E. We shall denote the largest possible value of λ by $J_{r,p}(E)$. Then, by induction, it is clear that

$$\left\| (\sum_{k=1}^n |x_k|^p)^{1/p} \right\| \geq \left(\|x_1\|^r + J_{r,p}(E) \sum_{k=2}^n \|x_k\|^r \right)^{1/r}$$

holds for every $x_1, \dots, x_n$ in E. This is an analogue of the formula in [5] concerning moduli of r-uniformly PL-convexity. This formula shows that if $M_1^E \succeq \epsilon^q$ for some $1 < q < \infty$, then E satisfies lower q-estimate. We shall use this fact in proof of Corollary 4.6.

We finish this section with the examples of moduli of monotonicity computed in $\mathbb{R}$ and L^p.

Example 2.6. Let E be a space of real numbers $\mathbb{R}$. Then

$$M_p^{\mathbb{R}}(\epsilon) = (1+\epsilon^p)^{1/p} \sim \epsilon^p$$

holds for every $\epsilon > 0$, and an easy calculation shows that $J_{q,p}(\mathbb{R}) = 1$ for all $1 \le p \le q < \infty$. Hence we cannot omit the power p in the inequality of Proposition 2.5.

Example 2.7. Let $1 \le p, q < \infty$ and E be an L^p-space over a measure space (Ω, Σ, μ). Suppose that $1 \le p \le q < \infty$ holds. Then the Minkowski inequality shows that for every $x, y \in E$,

$$\begin{aligned}\|(|x|^q + |y|^q)^{1/q}\|_p &= \left(\int_\Omega (|x(t)|^q + |y(t)|^q)^{p/q} dt\right)^{1/p} \\ &\ge (\|x\|_p^q + \|y\|_p^q)^{1/q} \ge (\|x\|_p^r + \|y\|_p^r)^{1/r}.\end{aligned}$$

Hence $M_q^{L^p}(\epsilon) \succeq \epsilon^q$ and $J_{r,q}(L^p) = 1$ hold for all $1 \le p \le q \le r < \infty$. Then $M_2^{L^p}(\epsilon) \succeq \epsilon^2$ for all $1 \le p \le 2$ and $M_2^{L^p}(\epsilon) \ge M_p^{L^p}(\epsilon) \succeq \epsilon^p$ for all $p \ge 2$.

3. Monotonicity and Complex convexity in Banach Lattices

Before we state the main results relating strict (uniform) convexity of a real Banach lattice E with the strict (uniform) complex convexity of its complexification $E^{\mathbb{C}}$ we need the following preliminary results.

Proposition 3.1. *If a complex Banach lattice $E^{\mathbb{C}}$ is strictly (resp. uniformly) complex convex, then E is strictly (resp. uniformly) monotone. In particular, for each $\epsilon > 0$,*

$$M_1^E(\epsilon) \ge H_\infty^E(\epsilon).$$

Proof. Suppose that E is not strictly monotone. Then there exist $0 \le y \in E$ and $x \in E$ such that

$$|x| < y \quad \text{and} \quad \|x\| = \|y\| = 1.$$

Now taking $z = y - |x| > 0$, for every $|\zeta| \le 1$, $\zeta \in \mathbb{C}$,

$$|x + \zeta z| \le |x| + |z| = |x| + y - |x| = y.$$

This yields that for every $|\zeta| \le 1$, $\zeta \in \mathbb{C}$,

$$\|x + \zeta z\| \le \|y\| = 1.$$

Therefore $E^{\mathbb{C}}$ is not strictly complex convex.

In the uniform case, it is easy to see that for each $x, y, \in E$ with $\|x\| = 1$, $\|y\| = \epsilon$,

$$\||x| + |y|\| \ge \sup\{\|x + \zeta y\| : |\zeta| \le 1\} \ge 1 + H_\infty^E(\epsilon).$$

Hence $M_1^E(\epsilon) \ge H_\infty^E(\epsilon)$, and the proof is finished. □

Proposition 3.2. *Let $E^{\mathbb{C}}$ be a complex Banach lattice. Then for all $x, y \in E^{\mathbb{C}}$,*

$$\frac{1}{2}\sup\{|x+\zeta y| + |x-\zeta y| : |\zeta| \le 1\} = (|x|^2 + |y|^2)^{1/2}.$$

$$\frac{1}{2\pi}\int_0^{2\pi} |x + e^{i\theta}y| \; d\theta \ge \left(|x|^2 + \frac{1}{2}|y|^2\right)^{1/2}.$$

Proof. The first equality reults from the Krivine functional calculus [12] for complex Banach lattices and the following identity on $\mathbb{C}$,

$$\frac{1}{2}\sup\{|z_1+\zeta z_2| + |z_1-\zeta z_2| : |\zeta| \le 1\} = (|z_1|^2 + |z_2|^2)^{1/2}.$$

For the second identity we refer to Theorem 7.1 in [5]. □

Now we can state the relations between moduli of monotonicity and moduli of complex convexity of Banach lattices.

Proposition 3.3. *If E is strictly (resp. uniformly) 2-monotone, then $E^{\mathbb{C}}$ is strictly (resp. uniformly) complex convex. In particular,*

$$M_1^E(\epsilon) \ge H_\infty^E(\epsilon) \ge H_1^E(\epsilon) \ge M_2^E(\epsilon/\sqrt{2}).$$

Proof. Suppose that E is strictly 2-monotone. Let $x \in S_{E^{\mathbb{C}}}$ and assume that there is $y \in E^{\mathbb{C}}$ such that $\|x + \zeta y\| \le 1$ for all $|\zeta| \le 1$, $\zeta \in \mathbb{C}$. Notice that for all $|\zeta| \le 1$, $\zeta \in \mathbb{C}$,

$$2 = 2\,\|x\| \le \||x+\zeta y| + |x-\zeta y|\| \le \|x+\zeta y\| + \|x-\zeta y\| \le 2.$$

By the strict monotonicity of E, for every $|\zeta| \le 1$, $\zeta \in \mathbb{C}$,

$$|x| = \frac{1}{2}\left(|x+\zeta y| + |x-\zeta y|\right).$$

By Proposition 3.2, we get

$$|x| = \frac{1}{2}\sup\{|x+\zeta y| + |x-\zeta y| : |\zeta| \le 1\} = (|x|^2 + |y|^2)^{1/2}.$$

Then the strict 2-monotonicity of E implies $y = 0$. Therefore strict 2-monotonicity of E implies strict complex convexity of $E^{\mathbb{C}}$.

For the converse, assume now that E is uniformly 2-monotone. Then by Proposition 3.2, for each $x \in S_{E^{\mathbb{C}}}$ and $y \in E^{\mathbb{C}}$ with $\|y\| \ge \epsilon$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} |x + e^{i\theta}y| \; d\theta \ge (|x|^2 + \frac{1}{2}|y|^2)^{1/2}.$$

Then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \|x + e^{i\theta}y\| \; d\theta &\ge \left\|\frac{1}{2\pi}\int_0^{2\pi} |x + e^{i\theta}y| \; d\theta\right\| \\ &\ge \left\|\left(|x|^2 + \frac{1}{2}|y|^2\right)^{1/2}\right\| \ge 1 + M_2^E(\epsilon/\sqrt{2}).\end{aligned}$$

□

As corollaries of Propositions 2.4, 2.5 and 3.3, we obtain the following two theorems. In case when E is a Köthe space they were proved in [10] by using quite different methods. The characterizations of strict and uniform complex convexity of Orlicz-Lorentz spaces are studied in [3] with application to density of norm-attaining operators.

Theorem 3.4. *Let E be a real Köthe function space or order continuous Banach lattice. Then E is strictly monotone if and only if $E^{\mathbb{C}}$ is strictly complex convex.*

Theorem 3.5. *Let E be a Banach lattice. E is uniformly monotone if and only if $E^{\mathbb{C}}$ is uniformly complex convex (i.e., uniformly PL-convex).*

It is well known [6] that unconditional convergence of the series $\sum_1^\infty x_j$ in a complex Banach space X implies that $\sum_{j=1}^\infty H_\infty^X(\|x_j\|)$ is convergent. Applying Proposition 3.3 we obtain immediately the following monotone version of this result.

Corollary 3.6. *Suppose that the series $\sum_1^\infty x_j$ is unconditionally convergent in a complex Banach lattice $E^{\mathbb{C}}$. Then*

$$\sum_{j=1}^{\infty} M_2^E(\|x_j\|) < \infty.$$

It is also shown in [6] that if $E^{\mathbb{C}}$ is a 2-uniformly smooth Banach space such that $\limsup_{\epsilon\to 0} H_1^E(\epsilon)/\epsilon^2 > 0$, then $E^{\mathbb{C}}$ is isomorphic to a Hilbert space. Again by Proposition 3.3 we obtain its monotone version as follows.

Corollary 3.7. *Suppose that $E^{\mathbb{C}}$ is a 2-uniformly smooth Banach lattice such that*

$$\limsup_{\epsilon\to 0} \frac{M_2^E(\epsilon)}{\epsilon^2} > 0.$$

Then $E^{\mathbb{C}}$ is isomorphic to a Hilbert space.

4. Relations with cotype

In this section we present some results relating cotype, lower q-estimate and q-uniform PL-convexity of a complex Banach lattice $E^{\mathbb{C}}$ as well as their relations to q-uniform monotonicity of a real Banach lattices E. For the notions of cotype, lower q-estimate, q-concavity, we refer to [12].

It is clear that a Banach lattice E satisfying a lower q-estimate for some $1 < q < \infty$ with constant one is uniformly monotone and $M_1^E \succeq \epsilon^q$ holds. Moreover, if a Banach lattice E is q-concave for some $2 \le q < \infty$ with q-concavity constant one, then $M_q^E \succeq \epsilon^q$ holds. Hence, using Proposition 3.3, we obtain the following corollaries.

Corollary 4.1. *Let $1 < q < \infty$ and E be a Banach lattice whose lower q-estimate constant is equal to one. Then E is uniformly monotone with $M_1^E \succeq \epsilon^q$ and $E^{\mathbb{C}}$ is $2q$-uniformly PL-convex.*

Corollary 4.2. *Let $2 \le q < \infty$ and E be a Banach lattice whose q-concavity constant is equal to one. Then E is uniformly q-monotone with $M_q^E \succeq \epsilon^q$ and $E^{\mathbb{C}}$ is q-uniformly PL-convex.*

In the proof of Proposition 2.2, a uniformly p-monotone Banach lattice cannot contain a lattice-isomorphic copy of c_0, hence it cannot contain an isomorphic copy of c_0(see [1]). Then Maurey-Pisier theorem gives the following (see Theorem 14.1 in [7]).

Theorem 4.3. *Let E be a uniformly monotone Banach lattice. Then E has finite cotype.*

Corollary 4.4. *Suppose that E is a Banach lattice. Then the following properties are equivalent:*

(1) *$E^{\mathbb{C}}$ is of cotype 2;*
(2) *E is of cotype 2;*
(3) *$E^{\mathbb{C}}$ is 2-concave;*
(4) *E is 2-concave;*
(5) *there is an equivalent uniformly monotone lattice norm on E with $M_2^E \succeq \epsilon^2$ under which $E^{\mathbb{C}}$ is 2-uniformly PL-convex.*

Proof. The equivalence of (1), (3) is proved in [5]. It is clear that (1) implies (2). The equivalence of (2) and (4) is well-known (see [12]). Both Theorem 1.d.8 in [12] and Corollary 4.2 show that (4) implies (5). Finally it is shown in [5] that (5) implies (1). □

Reviewing the proof of Theorem 7.3 in [5], we can obtain the following theorem so we omit the proof. Notice that for $q > 2$ it is a stronger result than Corollary 4.1.

Theorem 4.5. *Suppose that $2 < q < \infty$, and that a Banach lattice E whose lower q-estimate is equal to one. Then $E^{\mathbb{C}}$ is q-uniformly PL-convex.*

Corollary 4.6. *Suppose that $2 < q < \infty$ and that E is a Banach lattice. The following are equivalent:*

(1) *$E^{\mathbb{C}}$ is of cotype q;*
(2) *E is of cotype q;*
(3) *$E^{\mathbb{C}}$ satisfies a lower q-estimate;*
(4) *E satisfies a lower q-estimate;*
(5) *there is an equivalent lattice norm on E under which $E^{\mathbb{C}}$ is q-uniformly PL-convex;*
(6) *there is an equivalent lattice norm on E under which E is uniformly monotone with $M_1^E \succeq \epsilon^q$.*

Proof. The equivalence of (1), (3) and (5) is shown in [5]. It is obvious that (1) implies (2) and it is well-known that (2) is equivalent to (4) [12]. Recall that a Banach lattice which satisfies a lower q-estimate can be given an equivalent Banach lattice norm for which the lower q-estimate constant is one (see [12]). Then both Theorem 4.5 and the remark above Corollary 4.2 show that (4) implies (5) and (6). Finally, notice that if a Banach lattice E satisfies $M_1^E \succeq \epsilon^q$ for $1 < q < \infty$ then it satisfies a lower q-estimate. Hence (6) implies (4) and the proof is complete. □

5. Lifting properties of Monotonicity and Complex convexity

Let E be a nontrivial real Köthe space over a complete measure space (Ω, μ) and X be a nontrivial complex Banach space. Let $L^0(X)$ be the set of all X-valued strongly μ-measurable functions. The *Köthe-Bochner function space* $E(X)$ is a Banach space defined by

$$E(X) = \{f \in L^0(X) : t \mapsto \|f(t)\|_X \text{ is an element of } E\},$$

with the norm

$$\|f\| = \| \, \|f(\cdot)\|_X \|_E \, .$$

For more details of Köth-Bochner function spaces, see [11].

Notice that if we choose $g \in E$ and $a \in X$ such that $\|f\|_E = 1$ and $\|a\|_X = 1$, then both, the map $x \mapsto g(\cdot)x$ from X into $E(X)$ and the map $f \mapsto f(\cdot)a$ from E into $E(X)$, are isometries.

Theorem 5.1. *The Köthe-Bochner function space $E(X)$ is strictly complex convex if and only if E is strictly monotone and X is strictly complex convex.*

Proof. Suppose that Köthe-Bochner function space $E(X)$ is strictly complex convex and suppose for a contrary that E is not strictly complex convex. Choose a norm one element a in X. There exist $x, y \in E^+$ such that $\|x\| = \|y\| = 1$ and $0 < x < y$. Let $z = y - x > 0$. Then for every $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$,

$$\|(x(\cdot) + \zeta z(\cdot))a\|_X = |x(\cdot) + \zeta z(\cdot)| \leq |y(\cdot)|.$$

Hence $\|x \otimes a + \zeta z \otimes a\|_{E(X)} \leq \|y\|_E = 1$ for all $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$, but $\|x \otimes a\|_{E(X)} = 1$ where $z \otimes a \neq 0$. This is a contradiction to the fact that $E(X)$ is strictly complex convex. The isometric embedding of X into $E(X)$ implies that X is strictly complex convex if so is $E(X)$.

For the converse, suppose that E is strictly monotone and X is strictly complex convex. Let $f \in S_{E(X)}$. Assume that there is $g \in E(X)$ such that $\|f + \zeta g\| \leq 1$ for all $|\zeta| \leq 1$. Notice that for each $|\zeta| \leq 1$,

$$2\|f(\cdot)\|_X \leq \|f(\cdot) + \zeta g(\cdot)\|_X + \|f(\cdot) - \zeta g(\cdot)\|_X,$$

and

$$2\| \, \|f(\cdot)\|_X \|_E \leq \| \|f(\cdot) + \zeta g(\cdot)\|_X + \|f(\cdot) - \zeta g(\cdot)\|_X \|_E \leq 2.$$

By the strict monotonicity of E, we obtain that for each $|\zeta| \leq 1$,

$$2\|f(t)\|_X = \|f(t) + \zeta g(t)\|_X + \|f(t) - \zeta g(t)\|_X \quad \text{for } \mu\text{-a.e. } t.$$

Integrating, we get the following

$$\|f(t)\|_X = \frac{1}{2\pi} \int_0^{2\pi} \|f(t) + e^{i\theta} g(t)\|_X \, d\theta \quad \text{for } \mu\text{-a.e. } t.$$

The strict complex convexity of X and Theorem 1.1 show that $g(t) = 0$ for μ-*a.e* t and the proof is finished. □

Theorem 5.2. *The Köthe-Bochner function space $E(X)$ is uniformly complex convex if and only if E is uniformly monotone and X is uniformly complex convex.*

Proof. Suppose that $E(X)$ is uniformly complex convex and suppose for a contrary, that E is not uniformly monotone. So there are sequences (x_n), (y_n) in E and $\epsilon > 0$ such that

$$\|x_n\|_E = 1, \quad \|y_n\|_E = \epsilon, \quad \text{and} \quad \lim_n \| \, |x_n| + |y_n| \, \|_E = 1.$$

Let a be a norm one element of X. Then

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \|x_n \otimes a + e^{i\theta} y_n \otimes a\|_{E(X)} \, d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \|x_n + e^{i\theta} y_n\|_E \, d\theta \\ &\leq \| \, |x_n| + |y_n| \, \|_E \end{aligned}$$

holds for all $n \in \mathbb{N}$. Notice that $\|x_n \otimes a\|_{E(X)} = 1$ and $\|y_n \otimes a\|_{E(X)} = \epsilon$. Hence

$$\lim_n \frac{1}{2\pi} \int_0^{2\pi} \|x_n \otimes a + e^{i\theta} y_n \otimes a\|_{E(X)} \, d\theta = 1.$$

This contradicts the fact that $E(X)$ is uniformly complex convex. By the isometric embedding of X into $E(X)$, X is uniformly complex convex if $E(X)$ is uniformly complex convex.

For the converse, suppose that E is uniformly monotone and X is uniformly complex convex. Let $f, g \in E(X)$ with $\|f\| = 1$ and $\|g\| = 3\epsilon > 0$. It is clear that

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta} g\|_{E(X)}\ d\theta \geq \left\| \frac{1}{2\pi}\int_0^{2\pi} \|f(\cdot) + e^{i\theta} g(\cdot)\|_X\ d\theta \right\|_E .$$

Let

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_0^{2\pi} \|f(t) + e^{i\theta} g(t)\|_X\ d\theta \\ A_1 &= \{t : \|f(t)\| \geq \|g(t)\| \geq 0\}, \quad A_2 = \{t : \|f(t)\| = 0\}, \\ A_3 &= \{t : \|g(t)\| > \|f(t)\| > 0\}, \quad R = \text{support of } g. \end{aligned}$$

Then $g = g\chi_{A_1} + g\chi_{A_2} + g\chi_{A_3}$. So there is A_i such that $\|g\chi_{A_i}\| \geq \epsilon$.

Case (1): Assume $\|g\chi_{A_1}\| \geq \epsilon$. Let

$$C = \left\{ t : \|g(t)\| \geq \frac{\epsilon}{3}\|f(t)\| \right\}.$$

Then

$$\begin{aligned} h(t) &\geq \|f(t)\chi_{\Omega\setminus(A_1\cap R)}(t)\|_X + h(t)\chi_{A_1\cap R}(t) \\ &\geq \|f(t)\chi_{\Omega\setminus(A_1\cap R)}(t)\|_X + h(t)\chi_{A_1\cap R\cap C}(t) + h(t)\chi_{A_1\cap R\setminus C}(t) \\ &\geq \|f(t)\chi_{\Omega\setminus(A_1\cap R)}(t)\|_X + \|f(t)\|_X(1 + H_1^X(\epsilon/3))\chi_{A_1\cap R\cap C}(t) + \|f(t)\|_X\chi_{A_1\cap R\setminus C}(t) \\ &\geq \|f(t)\|_X + H_1^X(\epsilon/3)\|f(t)\|_X\chi_{A_1\cap R\cap C}(t). \end{aligned}$$

Notice also that

$$\begin{aligned} \|f\chi_{A_1\cap R\cap C}\| &\geq \|g\chi_{A_1\cap R\cap C}\| = \|g\chi_{A_1\cap C}\| \geq \|g\chi_{A_1}\| - \|g\chi_{A_1\setminus C}\| \\ &\geq \|g\chi_{A_1}\| - \frac{\epsilon}{3}\|f\chi_{A_1\setminus C}\| \geq \frac{2\epsilon}{3}. \end{aligned}$$

Now the uniform monotonicity of E implies that

$$\|h\|_E \geq \|\ \|f(\cdot)\|_X + H_1^X(\epsilon/3)\|f(\cdot)\|_X\chi_{A_1\cap R\cap C}\|_E \geq 1 + M_1^E\left(H_1^X\left(\frac{\epsilon}{3}\right)\frac{2\epsilon}{3}\right).$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta} g\|_{E(X)}\ d\theta \geq 1 + M_1^E\left(H_1^X\left(\frac{\epsilon}{3}\right)\frac{2\epsilon}{3}\right).$$

Case (2): Assume $\|g\chi_{A_2}\| \geq \epsilon$. Then

$$\begin{aligned} h(t) &\geq \|f(t)\chi_{\Omega\setminus(A_2\cap R)}(t)\|_X + h(t)\chi_{A_2\cap R}(t) \\ &= \|f(t)\chi_{\Omega\setminus(A_2\cap R)}(t)\|_X + (\|f(t)\|_X + \|g(t)\|_X)\chi_{A_2\cap R}(t) \\ &= \|f(t)\|_X + \|g(t)\|_X\chi_{A_2}(t). \end{aligned}$$

It is clear that the uniform monotonicity of E implies that

$$\|h\|_E \geq 1 + M_1^E(\epsilon).$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta} g\|_{E(X)}\ d\theta \geq 1 + M_1^E(\epsilon).$$

Case (3): Assume that $\|g\chi_{A_3}\| \geq \epsilon$. Then

$$h(t) \geq \|f(t)\|_X \chi_{\Omega\setminus A_3}(t) + h(t)\chi_{A_3}(t).$$

Let $\delta := \frac{1}{2}\min\{M_1^E(\epsilon), 1/2\}$. If $\|f\chi_{A_3}\| \leq \delta$ then $\|f\chi_{\Omega\setminus A_3}\| \geq 1-\delta$. Moreover

$$h(t) \geq \|f(t)\|_X \chi_{\Omega\setminus A_3}(t) + \|g(t)\|_X\chi_{A_3}(t).$$

Since the uniform monotonicity of E implies that

$$\|h\|_E \geq (1-\delta)(1+M_1^E(\epsilon))) = 1 + [M_1^E(\epsilon) - (1-M_1^E(\epsilon))\delta],$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_{E(X)}\ d\theta \geq 1 + [M_1^E(\epsilon) - (1-M_1^E(\epsilon))\delta].$$

If, on the other hand, $\|f\chi_{A_3}\| \geq \delta$, then

$$\begin{aligned} h(t) &\geq \|f(t)\|_X\chi_{\Omega\setminus A_3}(t) + (1+H_1^X(1))\|f(t)\|_X\chi_{A_3}(t) \\ &= \|f(t)\|_X + H_1^X(1)\|f(t)\|_X\chi_{A_3}(t) \end{aligned}$$

Thus by the uniform monotonicity of E,

$$\|h\|_E \geq 1 + M_1^E(H_1^X(1)\delta).$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_{E(X)}\ d\theta \geq 1 + M_1^E(H_1^X(1)\delta).$$

Combining these three cases and taking

$$\hat{\delta} = \min\left\{M_1^E\left(H_1\left(\frac{\epsilon}{3}\right)\frac{2\epsilon}{3}\right),\ M_1^E(\epsilon),\ M_1^E(\epsilon) - (1-M_1^E(\epsilon))\delta,\ M_1^E(H_1^E(1)\delta)\right\},$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} \|f + e^{i\theta}g\|_{E(X)}\ d\theta \geq 1 + \hat{\delta},$$

which completes the proof. □

It is known that L^p space is uniformly PL-convex for $0 < p < \infty$ [5]. Hence we can obtain the following corollaries.

Corollary 5.3. [5, 8] *Let $1 \leq p < \infty$. $L^p(X)$ is strictly complex convex (resp. uniformly PL-convex) if and only if X is strictly complex convex (resp. uniformly PL-convex).*

As the last topic of this paper we shall discuss the strict complex convexity of a generalized direct sums of complex Banach spaces (for the case of uniform complex convexity, see [6]).

Let $(X_n, \|\cdot\|_n)_{n=1}^\infty$ be a family of complex Banach spaces with corresponding moduli of complex convexity H_1^n, and let E be a real Banach sequence space. The vector space of sequences $x = (x_n)_{n=1}^\infty$, with $x_n \in X_n$ and with $(\|x_n\|)_{n=1}^\infty \in E$, becomes a complex Banach space when equipped with the norm $\|x\| = \|(\|x_n\|)_1^\infty\|_E$. This space shall be denoted by X_E. The natural inclusions $j_n : X_n \to X_E$ given by the mappings $x_n \mapsto (0, \cdots, 0, x_n, 0, \cdots,)$ are isometries.

Theorem 5.4. *Let $(X_n, \|\cdot\|_n)_{n=1}^\infty$ be a family of strictly complex convex Banach spaces. Suppose that E is strictly monotone Banach sequence space. Then X_E is strictly complex convex.*

Proof. Let $x = (x_n), y = (y_n) \in X_E$ with $\|x\| = 1$ and suppose that $\|x + \zeta y\| \leq 1$ for all $|\zeta| \leq 1$. Then

$$\begin{aligned} 1 &\geq \frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\| \, d\theta \\ &\geq \left\| \left(\frac{1}{2\pi} \int_0^{2\pi} \|x_n + e^{i\theta} y_n\|_{X_n} \, d\theta \right)_{n=1}^{\infty} \right\|_E \geq \| \, (\, \|x_n\|_{X_n})_{n=1}^{\infty} \|_E = 1. \end{aligned}$$

Notice that for each $n \in \mathbb{N}$,

$$\frac{1}{2\pi} \int_0^{2\pi} \|x_n + e^{i\theta} y_n\|_{X_n} \, d\theta \geq \|x_n\|_{X_n}.$$

Then the strict monotonicity of E implies that

$$\frac{1}{2\pi} \int_0^{2\pi} \|x_n + e^{i\theta} y_n\|_{X_n} \, d\theta = \|x_n\|_{X_n}.$$

Finally the strict complex convexity of X_n yields that $y_n = 0$ for each $n \in \mathbb{N}$, which completes the proof. □

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