A note on regular near polygons

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Abstract

In this note we prove several inequalities for regular near polygons.

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1 Introduction

The reader is referred to the next section for the definitions.

A regular near polygon of order (s,t) is a distance-regular graph of valency $s(t+1)$, diameter d and $a_i = c_i(s-1)$ for all $1 \le i \le d-1$ such that for any vertex x the subgraph induced by the neighbors of x is the disjoint union of $t+1$ complete graphs of size s.

Let Γ be a regular near polygon of order (s,t). We are looking at small t. If $t=0$, then it is clear that Γ is a complete graph. If $t=1$, then Γ is a line graph and they are classified, see [3] and [9]. In particular it was shown that the numerical girth is bounded by 12. In [6] we classified the regular near polygons of order $(s,2)$, and we showed that the numerical girth is bounded by 8 when $s>1$. Hence we may assume that $t \ge 3$.

In this note we prove several inequalities for regular near polygons of order (s,t), with $t \ge 3$. We will also discuss its implications for when t equals three.

Theorem 1 *Let Γ be a regular near polygon of order (s,t) with $t \ge 3$. Let d be the diameter of Γ and let $r := \max\{i \mid c_i = 1\}$. Assume $r+2 \le d$. Let $e := \min\{i \mid c_i = c_{d-1}\}$ and let q be an integer with $r+1 \le q \le e$ such that $2d < 2q+r+1$. Then $s < t^h$, where*

$$h := h(d,q,r) = \frac{2(q+d)-(r+1)}{2(q-d)+(r+1)}.$$

By putting $q = r+1$ in Theorem 1 we have the following corollary.

Corollary 2 *Let Γ be a regular near polygon of order (s,t) with $t \ge 3$. Let d be the diameter of Γ and let $r := \max\{i \mid c_i = 1\}$. Assume $r+2 \le d < \frac{3}{2}(r+1)$. Then $s < t^{h'}$, where*

$$h' := h'(d,r) = \frac{r+1+2d}{3(r+1)-2d}.$$

We remark that $h'(d,r) = h(d,r+1,r) \ge 3$ and that

$$\lim_{d \to r+1} h'(d,r) \;=\; 3.$$

It is well known that $s \le t^3$ for the generalized polygons of order (s,t). So our theorem is a generalization of this inequality for regular near polygons.

We also prove the following results. These are helpful for considering the regular near polygons of order $(s,3)$.

Theorem 3 *Let Γ be a regular near polygon of order (s,t) with $t \ge 3$. Let d be the diameter of Γ and let $r := \max\{i \mid c_i = 1\} \ge 3$. Suppose $c_{r+1} = \cdots = c_{d-1} \ge \frac{3}{4}(t+1)$. Then $s < t(t+1)$.*

Theorem 4 *Let Γ be a regular near polygon of order (s,t) with $t \ge 3$ and $s \ge t(t+1)$. Let d be the diameter of Γ and let $r := \max\{i \mid c_i = 1\} \ge 7$. Suppose $d \ge r+3$ and $c_{r+2} = \cdots = c_{d-1} = t$. Then $t = 3$. Moreover if $r \ge 13$, then $s \le 13$.*

In Section 2 we recall definitions and several known results. We prove Theorems 1, 3 and 4 in Sections 3, 4 and 5 respectively. In Section 6 we consider regular near polygons of order $(s,3)$.

2 Preliminary

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph without loops or multiple edges. For vertices x and y in Γ we denote by $\partial_\Gamma(x, y)$ the distance between x and y in Γ. The *diameter* of Γ, denoted by d, is the maximal distance of two vertices in Γ. We denote by $\Gamma_i(x)$ the set of vertices which are at distance i from x. A connected graph Γ with diameter d is called *distance-regular* if there are numbers

$$c_i \ (1 \leq i \leq d), \ a_i \ (0 \leq i \leq d) \ \text{ and } \ b_i \ (0 \leq i \leq d-1)$$

such that for any two vertices x and y in Γ at distance i the sets

$$\Gamma_{i-1}(x) \cap \Gamma_1(y), \Gamma_i(x) \cap \Gamma_1(y) \ \text{ and } \ \Gamma_{i+1}(x) \cap \Gamma_1(y)$$

have cardinalities c_i, a_i and b_i, respectively. Then Γ is regular with valency $k := b_0$.

Let Γ be a distance-regular graph with diameter d. The array

$$\iota(\Gamma) = \left\{ \begin{array}{ccccccc} * & c_1 & \cdots & c_i & \ldots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \ldots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \ldots & b_{d-1} & * \end{array} \right\}$$

is called the *intersection array of* Γ. Define $r = r(\Gamma) := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. The *numerical girth of* Γ is $2r + 2$ if $c_{r+1} \neq 1$ and $2r + 3$ if $c_{r+1} = 1$.

Let $k_i := |\Gamma_i(x)|$ for all $0 \leq i \leq d$ which does not depend on the choice of x. It is known that $k_i c_i = k_{i-1} b_{i-1}$ for all $1 \leq i \leq d$.

By an eigenvalue of Γ we will mean an eigenvalue of its adjacency matrix A. Its multiplicity is its multiplicity as eigenvalue of A. Define the polynomials $u_i(x)$ by

$$\begin{aligned} u_0(x) := 1, u_1(x) := x/k, \qquad & \text{and} \\ c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) = x u_i(x), \qquad & i = 1, 2, \ldots, d-1. \end{aligned}$$

Let θ be an eigenvalue of Γ with multiplicity $m(\theta)$. It is well-known that

$$m(\theta) = \frac{|V\Gamma|}{\sum_{i=0}^{d} k_i u_i(\theta)^2}.$$

For more information on distance-regular graphs we would like to refer to the books [1] [2], [3] and [4].

A graph Γ is said to be *of order* (s, t) if $\Gamma_1(x)$ is a disjoint union of $t + 1$ cliques of size s for every vertex x in Γ. In this case, Γ is a regular graph of valency $k = s(t + 1)$ and every edge lies on a clique of size $s + 1$.

A graph Γ is called (the collinearity graph of) *a regular near polygon of order* (s, t) if it is a distance-regular graph of order (s, t) with diameter d and $a_i = c_i(s - 1)$ for all $1 \leq i \leq d - 1$.

More information on regular near polygons can be found in [3, §6.4–6.6].

The rest of this section we collect several known results.

Lemma 5 *Let Γ be a distance-regular graph of diameter d. Let q be an integer with $1 \leq q \leq d-1$. Suppose $c_{q+1} < b_q$. Then*

$$|V\Gamma| \leq k_q \left(\frac{b_q}{c_{q+1}}\right)^{d-q} \left(\frac{b_q}{b_q - c_{q+1}}\right).$$

Proof. Let $\gamma = \left(\frac{b_q}{c_{q+1}}\right)$. Since $k_i c_i = b_{i-1} k_{i-1}$ for $1 \leq i \leq d$, we have

$$k_{q+j} = k_q \frac{b_q \cdots b_{q+j-1}}{c_{q+1} \cdots c_{q+j}} \leq k_q \left(\frac{b_q}{c_{q+1}}\right)^j = k_q \gamma^j$$

for all $1 \leq j \leq d-q$, and

$$k_{q-i} = k_q \frac{c_q \cdots c_{q-i+1}}{b_{q-1} \cdots b_{q-i}} \leq k_q \left(\frac{c_{q+1}}{b_q}\right)^i = k_q \gamma^{-i}$$

for all $1 \leq i \leq q$. It follows that

$$|V\Gamma| = \sum_{i=0}^{d} k_i \leq \sum_{i=0}^{d} k_q \gamma^{i-q} = k_q \gamma^{-q} \left(\frac{\gamma^{d+1} - 1}{\gamma - 1}\right) < k_q \gamma^{d-q} \left(\frac{\gamma}{\gamma - 1}\right).$$

The desired result is proved. ■

Proposition 6 [5, Proposition 3.3] *Let Γ be a distance-regular graph with valency k, numerical girth g such that each edge lies in an (a_1+2)-clique. Let h be a positive integer and let θ be an eigenvalue of Γ with multiplicity $m(\theta)$. Suppose $\theta \neq k, -\frac{k}{a_1+1}$. Then the following hold.*

(1) *If $g \geq 4h$, then $m(\theta) \geq k b_1^{h-1}$.*

(2) *If $g \geq 4h+2$ then $m(\theta) \geq (a_1+2) b_1^h$.*

Corollary 7 *Let Γ be a distance-regular graph of order (s,t) with $r := \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$. Let θ be an eigenvalue of Γ with the multiplicity $m(\theta)$. Suppose $s \geq t$, and $\theta \neq s(t+1), -t-1$. Then*

$$m(\theta) \geq \left(\frac{t+1}{t}\right) (st)^{\frac{r+1}{2}}.$$

Proof. We note that $a_1 = s-1, k = s(t+1)$ and $b_1 = st$.
If r is odd with $r = 2h-1$, then $g \geq 2r+2 = 4h$. Hence Proposition 6 (1) implies

$$m(\theta) \geq k b_1^{h-1} = \left(\frac{t+1}{t}\right) (st)^{\frac{r+1}{2}}.$$

If r is even with $r = 2h$, then $g \geq 2r+2 = 4h+2$. Hence Proposition 6 (2) implies

$$m(\theta) \geq (a_1+2) b_1^h = (s+1)(st)^{\frac{r}{2}}.$$

Since $s \geq t$, we have $\frac{(s+1)}{\sqrt{s}} \geq \frac{(t+1)}{\sqrt{t}}$. The desired result is proved. ■

Lemma 8 *Let Γ be a distance-regular graph of diameter d and let j be an integer with $1 \le j \le d$. Let x be a vertex of Γ and let Δ be the subgraph induced by $\Gamma_j(x)$. Suppose Δ is not connected. Then the second largest eigenvalue θ of Γ satisfies $\theta \ge a_j$.*

Proof. Δ is a_j-regular with at least two connected components. The assertion follows by interlacing. ∎

Remark. Let Γ be a regular near polygon of order (s, t) and let x be a vertex of Γ. Then $\Gamma_{d-1}(x)$ is not connected and thus the second largest eigenvalue θ of Γ satisfies $\theta \ge a_{d-1} = c_{d-1}(s-1)$.

The following proposition is an easy application of interlacing. (cf. [8, Theorem 6.2])

Proposition 9 *Let Γ be a distance-regular graph of diameter d. Let q and ℓ be positive integers with $q+\ell \le d$ such that $(c_q, a_q, b_q) = (c_{q+\ell-1}, a_{q+\ell-1}, b_{q+\ell-1})$. Then the second largest eigenvalue θ of Γ satisfies*

$$\theta \ge a_q + 2\sqrt{b_q c_q}\cos\left(\frac{2\pi}{\ell+1}\right).$$

∎

Proposition 10 ([7, Proposition 8].) *Let Γ be a distance-regular graph of diameter d and valency k. For any non-negative integer σ with $\sigma \le k$ let*

$$\delta := \delta(\sigma) = \min\{i \mid 1 \le i \le d, \sigma \le c_i + a_i\},$$

$$\beta_i := \beta_i(\sigma) = \sigma - c_i - a_i \quad \text{for } 0 \le i \le \delta,$$

$$\kappa_i := \kappa_i(\sigma) = \frac{\beta_0 \cdots \beta_{i-1}}{c_1 \cdots c_i} \quad \text{for } 1 \le i \le \delta$$

and

$$N(\sigma) := 1 + \kappa_1 + \cdots + \kappa_\delta.$$

Let h and j be positive integers with $h + j \le d$. Suppose $c_h = c_{h+j}$. Then

$$N(a_h) \le \frac{b_j \cdots b_{h+j-1}}{c_1 \cdots c_h}.$$

∎

To close this section we prove the following corollary of the above proposition.

Corollary 11 *Let Γ be a distance-regular graph of order (s, t) with $s > t \ge 3$. Let $r = \max\{i \mid (c_i, a_i, b_i) = (1, s-1, st)\} \ge 2$. Let q and ℓ be positive integer with $\ell < r < q$ and $q + \ell \le d$ such that $c_q = c_{q+\ell-1}$ and $a_q > c_{q-1} + a_{q-1}$. Then we have*

$$\left(\frac{st}{b_q}\right)^{\ell-1} < \frac{st(a_q - c_{q-1} - a_{q-1})}{a_q(c_q + a_q - c_{q-1} - a_{q-1})} \prod_{i=1}^{q-1} \left(\frac{b_i}{a_q - c_i - a_i}\right).$$

Proof. Let $a := a_q$. Put $h = q$ and $j = \ell - 1$ in Proposition 10. Then we have

$$a\left[1 + \frac{a - c_{q-1} - a_{q-1}}{c_q}\right] \prod_{i=1}^{q-2} \frac{(a - c_i - a_i)}{c_{i+1}} = \kappa_{q-1}(a) + \kappa_q(a) < N(a) \le \prod_{i=1}^{q} \frac{b_{\ell-2+i}}{c_i}.$$

The desired result is proved. ∎

3 Proof of Theorem 1

In this section we prove Theorem 1. We start from a proposition.

Proposition 12 *Let Γ be a distance-regular graph of diameter d. Let θ be the second largest eigenvalue of Γ. Let i be an integer with $1 \leq i \leq d-1$ such that $c_i \leq b_i$ and $\theta > a_i + 2\sqrt{b_i c_i}$. Let α_i be the largest root of the equation*

$$f_i(X) := b_i X^2 + (a_i - \theta)X + c_i = 0.$$

Then $u_{i+1}(\theta) > (\alpha_i)^{i+1}$.

Proof. The assertion follows from an easy induction. ■

Proposition 13 *Let Γ be a distance-regular graph of order (s,t) with $s > t \geq 3$. Let d be the diameter of Γ and $r := \max\{i \mid (c_i, a_i, b_i) = (1, s-1, st)\}$. Assume $d \geq r+2$. Let θ be the second largest eigenvalue of Γ. Let q be an integer with $1 \leq q \leq d-1$ such that $c_{q-1} \leq b_{q-1}$. Suppose there exists a real number β with $0 < \beta < 1$ such that $\theta > a_{q-1} + c_{q-1}\beta^{-1} + b_{q-1}\beta$. Then the following hold.*
(1) $u_q(\theta) > \beta^q$.
(2) If $c_{q+1} < b_q$, then

$$m(\theta) < \left(\frac{b_q}{c_{q+1}}\right)^{d-q} \left(\frac{b_q}{b_q - c_{q+1}}\right) \beta^{-2q}.$$

In particular,

$$\left(\sqrt{st}\right)^{r+1} < \frac{t b_q}{(t+1)(b_q - c_{q+1})} \left(\frac{b_q}{c_{q+1}}\right)^{d-q} \beta^{-2q}.$$

Proof. (1) Note that $\theta > a_{q-1} + c_{q-1}\beta^{-1} + b_{q-1}\beta \geq a_{q-1} + 2\sqrt{b_{q-1}c_{q-1}}$. Since $f_{q-1}(\beta) < 0$, we have $\alpha_{q-1} \geq \beta$. Thus the assertion follows from Proposition 12.
(2) We have

$$m(\theta) = \frac{|V\Gamma|}{\sum_{i=0}^{d} k_i u_i(\theta)^2} < \frac{|V\Gamma|}{k_q u_q(\theta)^2}.$$

The first assertion follows from (1) and Lemma 5.

The second assertion follows from the first and Corollary 7. ■

Proof of Theorem 1. It is easy to see that $h := h(d,q,r) \geq 3$. We may assume that $s \geq t^3$. Let θ be the second largest eigenvalue of Γ. Then we have

$$\theta \geq c_{d-1}(s-1) \geq (c_{q-1}+1)(s-1) > a_{q-1} + t c_{q-1} + \frac{b_{q-1}}{t}.$$

Since $b_q \geq s \geq t^3 > (t+1)c_{q+1}$, we have $t b_q \leq (t+1)(b_q - c_{q+1})$. Put $\beta = \dfrac{1}{t}$ in Proposition 13. Then

$$\left(\sqrt{st}\right)^{r+1} < \left(\frac{b_q}{c_{q+1}}\right)^{d-q} \beta^{-2q} < (st)^{d-q} t^{2q}.$$

The desired result is proved. ■

4 Proof of Theorem 3

Let Γ be a distance-regular graph of order (s,t) with $s > t \geq 3$. Let d be the diameter of Γ and $r = \max\{i \mid (c_i, a_i, b_i) = ((1, s-1, st)\} \geq 3$.

Assume $d \geq r+2$ and Γ has the following intersection array:

$$\iota(\Gamma) = \left\{ \begin{array}{ccccccccc} * & 1 & \cdots & 1 & c & \cdots & c & c_d \\ 0 & s-1 & \cdots & s-1 & a & \cdots & a & a_d \\ s(t+1) & st & \cdots & st & b & \cdots & b & * \end{array} \right\}.$$

Note that $a \geq c(s-1)$. Let θ be the second largest eigenvalue of Γ and let $\sigma := \left(\frac{b}{stc}\right)^{\frac{1}{4}}$.

Lemma 14 *Suppose $b \geq t(t+1)$ and $\theta \geq (s-1) + \sigma^{-1} + \sigma st$. Then $3r+4 \leq 2d$.*

Proof. Put $q = r+1$ in Proposition 13. Then we have $u_{r+1}(\theta) \geq \sigma^{r+1}$ and

$$\left(\frac{b}{c}\right)^{\frac{r+1}{2}} = \left(\sigma^2\sqrt{st}\right)^{r+1} < \frac{tb}{(t+1)(b-c_{r+2})}\left(\frac{b}{c_{r+2}}\right)^{d-r-1} \leq \left(\frac{b}{c}\right)^{d-r-1}.$$

The lemma is proved. ■

Proof of Theorem 3. Suppose $s \geq t(t+1)$ and derive a contradiction. Let $(c,a,b) := (c_{r+1}, a_{r+1}, b_{r+1})$ and $\sigma := \left(\frac{b}{stc}\right)^{\frac{1}{4}}$. Since $c \geq \frac{3}{4}(t+1)$, we have $b \leq \frac{1}{4}(t+1)s$. Thus

$$\frac{1}{\sqrt{st}} \leq \sigma \leq \frac{1}{\sqrt{3}}.$$

It follows by, Lemma 8, that

$$\theta \geq a_{d-1} = c(s-1) \geq \frac{3}{4}(t+1)(s-1) \geq (s-1) + \sigma^{-1} + \sigma st.$$

Since $b \geq s \geq t(t+1)$, we have $3r+4 \leq 2d$ from Lemma 14.

By putting $q = r+1$ and $\ell = d-r-1$ in Corollary 11 we have

$$\left(\frac{4t}{t+1}\right)^{\frac{r}{2}} \leq \left(\frac{st}{b}\right)^{\ell-1} < \frac{st}{a}\left(\frac{st}{a-s}\right)^{r} < \frac{4st}{3(t+1)(s-1)}\left(\frac{4st}{3st-3t-s-3}\right)^{r}$$

as $a \geq \frac{3}{4}(t+1)(s-1)$.

If $t \geq 4$, then we have

$$\left(\frac{16}{5}\right)^{\frac{r}{2}} < \frac{4}{3}\left(\frac{16s}{11s-9}\right)^{r}.$$

This is a contradiction as $s \geq t(t+1) \geq 20$.

If $t = 3$, then we have

$$(3)^{\frac{r}{2}} < \frac{s}{s-1}\left(\frac{3s}{2s-3}\right)^{r}.$$

This is a contradiction as $s \geq t(t+1) \geq 12$.

The theorem is proved. ■

5 Proof of Theorem 4

Throughout this section Γ denotes a regular near polygon of order (s,t) with $t \geq 3$ and $s \geq t(t+1)$. Let d be the diameter of Γ and $r = \max\{i \mid c_i = 1\} \geq 3$. Assume $d \geq r+3$ and $c_{r+1} < t = c_{r+2} = \cdots = c_{d-1}$. Let $\ell := d-r-2 = |\{i \mid c_i = t\}|$.

Let θ be the second largest eigenvalue of Γ and let $\alpha := \alpha_r$ be the largest root of the equation

$$f_r(X) := stX^2 + (s-1-\theta)X + 1 = 0.$$

Note that $\theta \geq a_{d-1} = t(s-1)$ by Lemma 8. Let β be the largest root of the equation

$$stX^2 - (s-1)(t-1)X + 1 = 0.$$

Then we have $\alpha \geq \beta$.

Lemma 15 *(1)*

$$\left(\alpha^2\sqrt{st}\right)^{r+1} < \frac{st(t-1)}{(t+1)(s-t)}\left(\frac{s}{t}\right)^{\ell+1}.$$

(2)

$$t^{\ell-1} < \frac{2s}{(s-1)}\left(\frac{st}{st-s-t}\right)^r.$$

(3) If $t \geq 4$, then $r \leq 2\ell$ and $r < 7$.

Proof. Put $q = r+2$ in Lemma 5. Then

$$|V\Gamma| \leq k_{r+2}\left(\frac{s}{t}\right)^{d-r-2}\left(\frac{s}{s-t}\right) = k_{r+1}\left(\frac{b_{r+1}}{s-t}\right)\left(\frac{s}{t}\right)^{d-r-1}.$$

It follows, by Corollary 7 and Proposition 12, that

$$\left(\frac{t+1}{t}\right)(st)^{\frac{r+1}{2}} \leq m(\theta) < \frac{|V\Gamma|}{k_{r+1}u_{r+1}(\theta)^2} < \frac{b_{r+1}}{(s-t)\alpha^{2(r+1)}}\left(\frac{s}{t}\right)^{\ell+1}.$$

Since $b_{r+1} = (t+1-c_{r+1})s \leq (t-1)s$, the assertion is proved.
(2) This follows by putting $q = r+2$ in Corollary 11.
(3) Suppose $2\ell+1 \leq r$. Then, using (1), we have

$$\left(\beta^2 t\right)^4 \leq \left(\alpha^2 t\right)^{r+1} < (t-1).$$

which is a contradiction. Hence $r \leq 2\ell$. It follows, by (2), that

$$\left(\frac{st-s-t}{s\sqrt{t}}\right)^r < \frac{2st}{(s-1)}.$$

This implies $r < 7$. The desired result is proved. ∎

Lemma 16 *Suppose $t = 3$ and $s \geq 14$. Then the following hold:*
(1) $r \leq 2\ell+1$.
(2) If $r \geq 8$, then $2\ell-2 \leq r \leq 2\ell$.
(3) If $r \geq 11$, then $s \geq 90$ and $2\ell = r$.

Proof. (1) Suppose $2\ell + 2 \le r$. Then Lemma 15 (1) implies that

$$1 < \left(3\beta^2\right)^{r+1} \le \frac{3\sqrt{3s}}{2(s-3)}.$$

which is a contradiction. Hence we have $r \le 2\ell + 1$.
(2) We have $\ell \ge 4$ from (1). It follows, by putting $q = r + 2$ in Proposition 9, that

$$\theta \ge t(s-1) + 2\sqrt{st}\cos\left(\frac{2\pi}{5}\right).$$

Since

$$f_r\left(\frac{2}{3}\right) = \frac{1}{3}\left\{7 - 4\sqrt{3s}\cos\left(\frac{2\pi}{5}\right)\right\} < 0,$$

we have $\alpha > \frac{2}{3}$. Suppose $r = 2\ell + 1$. Then it follows, by Lemma 15 (1), that

$$\left(\frac{4}{3}\right)^{11} \le \left(3\alpha^2\right)^{r+1} \le \frac{3s}{2(s-3)}.$$

Hence we have $r \le 2\ell$.

Suppose $r \le 2\ell - 3$. Then Lemma 15 (2) implies that

$$\left(\frac{2s-3}{s\sqrt{3}}\right)^r < \frac{2s\sqrt{3}}{3(s-1)}$$

which is a contradiction. The assertion is proved.
(3) We have $\ell \ge 6$ from (2). It follows, by putting $q = r + 2$ in Proposition 9, that

$$\theta \ge t(s-1) + 2\sqrt{st}\cos\left(\frac{2\pi}{7}\right).$$

Suppose $s \le 89$. Then we have

$$f_r\left(\frac{2\sqrt{30}}{15}\right) = \frac{1}{15}\left\{24s + 15 - 4\sqrt{30}(s-1) - 12\sqrt{10s}\cos\left(\frac{2\pi}{7}\right)\right\} < 0$$

and hence $\alpha > \frac{2\sqrt{30}}{15}$. It follows, by Lemma 15 (1), that

$$\left(\frac{8}{5}\right)^{12} \le \left(\alpha^2 t\right)^{r+1} < \frac{s^2\sqrt{3s}}{6(s-3)}$$

as $2\ell - 2 \le r$. This is a contradiction. Hence we have $s \ge 90$.

Suppose $r \le 2\ell - 1$. Then Lemma 15 (2) implies that

$$\left(\frac{2s-3}{s\sqrt{3}}\right)^r < \frac{2s\sqrt{3}}{(s-1)}$$

which is a contradiction. Hence we have $2\ell = r$. The desired result is proved. ∎

Proof of Theorem 4. Assume $r \ge 7$. Then we have $t = 3$ from Lemma 15 (3). Suppose $r \ge 13$ and $s \ge 14$ to derive a contradiction. Then we have $r = 2\ell$ from Lemma 16 (3). It follows, by Lemma 15 (2), that

$$\left(\frac{2s-3}{s\sqrt{3}}\right)^{13} < \frac{6s}{(s-1)}$$

This implies $s \leq 262$. It follows, by putting $q = r + 2$ in Proposition 9, that

$$\theta \geq t(s-1) + 2\sqrt{st}\cos\left(\frac{2\pi}{8}\right) = 3(s-1) + \sqrt{6s}$$

as $7 \leq \ell$. We have

$$f_r\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\left\{2 + 2\sqrt{2} - 2\sqrt{3s} + (3 - 2\sqrt{2})s\right\} < 0,$$

and thus $\alpha > \dfrac{1}{\sqrt{2}}$. Hence Lemma 15 (1) implies that

$$\left(\frac{3}{2}\right)^{14} \leq \left(\alpha^2 t\right)^{r+1} < \frac{s^2}{2(s-3)}$$

This is a contradiction as $s \leq 262$. The theorem is proved. ■

6 Concluding remarks

First we recall the following result.

Proposition 17 [6, Proposition 5] *Let* Γ *be a distance-regular graph with* $r = \max\{i \mid (c_i, a_i, b_i) = (c_1, a_1, b_1)\}$ *and* $(c_{r+1}, a_{r+1}) = (2, 2a_1)$. *If* $a_1 > 0$, *then* $c_{r+2} \neq 2$.

In this section we discuss the regular near polygons of order $(s, 3)$. By our theorems we have the following result.

Proposition 18 *Let* Γ *be a regular near polygon of order* $(s, 3)$. *Let* d *be the diameter of* Γ *and* $r = \max\{i \mid c_i = 1\}$. *The one of the following cases holds:*
(i) $d \leq r + 2$,
(ii) $r \leq 12$,
(iii) $s \leq 13$.

Proof. We may assume $d \geq r + 3$. Then $c_{r+1} \in \{2, 3\}$.

Suppose $c_{r+1} = 3$. Then we have $c_{r+1} = \cdots = c_{d-1} = 3$. This is the case of Theorem 3 and thus $s < 12$ if $r \geq 3$.

Suppose now $c_{r+1} = 2$. Then we have $c_{r+2} \neq 2$ from Proposition 17, and thus $c_{r+2} = \cdots = c_{d-1} = 3$. This is the case of Theorem 4, and hence $s \leq 13$ or $r \leq 12$ holds.

The desired result is proved. ■

In future work we will show that this claim implies that there exists a positive constant R such that all regular near polygons with order $(s, 3)$ have $r \leq R$, and hopefully this will lead to a classification of them.

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