EXTENSIONS OF POLYNOMIALS AND M-IDEAL PROPERTIES IN BANACH FUNCTION SPACES

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Abstract. We study the norm-preserving extension of norm-attaining n-homogeneous polynomials on Banach function spaces and M-ideal properties of function and sequence Marcinkiewicz spaces. We show for a large class of Banach function spaces that the extension of n-homogeneous polynomials does not need to be unique for $n \geq 2$ in real spaces and for $n \geq 3$ in complex spaces. We find further a geometric condition under which every norm-attaining 2-homogeneous polynomial on a complex symmetric sequence space X depends only on finitely many variables. This geometric condition yields that a unit ball in X does not possess any complex extreme points. In particular, if $X = m_\Psi$ is a Marcinkiewicz sequence space and m_Ψ^0 is its subspace of order continuous elements, we show that such properties as: every norm-attaining 2-homogeneous polynomial on m_Ψ^0 depends on finitely many variables, every norm-attaining 2-homogeneous polynomial on m_Ψ^0 has a unique norm preserving extension to its bidual m_Ψ, and no element of a unit sphere of m_Ψ^0 is a complex extreme point, are equivalent. Moreover, any of these properties is equivalent to the fact that Ψ is strictly increasing. As a corollary we obtain that m_Ψ is not rotund. We also find conditions when an order continuous subspace of either function or sequence Marcinkiewicz space is an M-ideal in its bidual. Finally we investigate the dual and M-ideal properties of $L^1 + L^\infty$, a particular example of Marcinkiewicz spaces.

1. Introduction and Preliminaries

Let X be a Banach space over a scalar field $\mathbb{F}$, where $\mathbb{F}$ is either the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. Let further B_X(resp. S_X) denote a unit ball(resp. unit sphere) in X. A bounded multi-linear form means an n-linear mapping $L : X^n \to \mathbb{F}$ for $n \in \mathbb{N}$, with finite norm $\|L\|$ defined as

$$\|L\| = \{|L(x_1, \cdots, x_n)| : x_i \in B_X, i = 1, \cdots, n\}.$$

Then a map $P(x) = L(x, \cdots, x) : X \to \mathbb{F}$ is called an *n-homogeneous polynomial* on X and its norm is defined by

$$\|P\| = \sup\{|P(x)| : x \in B_X\}.$$

Given a Banach space X, if $x \in X$ and $x^* \in X^*$ then $\langle x^*, x\rangle$ denotes $x^*(x)$. We also denote by $[x_1, \ldots, x_n]$ a linear span of vectors $\{x_i\}_{i=1}^n \subset X$. For each subset M of X, let $M^\perp$ be the set of all bounded linear functionals which vanish on M. A point x of a convex set K is an *extreme point* of K if $\{x + ty : -1 \leq t \leq 1\} \subset K$ for y in X implies that $y = 0$. If every point of S_X is an extreme point of B_X, X is called a *strictly convex(or, rotund)* space. A point x of a convex set K of a complex Banach space X is a *complex extreme point* of K if $\{x + \zeta y : |\zeta| \leq 1, \zeta \in \mathbb{C}\} \subset K$ for y in X implies that $y = 0$. It is easy to see that every extreme point of B_X is a complex extreme point of B_X when X is a complex space.

Let $(\Omega, \mu) = (\Omega, \mathcal{B}, \mu)$ be a measure space with a complete σ-finite measure μ on σ-algebra $\mathcal{B}$. Let $L^0(\mu)$ denote the space of all μ-equivalence classes of $\mathcal{B}$-measurable $\mathbb{F}$-valued functions on Ω with the topology of convergence in measure on μ-finite sets.

A Banach space $(X, \|\ \|)$ is said to be a *Banach function space* on (Ω, μ) if it is a subspace of $L^0(\mu)$ such that there is $h \in L^0(\mu)$ with $h > 0$ a.e. in Ω and it has the *ideal property* that is if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ a.e. then $f \in X$ and $\|f\| \leq \|g\|$. If in addition the unit ball B_X is closed in $L^0(\mu)$, then we say that X has the *Fatou property.* A Banach function space defined on $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with the counting measure μ is called a *Banach sequence space.* In this case $e_i \in X$ for all $i \in \mathbb{N}$, where e_i denotes a standard unit vector, that is $e_i = (0, \ldots, 0, 1, 0, \ldots)$ with 1 as the ith component.

A Banach function space X on (Ω, μ) is said to be *rearrangement invariant* (*r.i.,or symmetric*) if for every $f \in L^0(\mu)$ and $g \in X$ with $\mu_f = \mu_g$, we have $f \in X$ and $\|f\| = \|g\|$, where for any $h \in L^0(\mu)$, μ_h is a *distribution function* of h defined by

$$\mu_h(t) = \mu\{\omega \in \Omega : |h(\omega)| > t\}, \quad t \geq 0.$$

If X is a Banach function space on (Ω, μ), then the *associate space* X' of X is a Banach function space, which can be identified with the space of all functionals possessing an integral representation, that is,

$$X' = \{g \in L^0(\mu) : \|g\|_{X'} = \sup_{\|f\| \leq 1} \int_\Omega |fg| d\mu < \infty\}.$$

It is well known that if X has the Fatou property, then $(X'', \| \|_{X''})$ coincides with $(X, \| \|)$ [4, 11, 13].

An element $f \in X$ is said to be *order continuous* if $\|f_n\| \downarrow 0$ for every sequence $\{f_n\}$ with $|f_n| \leq |f|$ a.e. and $|f_n| \downarrow 0$ a.e. on Ω. A Banach function space X is said to be *order continuous* if every element of X is order continuous. It is well known that if X is an order continuous Banach function space, then X^* is order isometric to X', and this identification will be denoted by $X^* \simeq X'$.

Suppose for the moment that X is a Banach function space consisting of real valued functions. An element $\phi \in X^*$ is called an *integral functional* if for any $\{f_n\} \subset X$ with $0 \leq f_n \downarrow 0$ a.e., $\phi(f_n) \to 0$. A linear functional $\phi_s \in X^*$ is called a *positive singular linear functional* whenever $\phi_s(f) \geq 0$ holds for all non-negative f in X and for every integral linear functional ϕ, $0 \leq \phi(f) \leq \phi_s(f)$ for all non-negative f in X implies $\phi = 0$. A *singular linear functional* in X^* means the difference of two positive singular linear functionals in X^*. It is known that the space of integral linear functionals in X^* is order isometric to X' and a dual space X^* is order isometric to $X' \oplus X_s^*$, where X_s^* is the space of singular functionals on X [11, 13, 16].

Whenever X is a Banach function space, X_0 (or X^0) will denote the set of all order continuous elements of X. It is easy to show that X_0 is an *order ideal*, which means that it is a closed subspace with the ideal property. Note that X_0 is contained in the closure of the family of all simple functions in X with support of finite measure [4]. It is well known that if X is a Banach function space with the Fatou property and X_0 contains all simple functions with support of finite measure, then $(X_0)^* \simeq X'$. In this case $X^* \simeq (X_0)^* \oplus X_0^\perp$, where $X_0^\perp$ coincides with X_s^* when X is a Banach function space consisting of real valued functions (cf. Theorem 102.6, Theorem 102.7 in [13]).

Let Y be a closed subspace of a Banach space X. Y is called an *M-ideal* of X if there is a bounded projection $\mathcal{P} : X^* \to X^*$ with range $Y^\perp$ such that for each $x^* \in X^*$,

$$\|x^*\| = \|\mathcal{P}x^*\| + \|(I - \mathcal{P})x^*\|.$$

We can write this decomposition as $X^* = Y^\perp \oplus_1 Y^*$. A Banach space X is said to be *M-embedded* if X is an M-ideal of its bidual X^{**}. We will use the following facts about M-ideals [7].

Theorem 1.1. *Suppose Y is a closed subspace of a Banach space X.*

(i) *(The 3-ball property) Y is an M-ideal of X if and only if for all $y_1, y_2, y_3 \in B_Y$, all $x \in B_X$ and $\epsilon > 0$ there is $y \in Y$ satisfying*

$$\|x + y_i - y\| \leq 1 + \epsilon \quad \textit{for all } i = 1, 2, 3.$$

(ii) *A Banach space X is M-embedded if and only if every separable subspace of X is also M-embedded.*

(iii) *If X is an M-embedded space, then every separable subspace of X has a separable dual.*

For any real functions F and G, we say that F is equivalent to G and we write it as $F \approx G$ whenever there are constants $C_1, C_2 > 0$ such that $C_1|F(u)| \leq |G(u)| \leq C_2|F(u)|$ for all u in the domain of the functions. Recall also that for $z \in \mathbb{C}$, $\operatorname{sign} z = \overline{z}/|z|$ if $z \neq 0$ and $\operatorname{sign} z = 1$ if $z = 0$.

The Hahn-Banach type extension of n-homogeneous polynomials has been studied in a number of papers e.g. [1, 2, 3, 5, 6, 9]. In particular, it is known that every n-homogeneous polynomial on a Banach space X has a norm-preserving extension to its bidual [1, 2, 5]. Moreover, it is well known that if a subspace Y of X is an M-ideal, then every bounded linear functional (i.e., 1-homogeneous polynomial) on Y has a unique Hahn-Banach extension to X [7]. It has been shown in [3] that it is no longer true for polynomials. In fact they showed that the norm-preserving extension of an n-homogeneous polynomial on c_0 to ℓ_∞ does not need to be unique for $n \geq 2$ in real spaces and for $n \geq 3$ in complex spaces. They also showed that every norm-attaining 2-homogeneous polynomial on a complex c_0 must be finite, that is it depends only on finite many variables. These results have been further generalized to certain types of Marcinkiewicz spaces in [9].

In this paper, we investigate the Hahn-Banach type extension of norm-attaining n-homogeneous polynomials on Banach function spaces, generalizing in particular the results in [3, 9]. We also examine M-ideal properties of function and sequence Marcinkiewicz spaces, including the space $L^1 + L^\infty$.

Let us outline briefly the content of this article. In section 2 we show for a large class of Banach function spaces X that the norm preserving extension of n-homogeneous polynomials on a subspace of X does not need to be unique for $n \geq 2$ in real spaces and for $n \geq 3$ for complex spaces. In particular this statement holds true for any r.i. function space with the Fatou property or an arbitrary Banach sequence space.

In section 3 we study 2-homogeneous polynomials on a complex symmetric sequence space X. We define here a notion of a finite polynomial on X^{**} and a special geometric condition of X, under which any 2-homogeneous polynomial on X attains its norm if and only if it is finite. This geometric condition yields for instance that no point of S_X is a complex extreme point of B_X. As a corollary we obtain that every 2-homogeneous norm-attaining polynomial on X has a unique norm preserving extension to its bidual X^{**}. Finally in this section we present similar results for bounded functionals that are 1-homogeneous polynomials.

In section 4 we investigate Marcinkiewicz function spaces M_Ψ on $I = (0, 1)$ or $I = (0, \infty)$. After collecting some basic properties of M_Ψ and its order continuous subspace M_Ψ^0, we formulate conditions on Ψ when M_Ψ^0 is an M-ideal in M_Ψ, and we show for wide class of functions Ψ, that M_Ψ is a bidual of M_Ψ^0. It appears also that for $\Psi(t) = \max\{t, 1\}$, M_Ψ coincides with $\Sigma = L^1 + L^\infty$. It naturally leads to study M-ideal properties of Σ. We compute the dual norms of Σ equipped with two different traditional norms, and consequently we find out that Σ_0 is not an M-ideal in Σ under the Marcinkiewicz norm, while it is an M-ideal under the other norm. We also prove that Σ_0, under either norms, is not M-embedded. Thus Σ is the Marcinkiewicz space, which shows that without additional assumptions on Ψ we are not able to obtain the earlier results in this section.

Sections 5 and 6 are devoted to Marcinkiewicz sequence spaces m_Ψ. In section 5 we provide necessary and sufficient condition on Ψ for m_Ψ to be a bidual of m_Ψ^0, as well as for m_Ψ^0 to be an M-ideal of m_Ψ. Section 6 is a continuation of section 3 for Marcinkiewicz sequence spaces. The main result of this section, Theorem 6.8, states several equivalent conditions for the property that every norm-attaining 2-homogeneous polynomial on m_Ψ^0 is finite. It says among others that it is equivalent to the condition that no element of the unit sphere of m_Ψ^0 is a complex extreme point of a unit ball in m_Ψ. It is also equivalent to the fact that every norm-attaining 2-homogeneous polynomial on m_Ψ^0 has a unique norm preserving extension to m_Ψ. Finally any of these conditions is equivalent to the property that the sequence $\Psi = \{\Psi(n)\}$ is strictly increasing. We then partially extend Theorem 6.8 to a symmetric sequence space X, finding a connection between behaviour of the fundamental function of X and the existence of complex extreme points of B_X as well as the condition that 2-homogeneous polynomials are finite on X. We conclude the section with a corollary stating that m_Ψ^0 or m_Ψ are never rotund, and with an example of a symmetric sequence space showing that the fundamental function cannot fully determine extreme points of its unit ball.

2. Extensions of polynomials

Let X be a Banach space and Y a closed M-ideal in X. It is well known that a bounded linear functional on Y has a unique norm preserving extension to X [7]. With polynomials the situation is different. In [3] (see also [9] for some Marcinkiewicz sequence spaces), it has been shown that extension of n-homogeneous polynomials from c_0 to ℓ_∞ is not unique for $n \geq 2$ for real spaces and for $n \geq 3$ for complex spaces. We shall show a similar result for a large class of Banach function spaces.

Now let X be a Banach function space over (Ω, μ), and let Y be a proper closed subspace of X. Let's assume that there exist two disjoint sets $E_i \in \mathcal{B}$, $i = 1, 2$, such that

$$\Phi f = \varphi_1(f)\chi_{E_1} + \varphi_2(f)\chi_{E_2}, \quad f \in X,$$

is a norm-one projection on X, where for $i = 1, 2$, $\chi_{E_i} \in Y$ and

$$\varphi_i(f) = \frac{1}{\mu E_i}\int_{E_i} f, \quad f \in X.$$

If X is a real space and $n \geq 2$, then we can easily construct an n-homogeneous polynomial on Y which has two different norm preserving extensions to X. Indeed, let φ be a norm-one linear functional on X which vanishes on Y. Letting now $\alpha = \|\chi_{E_1}\|$, n-homogeneous polynomial $P(f) = (\alpha\varphi_1(f))^n$ on Y has norm one. It is clear that $P_1(f) = (\alpha\varphi_1(f))^n$ and $P_2(f) = (\alpha\varphi_1(f))^n - (\alpha\varphi_1(f)^{n-2}\varphi^2$ are two distinct norm preserving extensions of P on X.

In the complex case, we can find an n-homogeneous polynomial with two distinct norm preserving extensions if $n \geq 3$. In fact consider the set

$$S = \{(z_1, z_2) \in \mathbb{C}^2 : \|z_1\chi_{E_1} + z_2\chi_{E_2}\| \leq 1\},$$

and the function

$$\psi(z_1, z_2) = |z_1|^2 + |z_2|^2, \quad (z_1, z_2) \in S.$$

It is clear that ψ is continuous on compact set S, and so there exists $(u_1, u_2) \in S$ such that

$$\psi(u_1, u_2) = \max_{(z_1,z_2)\in S} \psi(z_1, z_2) = |u_1|^2 + |u_2|^2 = a^2 + b^2,$$

where $a = |u_1|$, $b = |u_2|$, $a^2 + b^2 \neq 0$, and $(a, b) \in S$. We have the following result.

Lemma 2.1. *There exists $(a, b) \in S$ such that for $n \geq 2$ and for all $(z_1, z_2) \in S$,*

$$|az_1 + bz_2|^n + |bz_1 - az_2|^n \leq (a^2 + b^2)^n.$$

In particular for $n \geq 2$ and $f \in B_X$,

$$|a\varphi_1(f) + b\varphi_2(f)|^n + |b\varphi_1(f) - a\varphi_2(f)|^n \leq (a^2 + b^2)^n,$$

and so

$$|a\varphi_1(f) + b\varphi_2(f)| \leq a^2 + b^2 \quad \textit{and} \quad |b\varphi_1(f) - a\varphi_2(f)| \leq a^2 + b^2.$$

Proof. For $n = 2$ and any $(z_1, z_2) \in S$ we have

$$\begin{aligned} |az_1 + bz_2|^2 + |bz_1 - az_2|^2 &= (az_1 + bz_2)(a\overline{z}_1 + b\overline{z}_2) + (bz_1 - az_2)(b\overline{z}_1 - a\overline{z}_2) \\ &= (a^2 + b^2)(|z_1|^2 + |z_2|^2) \leq (a^2 + b^2)^2. \end{aligned}$$

Hence $|az_1 + bz_2| \leq a^2 + b^2$ and $|bz_1 - az_2| \leq a^2 + b^2$ on S.

For $n > 2$ we apply induction. Assuming that the inequality is true for $n - 1 \geq 2$, we get for any $(z_1, z_2) \in S$,

$$|az_1 + bz_2|^n + |bz_1 - az_2|^n \leq (a^2 + b^2)\{|az_1 + bz_2|^{n-1} + |bz_1 - az_2|^{n-1}\} \leq (a^2 + b^2)^n.$$

Now, since Φ is a contraction, $\|\varphi_1(f)\chi_{E_1} + \varphi_2(f)\chi_{E_2}\| = \|\Phi f\| \leq 1$ for any $f \in B_X$. Thus $(\varphi_1(f), \varphi_2(f)) \in S$ and the proof is done. □

Now for $n \geq 3$ define a polynomial P on Y as

$$P(f) = (a\varphi_1(f) + b\varphi_2(f))^n.$$

It is clear that P is an n-homogeneous polynomial on Y with $\|P\| = (a^2 + b^2)^n$. In fact it follows from Lemma 2.1, since we have $|P(f)| \leq (a^2 + b^2)^n$ for $f \in B_X$, and also $P(a\chi_{E_1} + b\chi_{E_2}) = (a^2 + b^2)^n$. Then the following polynomials

$$\begin{aligned} &P_1(f) = (a\varphi_1(f) + b\varphi_2(f))^n, \\ &P_2(f) = (a\varphi_1(f) + b\varphi_2(f))^n + (a^2 + b^2)(b\varphi_1(f) - a\varphi_2(f))^{n-1}\varphi(f), \end{aligned}$$

are two distinct norm preserving extensions of P from Y to X, where $\varphi \in B_{X^*}$ is chosen in such a way that it vanishes on Y and $(b\varphi_1(f) - a\varphi_2(f))\varphi(f) \neq 0$ for some $f \in X$. In view of Lemma 2.1, it is clear that $\|P_1\| = (a^2+b^2)^n$. Moreover, again applying Lemma 2.1, we get for every $f \in B_X$

$$\begin{aligned} |P_2(f)| &\leq |a\varphi_1(f) + b\varphi_2(f)|^n + |a^2+b^2||b\varphi_1(f) - a\varphi_2(f)|^{n-1} \\ &\leq (|a\varphi_1(f) + b\varphi_2(f)|^{n-1} + |b\varphi_1(f) - a\varphi_2(f)|^{n-1})(a^2+b^2) \leq (a^2+b^2)^n, \end{aligned}$$

since $n \geq 3$. Since we also have $P_2(a\chi_{E_1} + b\chi_{E_2}) = (a^2+b^2)^n$, it follows that $\|P_2\| = (a^2+b^2)^n$.

As a conclusion of the above considerations we can state the following result.

Theorem 2.2. *Let X be a Banach function space such that there exist two disjoint sets E_i, $i = 1,2$, such that the projection*

$$\Phi f = \Big(\frac{1}{\mu E_1}\int_{E_1} f\Big)\chi_{E_1} + \Big(\frac{1}{\mu E_2}\int_{E_2} f\Big)\chi_{E_2}, \qquad f \in X,$$

is a contractive operator on X. Moreover, assume that Y is a proper closed subspace of X with $\chi_{E_i} \in Y$, $i = 1,2$.
If X is a real space then for $n \geq 2$, there exists a norm-attaining n-homogeneous polynomial P on Y which has at least two norm-preserving extensions to X. In the complex case the similar statement holds true for $n \geq 3$.

If X is a r.i. space with the Fatou property over non-atomic or counting measure then for any disjoint sets E_i, $i = 1,2$, the projection Φ on X has norm one [4]. It is also clear by the lattice properties, that for a Banach sequence space X, for any distinct $i, j \in \mathbb{N}$, the projection $\Phi(x) = x(i)e_i + x(j)e_j$ on X has also norm one. Thus the following corollaries are immediate consequences of the previous result.

Corollary 2.3. *If X is a r.i. space with the Fatou property over non-atomic or counting measure space, then the conclusion of Theorem 2.2 is valid in X for any proper closed subspace Y in X with $\chi_{E_i} \in Y$, $i = 1,2$.*

Corollary 2.4. *For any Banach sequence space X the conclusion of Theorem 2.2 is valid in X for any proper closed subspace Y in X with $e_i, e_j \in Y$.*

Example 2.5. In this example we will show that there is a non-symmetric function space with norm one projection in Theorem 2.2. Suppose that $p : \Omega \to [1, \infty)$ is a measurable function on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ and define the functional for each $f \in L^0$,

$$I(f) = \int_\Omega \frac{|f(t)|^{p(t)}}{p(t)}\, d\mu.$$

Then Nakano space $L^{p(t)}$ is the family of all measurable functions on Ω with the property $I(\lambda f) < \infty$ for some $\lambda > 0$ with norm

$$\|f\| = \inf\{\lambda > 0 : I(f/\lambda) \leq 1\}.$$

It is easy to show that Nakano space $L^{p(t)}$ is a Banach function space [14] but it is in general, not symmetric.

Suppose that $p(t)$ has constant values $a_i \geq 1$ on disjoint measurable sets E_i, $i = 1,2$, respectively with $0 < \mu E_1 = \mu E_2 < \infty$. Then the projection

$$\Phi f = \left(\frac{1}{\mu E_1}\int_{E_1} f\right)\chi_{E_1} + \left(\frac{1}{\mu E_2}\int_{E_2} f\right)\chi_{E_2}, \qquad f \in L^{p(t)},$$

is a contraction. Indeed, note that for any $\lambda > 0$,

$$\begin{aligned} I(\lambda\Phi f) &= \int_\Omega \frac{|\lambda\Phi f|^{p(t)}}{p(t)}\, d\mu \le \int_\Omega \left(\left(\frac{1}{\mu E_1} \int |\lambda f| \right)^{a_1} \frac{\chi_{E_1}}{a_1} + \left(\frac{1}{\mu E_2} \int |\lambda f| \right)^{a_2} \frac{\chi_{E_2}}{a_2} \right) d\mu \\ &\le \frac{1}{\mu E_1} \int_{E_1} |\lambda f|^{a_1} \int_{E_1} \frac{1}{a_1} + \frac{1}{\mu E_2} \int_{E_2} |\lambda f|^{a_2} \int_{E_2} \frac{1}{a_2} \\ &\le \int_{E_1} \frac{|\lambda f|^{a_1}}{a_1} + \int_{E_2} \frac{|\lambda f|^{a_2}}{a_2} \\ &\le \int_\Omega \frac{|\lambda f(t)|^{p(t)}}{p(t)}\, d\mu = I(\lambda f). \end{aligned}$$

This inequality yields that $\|\Phi f\| \le \|f\|$ for all $f \in L^{p(t)}$. Moreover we can see that $\|\chi_{E_i}\| = \left(\frac{\mu E_i}{a_i}\right)^{\frac{1}{a_i}}$, $i = 1, 2$. So if we further assume that $\left(\frac{\mu E_1}{a_1}\right)^{\frac{1}{a_1}} \neq \left(\frac{\mu E_2}{a_2}\right)^{\frac{1}{a_2}}$ then the norms of $\chi_{E_i}, i = 1, 2$, are different although they have the same distribution. Therefore we obtain a non-symmetric space with norm one projection Φ.

3. 2-homogeneous polynomials in r.i. sequence spaces

In view of the results of the previews section, our attention turns to 2-homogeneous polynomials on complex spaces. Let in this section X be a r.i. Banach sequence space. We say that n-homogeneous polynomial P on X^{**} is *finite* if there exists $m \in \mathbb{N}$ such that

$$P(x^{**}) = P\Big(\sum_{i=1}^m \langle x^{**}, e_i^* \rangle\, e_i \Big)$$

for all $x^{**} \in X^{**}$, where e_k^* are bounded linear functionals on X with $\langle e_k^*, x\rangle = x(k)$. By symmetry of X, each permutation σ of $\mathbb{N}$ induces an isometric isomorphism $T_\sigma : X \to X$ such that $T_\sigma x = (x(\sigma(1)), \cdots, x(\sigma(n)), \cdots)$ for every $x \in X$. Then $T_\sigma^{**} : X^{**} \to X^{**}$ is also an isometric isomorphism. Notice that the above definition of a finite polynomial is more general than the one used before (e.g. [3, 9]). In particular, it can be used for certain cases of non-sequence spaces, since a bidual X^{**} of a sequence space X may not be a sequence space itself.

We start with the following observation.

Proposition 3.1. *An n-homogeneous polynomial P on X^{**} is finite if and only if $P \circ T_\sigma^{**}$ is finite.*

Proof. Suppose that P is a finite n-homogeneous polynomial. Then the projection

$$\mathcal{R}x^{**} = \sum_{j=1}^m \left\langle x^{**}, e_j^* \right\rangle e_j$$

is such that $P\mathcal{R}x^{**} = Px^{**}$. Let $Q = P \circ T_\sigma^{**}$. Note that for every $k \in \mathbb{N}$, $\langle T_\sigma^* e_k^*, x\rangle = \langle e_k^*, T_\sigma x\rangle = x(\sigma(k))$, and so $T_\sigma^* e_k^* = e_{\sigma(k)}^*$. Therefore

$$\begin{aligned} Q(x^{**}) = P(T_\sigma^{**} x^{**}) = P(\mathcal{R} T_\sigma^{**} x^{**}) &= P\Big(\sum_{i=1}^m \langle T_\sigma^{**} x^{**}, e_i^* \rangle\, e_i \Big) \\ &= P\Big(\sum_{i=1}^m \langle x^{**}, T_\sigma^* e_i^* \rangle\, e_i \Big) \\ &= P\Big(\sum_{i=1}^m \Big\langle x^{**}, e_{\sigma(i)}^* \Big\rangle\, e_i \Big). \end{aligned}$$

Letting $s = \max\{\sigma(i) : i = 1, \cdots, m\}$, define

$$\mathcal{R}_s x^{**} = \sum_{j=1}^s \left\langle x^{**}, e_j^* \right\rangle e_j.$$

Clearly $s \geq m$ and in view of the above equations

$$\begin{aligned} Q(\mathcal{R}_s x^{**}) &= P\Big(\sum_{i=1}^{m} \Big\langle \mathcal{R}_s x^{**}, e^*_{\sigma(i)} \Big\rangle e_i\Big) \\ &= P\Big(\sum_{i=1}^{m} \sum_{j=1}^{s} \langle x^{**}, e^*_j \rangle \Big\langle e_j, e^*_{\sigma(i)} \Big\rangle e_i\Big) \\ &= P\Big(\sum_{i=1}^{m} \Big\langle x^{**}, e^*_{\sigma(i)} \Big\rangle e_i\Big) = Q(x^{**}). \end{aligned}$$

Hence $Q = P \circ T^{**}_\sigma$ is finite. The converse is clear since $P = P \circ T^{**}_\sigma \circ T^{**}_{\sigma^{-1}}$. □

In the case of 2-homogeneous norm-attaining polynomials we can state the following result.

Theorem 3.2. *Let X be a complex r.i. Banach sequence space. Suppose that for each $x \in B_X$, there are $n \in \mathbb{N}$ and $\epsilon > 0$ such that $X^{**} = [e_1, \cdots, e_n] \oplus G$ and*

$$x + \epsilon B_G \subset B_{X^{**}}. \tag{3.1}$$

Then 2-homogeneous polynomial P on X^{**} is norm-attaining on X, i.e., $P(x_0) = \|P\|$ for some $x_0 \in B_X$ if and only if P is finite.

Proof. Suppose that P is finite. Then the values of P are completely determined by the elements on finite dimensional subspace of X spanned by $\{e_1, \cdots, e_n\}$ for some n, which shows that P is norm-attaining on X.

Conversely suppose that $P(x_0) = \|P\| = 1$ for $x_0 \in B_X$. By the assumption, we can choose the following projection

$$\mathcal{R}_n x^{**} = \sum_{i=1}^{n} \langle x^{**}, e^*_i \rangle e_i.$$

Let $\mathcal{S}_n = I - \mathcal{R}_n$. Then

$$(\mathcal{R}_n|_X)^{**} = \mathcal{R}_n, \ (\mathcal{S}_n|_X)^{**} = \mathcal{S}_n,$$

and since both $\mathcal{R}_n|_X$ and $\mathcal{S}_n|_X$ are contractions, so $\|\mathcal{R}_n\| = \|\mathcal{S}_n\| = 1$. Thus

$$|P(x_0 + \lambda \mathcal{S}_n x^{**})| = |1 + 2\lambda \check{P}(x_0, \mathcal{S}_n x^{**}) + \lambda^2 P(\mathcal{S}_n x^{**})| \leq |P(x_0)| = 1,$$

for all $x^{**} \in B_{X^{**}}$, and for all $|\lambda| < \epsilon$, where $\check{P}$ is the unique symmetric bilinear form associated with P. By the Maximum Modulus Theorem,

$$\check{P}(x_0, \mathcal{S}_n x^{**}) = P(\mathcal{S}_n x^{**}) = 0 \text{ for } x^{**} \in B_{X^{**}}.$$

Take $y_0 = (0, \cdots, 0, x_0(n+1), x_0(n+2), \cdots)$. Then $y_0 \in B_X$ and $\mathcal{S}_n(y_0) = y_0$. Hence $P(y_0) = \check{P}(x_0, y_0) = 0$, which means that

$$P(x_0(1), \cdots, x_0(n), 0, \cdots) = P(x_0 - y_0) = P(x_0) + P(y_0) - 2\check{P}(x_0, y_0) = 1.$$

Let

$$N = \min\{|J| : P\Big(\sum_{i \in J} x_0(i) e_i\Big) = 1, \ J \subset \{1, \dots, n\}\},$$

where $|J|$ denotes cardinality of J. Now choose a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(\{1, \dots, n\}) = \{1, \dots, n\}$, $|x_0(\sigma(1))| \geq \cdots \geq |x_0(\sigma(n))|$ and $\sigma(i) = i$ for all $i \geq n+1$. Obviously $N \leq n$ and $|x_0(\sigma(N))| > 0$ and $|x_0(\sigma(k))| = 0$ for all $k \geq N+1$. Let $Q = P \circ T^{**}_{\sigma^{-1}}$. In view of Proposition 3.1 we need only to show that Q is finite.

Now take

$$v = (x_0(\sigma(1)), \dots, x_0(\sigma(N)), 0, \dots).$$

It is clear that $Q(v) = 1$ and $v \in B_X$. Thus by the assumption, there exist $m \in \mathbb{N}$ and $\epsilon > 0$ such that

$$|Q(v + \lambda \mathcal{S}_m x^{**})| = |Q(v) + 2\lambda \check{Q}(v, \mathcal{S}_m x^{**}) + \lambda^2 Q(\mathcal{S}_m x^{**})| \leq |Q(v)| = 1, \tag{3.2}$$

for all $x^{**} \in B_{X^{**}}$ and for all $|\lambda| < \epsilon$. Again the Maximum Modulus Theorem says that $\breve{Q}(v, \mathcal{S}_m x^{**}) = Q(\mathcal{S}_m x^{**}) = 0$ for all $x^{**} \in B_{X^{**}}$. If $m < N$, then applying the similar argument as above we could show that $Q(v_0) = 1$ where $v_0 = (v(1), \dots, v(m), 0, \dots)$. The latter however is a contradiction to the choice of N since

$$1 = Q(v_0) = P \circ T^{**}_{\sigma^{-1}}(v_0) = P\Big(\sum_{i \in M_0} x_0(i) e_i\Big)$$

for some $M_0 \subset \mathbb{N}$ with $|M_0| < N$. So $m \geq N$. Suppose that $m > N$. Then we know that for every $x \in B_X$, $|\lambda| < \epsilon$,

$$\|v + \lambda \mathcal{S}_m x\| \leq 1.$$

Since X is a r.i. Banach sequence space,

$$\|v + \lambda \mathcal{S}_N x\| \leq 1 \text{ for all } x \in B_X.$$

Note that $\mathcal{S}_N$ is weak*-to-weak* continuous. So weak*-lower semi-continuity of norm and density of B_X in $B_{X^{**}}$ in weak* topology, imply that

$$\|v + \lambda S_N x^{**}\| \leq 1 \text{ for all } x^{**} \in B_{X^{**}}. \tag{3.3}$$

So (3.2) holds for $m = N$. Therefore we may assume that $m = N$.

Now let $z_1 = (v(1), \cdots, v(m)), z_2 = (v(1), v(2) - mv(2), \cdots, v(m)), \cdots, z_m = (v(1), \cdots, v(m) - mv(m))$. And let $\tilde{z}_j = (z_j, 0, \cdots)$ for $1 \leq j \leq m$. Note that $\tilde{z}_1 = v$.

For any vector $x = (x(1), \dots, x(m)) \in \mathbb{C}^m$ we have the identity

$$\begin{aligned}(x(1), \cdots, x(m)) &= \frac{1}{m}\frac{x(1)}{t(1)}(z_1 + \cdots + z_m) + \sum_{j=2}^m \frac{1}{m}\frac{x(j)}{t(j)}(z_1 - z_j)\\ &= \frac{1}{m}\Big(\frac{x(1)}{t(1)} + \cdots + \frac{x(m)}{t(m)}\Big) z_1 + \frac{1}{m}\sum_{j=2}^m \Big(\frac{x(1)}{t(1)} - \frac{x(j)}{t(j)}\Big) z_j.\end{aligned}$$

Therefore for $x = (x(1), \cdots, x(m), 0 \cdots)$, and each $x^{**} \in B_{X^{**}}$,

$$\begin{aligned}Q(x + \mathcal{S}_m x^{**}) &= Q(x) + \frac{2}{m}\sum_{j=2}^m \Big(\frac{x(1)}{t(1)} - \frac{x(j)}{t(j)}\Big)\breve{Q}(\tilde{z}_j, \mathcal{S}_m x^{**})\\ &= Q(x) + \sum_{j=2}^m \Big(\frac{x(1)}{t(1)} - \frac{x(j)}{t(j)}\Big)\psi_j(\mathcal{S}_m x^{**}),\end{aligned}$$

where $\psi_j(\cdot) = \frac{2}{m}\breve{Q}(\tilde{z}_j, \cdot) \in X^{***}$.

For each $x^{**} \in B_{X^{**}}$ we will show that $\psi_j(\mathcal{S}_m x^{**}) = 0$. For such an x^{**}, for each $|\lambda| < \epsilon$, the similar argument as before (3.3) shows

$$\big\|v_\theta + \lambda e^{i\theta_1}\mathcal{S}_m x^{**}\big\| \leq 1,$$

and for each $\theta > 0$, there is an θ_1 such that

$$\begin{aligned}|Q(v_\theta + \lambda e^{i\theta_1}\mathcal{S}_m x^{**})| &= |Q(v_\theta) + (1 - e^{i\theta})\psi_2(\lambda e^{i\theta_1}\mathcal{S}_m x^{**})|\\ &= |Q(v_\theta)| + |1 - e^{i\theta}||\psi_2(\lambda \mathcal{S}_m x^{**})|\\ &\leq 1,\end{aligned}$$

where $v_\theta = (v(1), e^{i\theta}v(2), \cdots, v(m), 0, \cdots)$. Let now $f(\theta) = |Q(v_\theta)|$ and let $g(\theta) = |1 - e^{i\theta}| = 2\sin(\theta/2)$ for small $\theta > 0$. Then $|\psi_2(\lambda \mathcal{S}_m x^{**})| \leq \frac{1 - f(\theta)}{g(\theta)}$ for any $\lambda < \epsilon$. Therefore

$$\sup\{|\psi_2(\mathcal{S}_m x^{**})| : x^{**} \in \epsilon B_{X^{**}}\} \leq \lim_{\theta \downarrow 0}\frac{1 - f(\theta)}{g(\theta)} = \lim_{\theta \downarrow 0}\frac{-f'(\theta)}{g'(\theta)} = 0.$$

This implies that for $x^{**} \in B_{X^{**}}$, $\psi_2(\mathcal{S}_m x^{**}) = 0$. Similar calculations show that $\psi_3(\mathcal{S}_m x^{**}) = \cdots = \psi_m(\mathcal{S}_m x^{**}) = 0$. i.e., $Q(x + \mathcal{S}_m x^{**}) = Q(x)$. Taking $x = \mathcal{R}_m x^{**}$, $Q(x^{**}) = Q(\mathcal{R}_m x^{**} + \mathcal{S}_m x^{**}) = Q(\mathcal{R}_m x^{**})$, which shows that Q is finite and completes the proof. $\square$

The geometric assumption (3.1) on X^{**} in the above theorem says among others that no point of S_X is a complex extreme point of B_X.

Recall that every n-homogeneous polynomial on a Banach space X has a norm-preserving extension to its bidual X^{**} [1, 2, 5].

Corollary 3.3. *Suppose that X is a complex r.i. sequence space and X satisfies (3.1). Then a 2-homogeneous polynomial P on X attains its norm if and only if it is finite.*

Proof. Let Q be a norm-preserving extension of P to X^{**}. Then Q attains its norm on B_X, and by Theorem 3.2, Q is finite. So there is $m \in \mathbb{N}$ such that for every $x \in X$,

$$P(x) = Q(x) = Q\left(\sum_{i=1}^{m} x(i)e_i\right) = P\left(\sum_{i=1}^{m} x(i)e_i\right),$$

which completes the proof. □

Corollary 3.4. *Suppose that X is a complex r.i. sequence space and X satisfies (3.1). Every 2-homogeneous norm-attaining polynomial P on X has a unique norm-preserving extension to its bidual X^{**}.*

Proof. Let Q_1 and Q_2 be norm-preserving extensions of P from X to X^{**}. Then, by Theorem 3.2, Q_1 and Q_2 are finite. So there are $m_1, m_2 \in \mathbb{N}$ such that for each $x^{**} \in X^{**}$,

$$Q_1(x^{**}) = Q_1\left(\sum_{i=1}^{m_1} \langle x^{**}, e_i^*\rangle e_i\right) = \sum_{i=1}^{m_1}\sum_{j=1}^{i} a_{ij} \langle x^{**}, e_i^*\rangle \langle x^{**}, e_j^*\rangle,$$

$$Q_2(x^{**}) = Q_2\left(\sum_{s=1}^{m_2} \langle x^{**}, e_s^*\rangle e_s\right) = \sum_{s=1}^{m_2}\sum_{t=1}^{s} b_{st} \langle x^{**}, e_s^*\rangle \langle x^{**}, e_t^*\rangle,$$

for some complex numbers a_{ij}, b_{st}. They are equal on X so that there is $l \le \min\{m_1, m_2\}$ such that $a_{ij} = b_{ij}$ for all $1 \le j \le i \le l$ and $a_{ij} = 0 = b_{st}$ otherwise. So $Q_1(x^{**}) = Q_2(x^{**})$ for every $x^{**} \in X^{**}$. This completes the proof. □

It is easy to show that c_0 satisfies the assumptions of Theorem 3.2, and thus we get immediately by Corollary 3.4, the following result proved in [3].

Corollary 3.5. [3] *Every norm-attaining 2-homogeneous polynomial on a complex c_0 has a unique norm-preserving extension to ℓ_∞. In particular, the polynomial is finite.*

It is worth also to add here that at the end of section 6 we state a stronger result (Corollary 6.5) for some renormings of c_0 and ℓ_∞.

The following example shows that the assumptions on X in Theorem 3.2 are essential.

Example 3.6. Consider the space ℓ_∞ with the equivalent norm

$$\|x\| = |x(1)| + |x(2)| + \sup\{|x(n)| : n \ge 3\}.$$

It is not difficult to see that $(\ell_\infty, \| \ \|)$ is not symmetric and does not satisfy the assumption (3.1) of Theorem 3.2. It is also clear that c_0 is an order continuous subspace of $(\ell_\infty, \| \ \|)$. So $(c_0, \| \ \|)^{**} = (\ell_\infty, \| \ \|)$. Define on ℓ_∞, 2-homogeneous polynomials

$$P(x) = x(1)^2, \quad Q(x) = x(1)^2 + x(2)\sum_{k=3}^{\infty} \frac{x(k)}{2^{k-2}}.$$

Then P is norm-attaining on c_0 and $\|P\| = 1$. Note that for each $\|x\| \le 1$, $x \in \ell_\infty$,

$$Q(e_1) = 1 \quad \text{and} \quad |Q(x)| \le 1.$$

This shows that Q is norm-attaining on c_0 but it is not finite. In addition, choose a norm one linear functional φ on ℓ_∞ which vanishes on c_0. Letting

$$P_1(x) = x(1)^2 \quad \text{and} \quad P_2(x) = x(1)^2 + x(2)\varphi(x),$$

they are both norm-preserving extensions of P to ℓ_∞. Thus the conclusions of Theorem 3.2 and Corollary 3.4 are not valid for $(\ell_\infty, \| \ \|)$ and $(c_0, \| \ \|)$, respectively. We shall provide another example of this sort at the end of section 6 (cf. Example 6.6).

As for the norm-attaining bounded linear functional, we can obtain the following result.

Proposition 3.7. *Suppose X is a complex r.i. sequence space and X satisfies (3.1). Then a bounded linear functional φ on X attains its norm if and only if it is finite. Moreover, every norm-attaining bounded linear functional on X has a unique norm-preserving extension to X^{**}.*

Proof. If φ is finite, then it is clearly norm-attaining since its values depends only on a finite dimensional subspace of X.

Conversely, suppose that $\varphi(x_0) = \|\varphi\| = 1$ for some $x_0 \in B_X$. Then by the assumption, there are $n \in \mathbb{N}$ and $\epsilon > 0$ so that for every $|\lambda| < \epsilon$ and for every $y = (0, \cdots, 0, y(n+1), \cdots) \in B_X$,

$$|\varphi(x_0 + \lambda y)| = |\varphi(x_0) + \lambda\varphi(y)| \leq 1.$$

By the Maximum Modulus Theorem $\varphi(y) = 0$ for such an y. So for every $x \in X$, $\varphi(\mathcal{S}_n x) = 0$ and thus

$$\varphi(x) = \varphi(\mathcal{R}_n x) = \sum_{i=1}^{n} x_i \varphi(e_i) = \sum_{i=1}^{n} \langle e_i^*, x\rangle\, \varphi(e_i),$$

which means that φ is finite. Moreover it has a natural extension $\bar{\varphi}$ to X^{**}, defined by

$$\bar{\varphi}(x^{**}) = \sum_{i=1}^{n} \langle x^{**}, e_i^*\rangle\, \varphi(e_i).$$

Now, if φ has a norm-preserving extension ϕ to X^{**}, then similar arguments as above applied to ϕ and it shows that ϕ is finite. Since $\bar{\varphi}$ and ϕ are equal on X, so they must be equal on X^{**} too. The proof is done. $\square$

4. M-ideal properties of Marcinkiewicz function spaces M_Ψ, $L^1 + L^\infty$ and $L^1 \cap L^\infty$

Let $L^0 = L^0(I, \mathcal{B}, \mu)$ be the space of all Lebesgue measurable functions on I, where $I = (0, 1)$ or $I = (0, \infty)$, μ is the Lebesgue measure on σ-algebra $\mathcal{B}$ of the Lebesgue measurable subsets of I. For any $f \in L^0$ the *decreasing rearrangement* of f is the function f^* defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},$$

where μ_f is the distribution function of f.

Definition 4.1. Let $\Psi : [0, \infty) \to [0, \infty)$, $\Psi(0) = 0$, Ψ be increasing, and $\Psi(u) > 0$ for $u > 0$. Then the *Marcinkiewicz space* M_Ψ (called also *weak Lorentz space*) is the collection of all functions $f \in L^0$ such that

$$\|f\| = \|f\|_{M_\Psi} = \sup_{t>0} \frac{\int_0^t f^*}{\Psi(t)} < \infty.$$

Without loss of generality we can add (and we will) in the above definition the assumption that the function $\Psi(t)/t$ is decreasing on $(0, \infty)$. In fact, let's define

$$\widehat{\Psi}(t) = t \inf\{\Psi(s)/s : 0 < s \leq t\}, \quad t > 0.$$

Then it is not hard to show that $\widehat{\Psi}$ is increasing and $\widehat{\Psi}(t)/t$ is decreasing. For instance, if $0 < t_1 < t_2$ then

$$\begin{aligned}
\widehat{\Psi}(t_2) &= t_2 \min\{\inf\{\Psi(s)/s : 0 < s \leq t_1\}, \inf\{\Psi(s)/s : t_1 \leq s \leq t_2\}\} \\
&= \min\{t_2 \inf\{\Psi(s)/s : 0 < s \leq t_1\}, \Psi(t_1)\} \\
&\geq t_1 \min\{\inf\{\Psi(s)/s : 0 < s \leq t_1\}, \Psi(t_1)/t_1\} = \widehat{\Psi}(t_1).
\end{aligned}$$

Notice also that M_Ψ is not trivial if and only if $\widehat{\Psi}(t) > 0$ for $t > 0$. Finally we have that $M_\Psi = M_{\widehat{\Psi}}$ with equality of norms. In fact, since $\widehat{\Psi}(t) \leq \Psi(t)$, $\|f\|_{M_\Psi} \leq \|f\|_{M_{\widehat{\Psi}}}$. On the other hand for any $0 < s \leq t$,

$$t\,\frac{1}{s}\int_0^s f^* = t\,\frac{\Psi(s)}{s}\frac{\int_0^s f^*}{\Psi(s)} \leq t\,\frac{\Psi(s)}{s}\|f\|_{M_\Psi},$$

and so

$$\int_0^t f^* = t \inf\{\frac{1}{s}\int_0^s f^* : 0 < s \le t\} \le t \inf_{0<s\le t} \frac{\Psi(s)}{s}\, \|f\|_{M_\Psi} = \widehat{\Psi}(t)\|f\|_{M_\Psi},$$

which yields $\|f\|_{M_{\widehat{\Psi}}} \le \|f\|_{M_\Psi}$.

In view of the above remarks we shall assume further in this section that $\Psi : [0,\infty) \to [0,\infty)$, $\Psi(0) = 0$, $\Psi(t) > 0$ for $t > 0$, Ψ is increasing and $\Psi(t)/t$ is decreasing on $(0,\infty)$ i.e., Ψ is *quasi-concave.*

Definition 4.2. M_Ψ^0 is a subspace of M_Ψ consisting of all $f \in M_\Psi$ satisfying

$$\lim_{t\to 0^+} \frac{\int_0^t f^*}{\Psi(t)} = 0 \quad \text{in case when} \quad I = (0,1),$$

and

$$\lim_{t\to 0^+,\infty} \frac{\int_0^t f^*}{\Psi(t)} = 0 \quad \text{in case when} \quad I = (0,\infty).$$

Most of the following basic facts about M_Ψ and M_Ψ^0 are well known (cf. [11]). We collect them here for the sake of completeness.

Theorem 4.3. (1) *M_Ψ is a r.i. Banach function space with the Fatou property.*

(2) *$M_\Psi^0 \ne \{0\}$ if and only if $\inf_{t>0} \frac{t}{\Psi(t)} = 0$. If $I = (0,1)$ (resp. $I = (0,\infty)$) then the support of M_Ψ^0 is equal to $(0,1)$ (resp. $(0,\infty)$), that is there exists $h \in M_\Psi^0$ with $h > 0$ a.e. in I, if and only if*

$$\inf_{t>0} \frac{t}{\Psi(t)} = 0 \tag{4.1}$$

$$\Big(\textit{resp.} \quad \inf_{t>0} \frac{t}{\Psi(t)} = 0 \quad \textit{and} \quad \sup_{t>0} \Psi(t) = \infty\Big). \tag{4.2}$$

(3) *If Ψ satisfies condition (4.1) when $I = (0,1)$ (resp. (4.2) when $I = (0,\infty)$), then M_Ψ^0 is the subspace of all order continuous elements of M_Ψ.*

(4) *If Ψ satisfies condition (4.1) when $I = (0,1)$ (resp. (4.2) when $I = (0,\infty)$), then M_Ψ^0 is the closure of all simple (or bounded) functions with support of finite measure.*

Proof. (1) It can be shown directly by definition and the properties of decreasing rearrangement f^* (cf. [4]). For (2), it is enough to observe that if $\chi_{(0,a)} \in M_\Psi^0$ then for $0 < t < a$

$$\frac{\int_0^t \chi_{(0,a)}}{\Psi(t)} = \frac{t}{\Psi(t)},$$

and for $t > a$

$$\frac{\int_0^t \chi_{(0,a)}}{\Psi(t)} = \frac{a}{\Psi(t)}.$$

We shall show (3), (4) only in the case when $I = (0,\infty)$. Let $0 < f_n \le f \in M_\Psi^0$ and $f_n \downarrow 0$. Given $\epsilon > 0$, there exist $0 < t_0 < t_1 < \infty$ such that

$$\sup_{0<t<t_0} \frac{\int_0^t f^*}{\Psi(t)} < \epsilon \quad \text{and} \quad \sup_{t_1<t<\infty} \frac{\int_0^t f^*}{\Psi(t)} < \epsilon.$$

By the dominated Lebesgue theorem, there exists N such that for all $n > N$

$$\int_0^{t_1} f_n^* < \epsilon\Psi(t_0).$$

Hence for $n > N$,

$$\|f_n\| \le \sup_{0<t<t_0} \frac{\int_0^t f^*}{\Psi(t)} + \sup_{t_1<t<\infty} \frac{\int_0^t f^*}{\Psi(t)} + \frac{\int_0^{t_1} f_n^*}{\Psi(t_0)} < 3\epsilon.$$

So every element in M_Ψ^0 is order continuous. This means that M_Ψ^0 is contained in the closure of all simple (or bounded) functions with support of finite measure. If conditions (4.2) are satisfied, then the closure of the set of all simple functions with support of finite measure is M_Ψ^0. This proves

(4). Moreover M^0_Ψ is the subspace of all order continuous elements in M_Ψ. This shows (3) and completes the proof. □

Now, we investigate when M^0_Ψ is an M-ideal in M_Ψ. The next theorem extends the already known result for some functions Ψ (cf. [7]).

Theorem 4.4. *If $I=(0,1)$ and Ψ satisfies condition (4.1), then M^0_Ψ is an M-ideal in M_Ψ.*

If $I=(0,\infty)$ and Ψ satisfies conditions (4.2) and additional condition $\inf_{t>0}\Psi(t)/t=0$, then M^0_Ψ is an M-ideal in M_Ψ.

Proof. In the proof we shall use the 3-ball property (see Theorem 1.1), that is we show that for every $f\in B_{M_\Psi}$, every $f_i\in B_{M^0_\Psi}$, $i=1,2,3$, and $\epsilon>0$ there exists $g\in B_{M^0_\Psi}$ such that $\|f+f_i-g\|\le 1+\epsilon$, $i=1,2,3$.

Let first $I=(0,1)$. By density of bounded functions in M^0_Ψ we can take f_i bounded. By the assumption $\inf_{t>0} t/\Psi(t)=0$, there exists $b>0$ such that for all $0<t\le b$

$$\frac{\int_0^t f_i^*}{\Psi(t)}\le\frac{Mt}{\Psi(t)}\le\frac{Mb}{\Psi(b)}<\epsilon,$$

where $|f_i(x)|\le M$, $x\in(0,1)$, $i=1,2,3$. Also we choose $0<c\le b$ such that

$$\frac{\int_0^c f^*}{\Psi(b)}\le\epsilon.$$

Setting

$$g=f\chi_{\{s:|f(s)|\le f^*(c)\}},$$

it is clear that $g\in B_{M^0_\Psi}$. Moreover, for $0<t\le b$, $i=1,2,3$

$$\frac{\int_0^t(f_i+f-g)^*}{\Psi(t)}\le\frac{\int_0^t f_i^*}{\Psi(t)}+\frac{\int_0^t(f-g)^*}{\Psi(t)}\le\epsilon+\frac{\int_0^t f^*}{\Psi(t)}\le 1+\epsilon.$$

We also have

$$(f-g)^*(s)\le f^*\chi_{(0,c)}(s),\quad s\in I.$$

Hence for $t\ge b, i=1,2,3$

$$\frac{\int_0^t(f_i+f-g)^*}{\Psi(t)}\le\|f_i\|+\frac{\int_0^c f^*}{\Psi(b)}\le 1+\epsilon.$$

Combining the above inequalities we get $\|f_i+f-g\|\le 1+\epsilon$.

Now let $I=(0,\infty)$. Note that for every $f\in M_\Psi$

$$\limsup_{t\to\infty}\frac{\int_0^t f^*}{\Psi(t)}=\limsup_{t\to\infty}\frac{\frac{1}{t}\int_0^t f^*}{\frac{\Psi(t)}{t}}\le\sup_{t>0}\frac{\int_0^t f^*}{\Psi(t)}<\infty,$$

which means that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f^*=\lim_{t\to\infty}f^*(t)=0.$$

Since $f_i\in M^0_\Psi$, there are $0<b_1<b_2$ such that for all $t<b_1$ or all $t>b_2$,

$$\frac{\int_0^t f_i^*}{\Psi(t)}<\epsilon,$$

for $i=1,2,3$. Choose $\eta>0$ so small that $\eta\frac{b_2}{\Psi(b_1)}<\epsilon$ and take $0<c\le b_1$ for which

$$\frac{\int_0^c f^*}{\Psi(b_1)}\le\epsilon.$$

Setting

$$g=f\chi_{\{s:\eta<|f(s)|\le f^*(c)\}},$$

we have $g\in M^0_\Psi$. Indeed, there is $T>0$ such that

$$f^*(T)=\inf\{s>0:\mu_f(s)\le T\}<\eta.$$

So there is $0 < s < \eta$ such that $\mu_f(s) \le T$. Hence $\mu_f(\eta) = \mu\{|f| > \eta\} \le T$ and

$$\lim_{t\to\infty} \frac{\int_0^t g^*}{\Psi(t)} \le \lim_{t\to\infty} \frac{\int_0^T f^*}{\Psi(t)} = 0.$$

Moreover,

$$\lim_{t\to 0^+} \frac{\int_0^t g^*}{\Psi(t)} \le \lim_{t\to 0^+} \frac{tf^*(c)}{\Psi(t)} = 0.$$

For $i = 1, 2, 3$ and $0 < t \le b_1$ or $t \ge b_2$,

$$\frac{\int_0^t (f_i + f - g)^*}{\Psi(t)} \le \frac{\int_0^t f_i^*}{\Psi(t)} + \frac{\int_0^t f^*}{\Psi(t)} \le 1 + \epsilon.$$

For $i = 1, 2, 3$ and $b_1 \le t \le b_2$,

$$\begin{aligned}
\frac{\int_0^t (f_i + f - g)^*}{\Psi(t)} &\le \frac{\int_0^t (f_i + f\chi_{\{|f|\le\eta\}\cup\{|f|>f^*(c)\}})^*}{\Psi(t)} \le \frac{\int_0^t f_i^* + \int_0^t (f\chi_{\{|f|\le\eta\}})^* + \int_0^t (f\chi_{\{|f|>f^*(c)\}})^*}{\Psi(t)} \\
&\le \frac{\int_0^{b_1} f_i^* + \int_{b_1}^t f_i^* + \int_0^c f^* + t\eta}{\Psi(t)} \le \frac{\int_0^{b_1} f_i^* + b_1\eta}{\Psi(t)} + \frac{\int_{b_1}^t f_i^* + (t-b_1)\eta}{\Psi(t)} + \frac{\int_0^c f^*}{\Psi(b_1)} \\
&\le \frac{\int_0^{b_1} f_i^* + b_1\eta}{\Psi(b_1)} + \frac{\int_{b_1}^t f_i^* + \eta(b_2 - b_1)}{\Psi(t)} + \epsilon \le \epsilon + \eta\frac{b_1}{\Psi(b_1)} + 1 + \eta\frac{(b_2-b_1)}{\Psi(b_1)} + \epsilon \\
&< 1 + 4\epsilon.
\end{aligned}$$

These inequalities complete the proof. □

We will see later (Remark 4.8) that the assumption $\inf_{t>0} \Psi(t)/t = 0$ in the case of $I = (0, \infty)$ cannot be skipped.

It is well known that if Ψ is quasi-concave, then there is an increasing concave function $\widetilde{\Psi}$ on I such that $\Psi(t) \le \widetilde{\Psi}(t) \le 2\Psi(t)$ on I (cf. Proposition 5.10 in [4]). It is easy to show that $\| \|_{M_{\widetilde{\Psi}}} \approx \| \|_{M_\Psi}$. So we can obtain an equivalent norm on M_Ψ, which is induced by an increasing concave function on I.

Theorem 4.5. *If Ψ satisfies (4.1) in the case when $I = (0, 1)$ (resp. (4.2) and $\inf_{t>0} \Psi(t) = 0$ in the case when $I = (0, \infty)$), then M_Ψ is the bidual of M_Ψ^0.*

Proof. Assume first that Ψ is concave. By conditions (4.1) (resp. (4.2)), M_Ψ^0 is the set of all order continuous elements of M_Ψ and contains all characteristic functions with support of finite measure. It follows that $(M_\Psi^0)^* = (M_\Psi)'$, where $(M_\Psi)'$ is the associate space of M_Ψ.

If $\|f\|_{M_\Psi} \le 1$, then for all $t > 0$,

$$\int_0^t f^* \le \Psi(t).$$

Take a simple function $g* = \sum_{i=1}^n a_i\chi_{(0,t_i]}$, where $0 < t_1 < \cdots < t_n$, and $a_i \ge 0$. Then

$$\int_I g^* f^* \le \sum_{i=1}^n a_i\Psi(t_i) = \int_I g^* d\Psi,$$

where the Lebesgue-Stieltjes integral is well-defined since Ψ is continuous on $[0, \infty)$. By the Fatou property for all g in L^0,

$$\|g\|_{(M_\Psi)'} \le \int_I g^* d\Psi.$$

Since $\Psi(t)$ is a continuous concave function on I, there is an integral representation

$$\Psi(t) = \int_0^t h^*(s)ds,$$

on I for some h in L^0 [4]. Then $\|h\|_{M_\Psi} \le 1$, and for nay $g \in L^0$,

$$\int_I h^* g^* = \int_I g^* d\Psi.$$

So we get the reverse inequality

$$\|g\|_{(M_\Psi)'} \geq \int_I g^* d\Psi.$$

Therefore the associate space

$$(M_\Psi)' = \left\{ g \in L^0 : \int_I g^* d\Psi < \infty \right\},$$

which is a Lorentz space, must be order continuous [11]. In general, if Ψ is not concave then $\| \ \|_{M_\Psi} \approx \| \ \|_{M_{\tilde{\Psi}}}$, and hence $\| \ \|_{(M_\Psi)'} \approx \| \ \|_{(M_{\tilde{\Psi}})'}$. Since $(M_{\tilde{\Psi}})'$ is order continuous, $(M_\Psi)'$ is order continuous too. Then order continuity of $(M_\Psi)'$ implies $(M_\Psi^0)^{**} = (M_\Psi)'^* = (M_\Psi)'' = M_\Psi$, by the Fatou property of $\| \ \|_{M_\Psi}$. This completes the proof. □

Notice that the assumption $\inf_{t>0} \Psi(t) = 0$ cannot be skipped in the above theorem (cf. Remark 4.10).

Now let's turn our attention to spaces

$$\Sigma = L^1 + L^\infty \quad \text{and} \quad \Delta = L^1 \cap L^\infty,$$

on $I = (0, \infty)$. They are equipped with the following norms.

$$\|f\|_\Sigma = \inf\{\|g\|_1 + \|h\|_\infty : f = g + h, g \in L^1, h \in L^\infty\} = \int_0^1 f^*, \tag{4.3}$$

$$|||f|||_\Sigma = \inf\{\max\{\|g\|_1, \|h\|_\infty\} : f = g + h, g \in L^1, h \in L^\infty\},$$

$$\|f\|_\Delta = \max\{\|f\|_1, \|f\|_\infty\},$$

$$|||f|||_\Delta = \|f\|_1 + \|f\|_\infty.$$

It is obvious that $\| \ \|$ and $||| \ |||$ are equivalent. The equality in (4.3) is well known and can be found e.g. in [4]. It is also well known [8] that $(\Sigma, \| \ \|_\Sigma)' = (\Delta, \| \ \|_\Delta)$ and $(\Sigma, ||| \ |||_\Sigma)' = (\Delta, ||| \ |||_\Delta)$. Moreover,

$$\Sigma_0 = \{f \in \Sigma : \lim_{t\to\infty} f^*(t) = 0\},$$

where Σ_0 is a subspace of all order continuous elements of Σ (cf. [4, 11]).

It appears that for certain choice of Ψ, the Marcinkiewicz space M_Ψ coincides with Σ, and M_Ψ^0 with Σ_0. In fact we have the following result.

Proposition 4.6. *The norms $\| \ \|_{M_\Psi}$ and $\| \ \|_\Sigma$ are equal if and only if for $t > 0$*

$$\Psi(0) = 0 \quad \text{and} \quad \Psi(t) = \max\{t, 1\},$$

and they are equivalent if and only if for $t > 0$

$$\Psi(0) = 0 \quad \text{and} \quad \Psi(t) \approx \max\{t, 1\}.$$

Consequently if $I = (0, \infty)$ and $\lim_{t\to 0+} \Psi(t) = \alpha > 0$ and $\lim_{t\to\infty} \Psi(t)/t = \beta > 0$ then the spaces M_Ψ^0 and Σ_0 coincide as sets with equivalent norms.

Proof. If $\| \ \|_{M_\Psi}$ and $\| \ \|_\Sigma$ are equal, then for $t > 0$,

$$\|\chi_{(0,t)}\|_\Psi = \frac{t}{\Psi(t)} = \|\chi_{(0,t)}\|_\Sigma = \min\{t, 1\}.$$

Hence $\Psi(t) = \max\{t, 1\}$, for $t > 0$. Conversely suppose that $\Psi(t) = \max\{t, 1\}$ for $t > 0$. Then

$$\sup_{t>0} \frac{\int_0^t f^*}{\max\{t, 1\}} = \max\left\{ \sup_{0<t\leq 1} \int_0^t f^*, \sup_{t>1} \frac{1}{t} \int_0^t f^* \right\} = \int_0^1 f^*,$$

which shows that the two norms are equal. The similar calculation shows the condition for the equivalence of the norms. □

Let $\| \ \|$ be an equivalent norm to $\| \ \|_\Sigma$ or to $\|\!| \ \|\!|_\Sigma$. Then it is not difficult to see that ℓ_1 is isomorphically embedded in $(\Sigma_0, \| \ \|)$. Therefore (see Theorem 1.1) $(\Sigma_0, \| \ \|)$ is not an M-embedded space.

In the next two propositions we calculate the exact norms of the duals $(\Sigma, \| \ \|_\Sigma)^*$ and $(\Sigma, \|\!| \ \|\!|_\Sigma)^*$, which provide the answer to the question when Σ_0 is an M-ideal in Σ. In the sequel $\| \ \|_1$ and $\| \ \|_\infty$ will denote the norms in L^1 or L^∞, respectively.

Proposition 4.7. *The following equalities hold true.*

$$(\Sigma, \| \ \|_\Sigma)^* = \Sigma_0^* \oplus \Sigma_0^\perp \simeq (\Delta, \| \ \|_\Delta) \oplus \Sigma_0^\perp.$$

Moreover for any $F \in \Sigma^*$,

$$F = F_1 + F_2,$$

with $F_2 \in \Sigma_0^\perp$ and

$$F_1(g) = \int g f_1,$$

for some $f_1 \in (\Delta, \| \ \|_\Delta)$, and

$$\|F\| = \max\{\|f_1\|_\infty, \|f_1\|_1 + \|F_2\|\}.$$

Consequently, Σ_0 is not an M-ideal of $(\Sigma, \| \ \|_\Sigma)$.

Proof. The equalities $(\Sigma, \| \ \|_\Sigma)^* = \Sigma_0^* \oplus \Sigma_0^\perp \simeq (\Delta, \| \ \|_\Delta) \oplus \Sigma_0^\perp$ up to equivalence in norms is a consequence of the well known results on duals in Banach function spaces (cf. Theorem 102.6, Theorem 102.7 in [13]).

Now let $F \in \Sigma^*$ and let $\tilde{F}_1 = F|_{\Sigma_0}$. Then there is $f_1 \in \Sigma_0^* = \Sigma'$ such that $\tilde{F}_1(g) = \int f_1 g$ for all $g \in \Sigma_0$ and $\|\tilde{F}_1\| = \|f_1\|_{\Sigma'} = \|f_1\|_\Delta$. Then define $F_1(g) = \int f_1 g$ for all $g \in \Sigma$, and let $F_2 = F - F_1$. Then $F_2|_{\Sigma_0} = 0$ and $\|\tilde{F}_1\| = \|F_1\|$.

For each $f = g + h$ with $g \in L^1$ and $h \in L^\infty$, we have $F_2(g) = 0$, and so

$$\begin{aligned}
|F(g+h)| &\le \left|\int f_1 g\right| + \left|\int f_1 h\right| + |F_2(h)| \\
&\le \|f_1\|_\infty \|g\|_1 + \|f_1\|_1 \|h\|_\infty + \|F_2\| \|h\|_\Sigma \\
&\le \|f_1\|_\infty \|g\|_1 + (\|f_1\|_1 + \|F_2\|)\|h\|_\infty \\
&\le (\|g\|_1 + \|h\|_\infty) \max\{\|f_1\|_\infty, \|f_1\|_1 + \|F_2\|\}
\end{aligned}$$

Therefore, $\|F\| \le \max\{\|f_1\|_\infty, \|f_1\|_1 + \|F_2\|\}$.

Conversely, given $\epsilon > 0$ there exist $g \in L^1$, $h \in L^\infty$ such that $\|g\|_1 + \|h\|_\infty \le 1 + \epsilon$ and $\|F_2\| \le \operatorname{Re} F_2(h) + \epsilon$. For each $N \ge 1$, Let $f = \operatorname{sign}(f_1)\chi_{[0,N)} + h\chi_{[N,\infty)}$. Then $|f| = \chi_{[0,N)} + |h|\chi_{[N,\infty)}$, and so $\|f\|_\Sigma = \int_0^1 f^* \le 1 + \epsilon$. Thus

$$\begin{aligned}
\operatorname{Re} F(f) &= \int_0^N |f_1| + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \operatorname{Re} F_2(\operatorname{sign}(f_1)\chi_{[0,N)} + h\chi_{[N,\infty)}) \\
&= \int_0^N |f_1| + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \operatorname{Re} F_2(h) \\
&\ge \int_0^N |f_1| + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \|F_2\| - \epsilon
\end{aligned}$$

Therefore

$$\|F\| \ge \frac{1}{1+\epsilon}(\|F_2\| - \epsilon + \operatorname{Re}\Big(\int_N^\infty f_1 h\Big) + \int_0^N |f_1|)$$

for all $\epsilon > 0$ and all $N \ge 1$. Since $\int_N^\infty f_1 h \to 0$ as $N \to \infty$, so $\|F\| \ge \|F_2\| + \|f_1\|_1$. Clearly, $\|F\| \ge \|\tilde{F}_1\| = \|f_1\|_\Delta \ge \|f_1\|_\infty$. Hence $\|F\| = \max\{\|f\|_\infty, \|f_1\|_1 + \|F_2\|\}$.

Now suppose that Σ_0 is an M-ideal of Σ. Then there is a projection $P : \Sigma^* \to \Sigma^*$ such that the range of P is $\Sigma_0^\perp$ and for each $F \in \Sigma^*$, $\|F\| = \|PF\| + \|(I-P)F\|$. Note that $PF = F_2$ and $(I-P)F = F_1$ so that we can choose $f_1 = \chi_{[0,1/2)}$ and F_2 with $\|F_2\| = 1$. Then by the above calculations $\|F\| = 3/2$. But on the other hand we must have $\|F\| = \|PF\| + \|(I-P)F\| = \|F_2\| + \|f_1\|_\Delta = 2$, which is a contradiction. $\square$

Remark 4.8. By Proposition 4.6, $(\Sigma, \|\ \|_\Sigma) = M_\Psi$, where $\Psi(t) = \max\{t, 1\}$, $t > 0$. Thus $\inf_{t>0} \Psi(t)/t = 1$, and so the assumption in Theorem 4.4 is not satisfied. Since Σ_0 is not an M-ideal in $(\Sigma, \|\ \|_\Sigma)$, we see that the assumption $\inf_{t>0} \Psi(t)/t = 0$ cannot be omitted in Theorem 4.4.

The following proposition shows that if we use another equivalent norm in Σ, the M-ideal properties are remarkably changed.

Proposition 4.9. *The following equalities are satisfied*

$$(\Sigma, \|\!|\ \|\!|_\Sigma)^* = \Sigma_0^* \oplus \Sigma_0^\perp = (\Delta, \|\!|\ \|\!|_\Delta) \oplus_1 \Sigma_0^\perp .$$

Moreover for $F \in \Sigma^*$,

$$F = F_1 + F_2,$$

where $F_2 \in \Sigma_0^\perp$ *and*

$$F_1(g) = \int g f_1,$$

for some $f_1 \in (\Delta, \|\!|\ \|\!|_\Delta)$, *and*

$$\|F\| = \|F_1\| + \|F_2\| = \|f_1\|_\infty + \|f_1\|_1 + \|F_2\| .$$

Therefore Σ_0 *is an* M*-ideal of* $(\Sigma, \|\!|\ \|\!|_\Sigma)$.

Proof. By the same method as in the proof of the previous proposition, we can get a decomposition $F = F_1 + F_2$ with $F_2|_{\Sigma_0} = 0$, $F_1(g) = \int f_1 g$ for all $g \in \Sigma$, and $\|F_1\| = \|\!|f_1\|\!|_\Delta$.

For each $f = g + h \in \Sigma$ with $g \in L^1$ and $h \in L^\infty$,

$$\begin{aligned}
|F(g+h)| &\le \left| \int f_1(g+h) \right| + |F_2(h)| \\
&\le (\|f\|_1 + \|f_1\|_\infty) \max\{\|g\|_1, \|h\|_\infty\} + \|F_2\| \, \|\!|h\|\!|_\Sigma \\
&\le (\|f\|_1 + \|f_1\|_\infty) \max\{\|g\|_1, \|h\|_\infty\} + \|F_2\| \|h\|_\infty \\
&\le \max\{\|g\|_1, \|h\|_\infty\} (\|f_1\|_\infty + \|f_1\|_1 + \|F_2\|)
\end{aligned}$$

Therefore $\|F\| \le \|f_1\|_\infty + \|f_1\|_1 + \|F_2\|$.

Conversely suppose that $\|f_1\|_\infty \neq 0$. For large enough $n \in \mathbb{N}$, choose $E_n \subset \{|f_1| > \|f_1\|_\infty - 1/n\}$ with $0 < \mu E_n < \infty$. Let

$$g_n = \operatorname{sign}(f_1) \frac{\chi_{E_n}}{\mu E_n}.$$

Given $\epsilon > 0$, choose $g \in L^1$ and $h \in L^\infty$ so that $\max\{\|g\|_1, \|h\|_\infty\} \le 1 + \epsilon$ and $\|F_2\| \le \operatorname{Re} F_2(h) + \epsilon$. Let

$$h_n = h\chi_{[n,\infty)} + \operatorname{sign}(f_1)\chi_{[0,n)}.$$

Then $\|h_n\|_\infty \le 1 + \epsilon$ and $\|g_n\|_1 \le 1$. Hence $f_n = g_n + h_n$ we have $\|\!|f_n\|\!|_\Sigma \le 1 + \epsilon$. Consequently

$$\begin{aligned}
\operatorname{Re} F(f_n) &= \operatorname{Re} \int f_1 g_n + \operatorname{Re} \int f_1 h_n + \operatorname{Re} F_2(h_n) \\
&= \int_{E_n} \frac{|f_1|}{\mu E_n} + \int_0^n |f_1| + \operatorname{Re} \int_n^\infty f_1 h + \operatorname{Re} F_2(h_n - \operatorname{sign}(f_1)\chi_{[0,n)} + h\chi_{[0,n)}) \\
&\ge \|f\|_\infty - 1/n + \int_0^n |f_1| + \operatorname{Re} \int_n^\infty f_1 h + \operatorname{Re} F_2(h) \\
&\ge \|f\|_\infty - 1/n + \int_0^n |f_1| + \operatorname{Re} \int_n^\infty f_1 h + \|F_2\| - \epsilon.
\end{aligned}$$

Therefore $\|F\| \ge \frac{1}{1+\epsilon}(\|f\|_\infty - \frac{1}{n} + \int_0^n |f_1| + \operatorname{Re} \int_n^\infty f_1 h + \|F_2\| - \epsilon)$. Note that h is independent of n. Since $\lim_{n\to\infty} \int_n^\infty f_1 h = 0$ and ϵ is arbitrary we obtain $\|F\| \ge \|f_1\|_\infty + \|f_1\|_1 + \|F_2\|$, and this completes the proof. $\square$

Remark 4.10. Note that we have the following equalities (with equivalence of norms)

$$\Sigma_0^{**} \simeq (\Sigma')^* = \Delta^* \simeq \Delta' \oplus \Delta_s^* = \Sigma \oplus \Delta_s^*,$$

where $\Delta_s^* \neq \{0\}$ since Δ is not order continuous. Thus the bidual of $\Sigma_0 = M_\Psi^0$ with $\Psi(t) = \max\{t, 1\}$, $t > 0$, is not equal to $\Sigma = M_\Psi$. It shows that the assumption $\inf_{t>0} \Psi(t) = 0$ in Theorem 4.5 cannot be omitted.

It is also interesting to observe that if we define Δ_b as the closure of all simple functions with support of finite measure in $(\Delta, \| \,\|_\Delta)$, then Δ_b also contains an isomorphic copy of ℓ_1 which has a non-separable dual. Therefore Δ_b with any equivalent norm to $\| \,\|_\Delta$ is not M-embedded.

5. M-ideal properties of Marcinkiewicz sequence spaces

In this section we will consider Marcinkiewicz sequence spaces. Assume further that $\Psi = \{\Psi(n)\} = \{\Psi(n)\}_{n=0}^\infty$ is a sequence such that $\Psi(0) = 0$, $\{\Psi(n)\}$ is increasing, $\Psi(n) > 0$ for $n > 0$ and $\{\Psi(n)/n\}$ is decreasing.

Definition 5.1. By analogy to the function spaces, the *Marcinkiewicz sequence space* m_Ψ consists of all sequences $x = \{x(n)\} = \{x(n)\}_{n=1}^\infty$ such that

$$\|x\| = \|x\|_{m_\Psi} = \sup_{n\geq 1} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)},$$

where $x^* = \{x^*(n)\}$ is a decreasing rearrangement of $\{x(n)\}$.

Similarly define m_Ψ^0 as a subspace of m_Ψ consisting of all $x \in m_\Psi$ satisfying

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} = 0.$$

Notice that reasoning analogously as in the previous section for function spaces, the assumption that $\{\Psi(n)/n\}$ is a decreasing sequence is not a real restriction.

We have the following basic facts about m_Ψ and m_Ψ^0.

Theorem 5.2. (1) *m_Ψ is a r.i. Banach function space with the Fatou property.*

(2) *$m_\Psi^0 \neq \{0\}$ if and only if $\lim_{n\to\infty} \Psi(n) = \infty$.*

(3) *If $\lim_{n\to\infty} \Psi(n) = \infty$, then m_Ψ^0 is a non-trivial subspace of all order continuous elements of m_Ψ.*

(4) *The following conditions are equivalent.*

   (*a*) *$\|x\|_{m_\Psi} = \|x\|_\infty$ for all $x \in \ell_\infty$ (resp. $\|x\|_{m_\Psi} \approx \|x\|_\infty$ for all $x \in \ell_\infty$).*

   (*b*) *$\|x\|_{m_\Psi} = \|x\|_\infty$ for all $x \in c_0$ (resp. $\|x\|_{m_\Psi} \approx \|x\|_\infty$ for all $x \in c_0$).*

   (*c*) *$\Psi(n) = n$ for all $n \in \mathbb{N}$ (resp. $\Psi(n) \approx n$ for all $n \in \mathbb{N}$).*

Proof. Condition (1) is immediate and (2) is clear if we note that $e_1 \in m_\Psi^0$ is equivalent to $\lim_{n\to\infty} 1/\Psi(n) = 0$. For (3), note that m_Ψ^0 contains all characteristic functions with support of finite measure by (2), so it contains all order continuous elements [4]. The proof that any $x \in m_\Psi^0$ is order continuous is very similar to the function case, so we omit it. Finally we shall prove that $4(a)$ is equivalent to $4(c)$. Let's assume first that two norms are equal. Then for $n \in \mathbb{N}$,

$$\|e_1 + \cdots + e_n\|_{m_\Psi} = \frac{n}{\Psi(n)} = 1.$$

For the converse, if we assume $\Psi(n) = n$ for $n \in \mathbb{N}$, then for any $x \in \ell_\infty$,

$$\|x\|_\infty = x^*(1) = \sup_{n\geq 1} \frac{1}{n} \sum_{k=1}^n x^*(k) = \|x\|_{m_\Psi}.$$

The remaining equivalences can be proved in a similar way. $\square$

Given the sequence $\{\Psi(n)\}$ define the function $\Psi(t) = \sum_{i=0}^\infty \Psi(i)\chi_{[i,i+1)}(t)$ on $[0,\infty)$. Obviously $\Psi|_{\mathbb{N}\cup\{0\}}$ coincides with $\{\Psi(n)\}$. The following result we shall use further.

Lemma 5.3. *There is a concave continuous function $\widetilde{\Psi}$ on $[0,\infty)$ such that $\Psi \leq \widetilde{\Psi} \leq 3\Psi$ on $[1,\infty)$ and $\widetilde{\Psi}(0) = 0$.*

Proof. Fix $x \geq 1$. For $0 < t \leq x$,

$$\frac{\Psi(t)}{t} \leq \frac{\Psi(x)}{t},$$

and for $[x] \leq [t]$,

$$\frac{\Psi(t)}{t} \leq \frac{\Psi([t])}{[t]} \leq \frac{\Psi([x])}{[x]} = \frac{x}{[x]}\frac{\Psi(x)}{x} \leq 2\frac{\Psi(x)}{x},$$

where for real $y \in \mathbb{R}$, $[y]$ is the greatest integer less than or equal to y. Hence for every $t \geq 0$ and $x \geq 1$,

$$\Psi(t) \leq (1 + \frac{2t}{x})\Psi(x) \quad \text{and} \quad \Psi(t) \leq t\Psi(1).$$

Therefore there is a minimal concave function $\widetilde{\Psi}$ such that for each $t \geq 0$, $x \geq 1$,

$$\Psi(t) \leq \widetilde{\Psi}(t) \leq \ \min\{(1 + \frac{2t}{x})\Psi(x), t\Psi(1)\}.$$

Then for every $x \geq 1$ and $t > 0$,

$$\widetilde{\Psi}(x) \leq (1 + \frac{2x}{x})\Psi(x) = 3\Psi(x) \quad \text{and} \quad \widetilde{\Psi}(t) \leq t\Psi(1).$$

So $\lim_{t\to 0^+} \widetilde{\Psi}(t) = 0$. Therefore $\widetilde{\Psi}(t)$ is a continuous concave function on $[0, \infty)$. □

Now, we are ready to investigate when m_Ψ is the bidual of m_Ψ^0 and when m_Ψ^0 is an M-ideal of m_Ψ. The following theorems show that the situation is simpler than that of the non-atomic case.

Theorem 5.4. *The space m_Ψ is the bidual of m_Ψ^0 if and only if $\lim_{n\to\infty} \Psi(n) = \infty$.*

Proof. If $\lim_{n\to\infty} \Psi(n) < \infty$, then by Theorem 5.2 (2), $m_\Psi^0 = \{0\}$. So m_Ψ cannot be the bidual of m_Ψ^0 since $m_\Psi \neq \{0\}$.

For the converse, suppose that $\lim_{n\to\infty} \Psi(n) = \infty$. Then by Theorem 5.2 (2) and (3), m_Ψ^0 is the order continuous subspace of m_Ψ and it contains all simple functions with support of finite measure. Hence $(m_\Psi^0)^* \simeq (m_\Psi)'$. So if we show that $(m_\Psi)'$ is order continuous, then $(m_\Psi^0)^{**} \simeq ((m_\Psi)')^* \simeq (m_\Psi)'' = m_\Psi$, and the proof is done.

Note that by Lemma 5.3, there is an equivalent norm induced by the concave function $\widetilde{\Psi}$, that is

$$\|x\|_{m_{\widetilde{\Psi}}} = \sup_{n\geq 1} \frac{\sum_{k=1}^n x^*(k)}{\widetilde{\Psi}(n)}.$$

If $\|x\|_{m_{\widetilde{\Psi}}} \leq 1$, then

$$\sum_{k=1}^n x^*(k) \leq \widetilde{\Psi}(n),$$

for all $n \geq 1$. For any decreasing sequence

$$y^* = (y^*(1), \cdots, y^*(n), 0, \cdots),$$

the summation by parts shows that

$$\sum_{k=1}^n x^*(k)y^*(k) \leq \sum_{k=1}^n y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)).$$

Then by the Fatou property, for any $y = \{y(k)\}$,

$$\|y\|_{(m_{\widetilde{\Psi}})'} \leq \sum_{k=1}^\infty y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)).$$

Note that there is an integral representation $\widetilde{\Psi}(t) = \int_0^t h^*(s)ds$ for some $h \in L^0$. This shows that, if we take $x(k) = \widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)$ for all $k \in \mathbb{N}$, then the sequence $\{x(k)\}$ is decreasing and for each $n \in \mathbb{N}$,

$$\frac{\sum_{k=1}^n x^*(k)}{\widetilde{\Psi}(n)} = \frac{\widetilde{\Psi}(n)}{\widetilde{\Psi}(n)} = 1.$$

This means that $\|x\| = 1$ and for all y,

$$\sum_{k=1}^{\infty} x^*(k)y^*(k) = \sum_{k=1}^{\infty} y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)).$$

Hence

$$\|y\|_{(m_{\widetilde{\Psi}})'} \geq \sum_{k=1}^{\infty} y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)),$$

for all y. Therefore we obtain the following formula

$$\|y\|_{(m_{\widetilde{\Psi}})'} = \sum_{k=1}^{\infty} y^*(k)(\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1))$$

and this implies that $(m_{\widetilde{\Psi}})'$ and hence $(m_{\Psi})'$ is order continuous [11]. $\square$

In view of Theorem 5.2 (4), if $\Psi(n) = n$, then $m_{\Psi}^0 = c_0$ and $m_{\psi} = \ell_{\infty}$ with equality of norms, and thus m_{Ψ}^0 is an M-ideal of m_{Ψ} [7]. The next theorem extends this result to a broader class of functions Ψ and improves already existing results in certain class of m_{Ψ} (cf. [7]).

Theorem 5.5. *Assume that* $\lim_{n\to\infty} \frac{\Psi(n)}{n} = 0$ *and* $\lim_{n\to\infty} \Psi(n) = \infty$. *Then* m_{Ψ}^0 *is an M-ideal in its bidual* m_{Ψ}.

Proof. First observe that if $x \in m_{\Psi}$, then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} = \limsup_{n\to\infty} \frac{\frac{1}{n}\sum_{k=1}^n x^*(k)}{\frac{1}{n}\Psi(n)} \leq \sup_n \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} < \infty,$$

and in view of the assumption $\lim_{n\to\infty} \frac{\Psi(n)}{n} = 0$,

$$\lim_{n\to\infty} x^*(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x^*(k) = 0.$$

In the proof we shall use the 3-ball property (cf. Theorem 1.1) and the same technique as in [7], that is we show that for every $x = \{a(n)\} \in B_{m_{\Psi}}$, every $x_i = \{x_i(n)\} \in B_{m_{\Psi}^0}$ with finite support, $i = 1, 2, 3$, and $\epsilon > 0$ there is $y \in m_{\Psi}^0$ such that $\|x + x_i - y\| \leq 1 + \epsilon$, $i = 1, 2, 3$. First assume that for all $i = 1, 2, 3$,

$$\max\{j : x_i^*(j) \neq 0\} =: k_i = k,$$

and

$$\sum_{j=1}^{k} x_i^*(j) \leq \sum_{j=1}^{k} a^*(j).$$

Next pick up N such that for all $n \geq N$, $x_i(n) = 0$ and

$$|a(n)| \leq \min\{\delta, a^*(k)\},$$

where $\delta = \min_i x_i^*(k)$. Then define the sequence $y = \{y(n)\}$ by $y(n) = a(n)$ if $n \leq N$ and $y(n) = 0$ otherwise. If $z_i(n) = a(n) + x_i(n) - y(n)$, then $z_i^*(j) = x_i^*(j)$ for $j \leq k$ and $z_i^*(j) \leq a^*(j)$ for $j > k$. Hence for $n \leq k$,

$$\frac{\sum_{j=1}^n z_i^*(j)}{\Psi(n)} \leq 1,$$

and for $n > k$,

$$\frac{\sum_{j=1}^n z_i^*(j)}{\Psi(n)} \leq \frac{\sum_{j=1}^n a^*(j)}{\Psi(n)} \leq 1.$$

Therefore $\|x + x_i - y\| \leq 1$.

In general case, we may assume that x is not an element of m_{Ψ}^0. In this case, we cannot have $x \in \ell_1$. Hence we can find $l \geq k_i$ for all $i = 1, 2, 3$, such that

$$\sum_{j=1}^{k_i} x_i^*(j) < \sum_{j=1}^{l} a^*(j).$$

Define ξ as follows: If $x_i(n) \neq 0$ then let $\xi_i(n) = x_i(n)$. At $l - k_i$ indices where $x_i(n) = 0$, let $\xi_i(n) = \alpha$ ($\alpha > 0$ is chosen later), otherwise let $\xi_i(n) = 0$. The number α should be chosen so small that for all $i = 1, 2, 3$, $\|x_i - \xi_i\| \leq \epsilon$ and

$$\sum_{j=1}^{n} \xi_i^*(j) \leq \sum_{j=1}^{l} a^*(j).$$

By the first part of the proof, there exists $y \in m_\Psi^0$ such that

$$\|x + \frac{\xi_i}{1+\epsilon} - y\| \leq 1.$$

Hence $\|x + x_i - y\| \leq 1 + 2\epsilon$, which completes the proof. □

Theorem 5.2 (4) shows that $\lim_{n\to\infty} \frac{\Psi(n)}{n} = \beta > 0$ if and only if $m_\Psi^0 = c_0$ up to equivalent norms. Therefore if $\lim_{n\to\infty} \frac{\Psi(n)}{n} = \beta > 0$, then m_Ψ can be renormed so that m_Ψ^0 is an M-ideal of its bidual m_Ψ, since c_0 is an M-ideal of ℓ_∞. But m_Ψ^0 with its original norm does not need to be an M-ideal of m_Ψ if we drop the assumption $\lim_{n\to\infty} \Psi(n)/n = 0$, as we can see in the following example.

Example 5.6. Let $\Psi(0) = 0$, $\Psi(n) = \max\{\frac{2n}{3}, 1\}$ for $n \in \mathbb{N}$. Then $m_\Psi = \ell_\infty$ with norm

$$\|x\|_\Psi = \sup\left\{x^*(1), \frac{3(x^*(1) + x^*(2))}{4}, \cdots, \frac{3\sum_{k=1}^n x^*(k)}{2n}, \cdots\right\}$$

that is equivalent to $\| \ \|_\infty$-norm. Then $(c_0, \| \ \|_\Psi)$ is not an M-ideal of $(\ell_\infty, \| \ \|_\Psi)$.

Proof. Let $x_1 = e_1 + \frac{1}{3}e_2$, $x_2 = e_1 - \frac{1}{3}e_2$, $x_3 = -e_1 + \frac{1}{3}e_2$, and let $x \equiv 2/3$. Note that $\|x_i\| = \|x\| = 1$. Then there is no $y \in c_0$ such that $\|x_i + x - y\|_\Psi < \frac{5}{4}$. Observe the following formulas for any $y \in c_0$,

$$|x_1 + x - y| = (|5/3 - y(1)|, |1 - y(2)|, |2/3 - y(3)|, \ldots),$$
$$|x_2 + x - y| = (|5/3 - y(1)|, |0 - y(2)|, |2/3 - y(3)|, \ldots),$$
$$|x_3 + x - y| = (|1/3 + y(1)|, |1 - y(2)|, |2/3 - y(3)|, \ldots).$$

Then $\max\{|5/3 - y(1)|, |1/3 + y(1)|\} \geq 1$ for all scalars $y(1)$. Therefore for each $y \in c_0$ there is i such that $(x_i + x - y)^*(1) \geq 1$ and note that $\lim_{n\to\infty} |2/3 - y(n)| = 2/3$, so that $(x_i + x - y)^*(2) \geq 2/3$ for all $i = 1, 2, 3$. This means that for every $y \in c_0$ there is some i such that $\|x_i + x - y\|_\Psi \geq 3/4(1 + 2/3) = 5/4$. This completes the proof. □

This example shows that we cannot omit the additional conditions in Theorem 5.5.

6. Polynomials on Marcinkiewicz sequence spaces

This section is a continuation of section 3 in the case of Marcinkiewicz sequence spaces. Let $\Psi = \{\Psi(n)\} = \{\Psi(n)\}_{n=0}^\infty$ be like in section 5, that is $\Psi(0) = 0$, Ψ is increasing, $\Psi(n) > 0$ for $n > 0$ and $\{\Psi(n)/n\}$ is decreasing. Note first that if $\lim_{n\to\infty} \Psi(n) = \infty$, then m_Ψ^0 is a non-trivial proper ideal of m_Ψ. Indeed, for $\widetilde{\Psi}$ from Lemma 5.3,

$$\{\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)\}_{k\geq 1} \in m_{\widetilde{\Psi}} = m_\Psi \quad \text{but} \quad \{\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)\}_{k\geq 1} \notin m_{\widetilde{\Psi}}^0 = m_\Psi^0.$$

Notice that $\{\widetilde{\Psi}(k) - \widetilde{\Psi}(k-1)\}_{k\geq 1}$ is a decreasing sequence, since $\widetilde{\Psi}$ is concave. In section 5 we also showed that if in addition Ψ satisfies one of the conditions

$$\Psi(n) = n \quad \text{or} \quad \lim_{n\to\infty} \frac{\Psi(n)}{n} = 0,$$

then m_Ψ^0 is an M-ideal of its bidual m_Ψ. As we know, this implies the uniqueness of the Hahn-Banach extension of bounded linear functionals from m_Ψ^0 to m_Ψ [7].

On the other hand, the M-ideal property does not affect too much the uniqueness of n-homogeneous polynomial norm-preserving extension when $n \geq 2$. In section 2, we showed that in real case, for every $n \geq 2$, we could construct an n-homogeneous polynomial on m_Ψ^0 which had two different norm-preserving extensions to m_Ψ, and in complex case, we could find an n-homogeneous polynomial with two distinct norm-preserving extensions if $n \geq 3$. In the following lemma, we state the conditions when m_Ψ^0 satisfies the assumptions of Theorem 3.2. This in turn gives interesting conclusions about norm-preserving extension to m_Ψ of norm-attaining 2-homogeneous polynomials on m_Ψ^0.

Lemma 6.1. *Assume that $\lim_{n\to\infty}\Psi(n)=\infty$ and $\{\Psi(n)\}$ is strictly increasing. Then for each $x\in B_{m_\Psi^0}$, there exist $n\in\mathbb{N}$ and $\epsilon>0$ such that for each $y\in B_{m_\Psi}$, $y=(0,\cdots,0,y(n+1),y(n+2),\cdots)$, and for each $|\lambda|\le\epsilon$, $\|x+\lambda y\|\le 1$ holds.*

Proof. We may assume that $\|x\|=1$. Since $\lim_{k\to\infty}\frac{\sum_{i=1}^k x^*(i)}{\Psi(k)}=0$, we can find the maximum integer $n_1\in\mathbb{N}$ such that

$$\|x\|=1=\frac{\sum_{i=1}^{n_1}x^*(i)}{\Psi(n_1)}.$$

Thus for every $k\ge n_1+1$,

$$\sum_{i=1}^{n_1}x^*(i)=\Psi(n_1)\quad\text{and}\quad\sum_{i=1}^{k}x^*(i)<\Psi(k).$$

Take

$$a=1-\max\left\{\frac{\sum_{i=1}^k x^*(i)}{\Psi(k)}:k\ge n_1+1\right\}>0.$$

We note that $x^*(n_1)\ne 0$. Indeed, if we suppose that $x^*(n_1)=0$, then

$$\sum_{i=1}^{n_1}x^*(i)=\sum_{i=1}^{n_1-1}x^*(i)=\Psi(n_1)\le\Psi(n_1-1),$$

which is a contradiction to the fact that Ψ is strictly increasing.

Note that for $x\in m_\Psi^0$,

$$\lim_{n\to\infty}\frac{\sum_{k=1}^n x^*(k)}{\Psi(n)}=\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{k=1}^n x^*(k)}{\frac{1}{n}\Psi(n)}=0,$$

which yields

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n x^*(k)=\lim_{n\to\infty}x^*(n)=\lim_{i\to\infty}|x(i)|=0.$$

Thus we can choose $n>n_1$ so that for all $i\ge n+1$,

$$|x(i)|<\frac{1}{2}x^*(n_1).$$

Take $\epsilon=\min\{\frac{x^*(n_1)\|e_1\|}{2},a\}>0$ and let $y=(0,\cdots,0,y(n+1),y(n+2),\cdots)\in B_{m_\Psi}$. Fix λ with $|\lambda|<\epsilon$. Then for $i\ge n+1$, $\|e_i\|\,|y(i)|\le 1$ and so

$$|x(i)+\lambda y(i)|<\frac{x^*(n_1)}{2}+\frac{x^*(n_1)|y(i)|\,\|e_1\|}{2}\le x^*(n_1).$$

Thus for each $k\le n_1$,

$$\sum_{i=1}^k(x+\lambda y)^*(i)=\sum_{i=1}^k x^*(i)\le\Psi(k),$$

and for each $k>n_1$,

$$\sum_{i=1}^k(x+\lambda y)^*(i)\le\sum_{i=1}^k x^*(i)+a\sum_{i=1}^k y^*(i)\le(1-a)\Psi(k)+a\Psi(k)=\Psi(k).$$

Therefore $\|x+\lambda y\|\le 1$ and the proof is completed. □

Lemma 6.1 and Theorem 3.2 imply the following result.

Theorem 6.2. *Let $\{\Psi(n)\}$ satisfy the assumptions of Lemma 6.1. Let P be a 2-homogeneous polynomial on complex m_Ψ. Then there exists $x_0\in B_{m_\Psi^0}$ such that $P(x_0)=\|P\|$ if and only if P is finite.*

So Corollaries 3.3 and 3.4 can be applied to m^0_Ψ. The next corollary is a generalization of the analogous result in [9] for the spaces m_Ψ with strictly concave Ψ.

Corollary 6.3. *Let $\{\Psi(n)\}$ satisfy the assumptions of Lemma 6.1. A 2-homogeneous polynomial on complex space m^0_Ψ attains its norm if and only if it is a finite polynomial. Furthermore it has a unique norm-preserving extension to its bidual m_Ψ.*

The next result on norm-attaining bounded linear functionals on m^0_Ψ, follows from Lemma 6.1 and Proposition 3.7.

Corollary 6.4. *Let $\{\Psi(n)\}$ satisfy the assumptions of Lemma 6.1. A bounded linear functional on complex space m^0_Ψ attains its norm if and only if it is a finite polynomial. Furthermore it has a unique norm-preserving extension to its bidual m_Ψ.*

In view of Theorem 5.2 and the preceding corollaries, we get the following result.

Corollary 6.5. *If $\lim_{n\to\infty}\Psi(n)/n > 0$, then $m^0_\Psi = c_0$ and $m_\Psi = \ell_\infty$ up to norm equivalence. Suppose that $\{\Psi(n)\}$ satisfies the assumptions of Lemma 6.1 and $\lim_{n\to\infty}\Psi(n)/n > 0$. Then every norm-attaining bounded linear functional on complex $(c_0, \|\ \|_{m_\Psi})$ is finite and has a unique extension to its bidual $(\ell_\infty, \|\ \|_{m_\Psi})$. Moreover, every 2-homogeneous norm-attaining polynomial on complex $(c_0, \|\ \|_{m_\Psi})$ is finite and has a unique extension to its bidual $(\ell_\infty, \|\ \|_{m_\Psi})$.*

Note that m^0_Ψ in the above corollary may not be an M-ideal of m_Ψ as we could see in Example 5.6. Moreover, not every renorming of c_0 and ℓ_∞ guarantees the hypothesis of Corollary 6.5. In fact, in Example 3.6 we constructed a non-symmetric norm $\|\ \|$ equivalent to $\|\ \|_\infty$ such that the last conclusion of Corollary 6.5 failed. However we can ask another question, whether or not, in c_0 equipped with an equivalent symmetric norm, every 2-homogeneous norm-attaining polynomial is finite and has a unique extension to its bidual ℓ_∞? But, as we see below, both answers are negative.

Example 6.6. Let $\Psi(0) = 0$, $\Psi(n) = \max\{n, 2\}$ for $n \in \mathbb{N}$. Then, by Theorem 5.2(4), $m_\Psi = \ell_\infty$ and $m^0_\Psi = c_0$ with norm $\|x\| = \frac{x^*(1)+x^*(2)}{2}$, which is equivalent to $\|\ \|_\infty$-norm. Consider 2-homogeneous polynomials on ℓ_∞,

$$P(x) = \frac{x(1)^2}{4} \quad \text{and} \quad Q(x) = \frac{x(1)^2}{4} + \frac{x(2)}{2}\sum_{k=2}^{\infty}\frac{x(2k-1)+x(2k)}{2^k}.$$

Clearly, P is a norm-attaining polynomial at $x = 2e_1$ and $\|P\| = 1$. Moreover, for every $x \in B_{(\ell_\infty, \|\ \|_{m_\Psi})}$,

$$\begin{aligned}
|Q(x)| &\le \left|\frac{x(1)}{2}\right|^2 + \left|\frac{x(2)}{2}\right|\sum_{k=2}^{\infty}\frac{|x(2k-1)|+|x(2k)|}{2^k}\\
&\le \frac{|x(1)|}{2} + \frac{|x(2)|}{2}\sum_{k=2}^{\infty}\frac{x^*(1)+x^*(2)}{2^k}\\
&\le \frac{|x(1)|+|x(2)|}{2} \le \frac{x^*(1)+x^*(2)}{2} \le 1.
\end{aligned}$$

Therefore Q is also norm-attaining at $2e_1 \in B_{(c_0, \|\ \|_{m_\Psi})}$ and $\|Q\| = 1$. But Q is not finite. Furthermore, choose a norm one linear functional φ on $(\ell_\infty, \|\ \|_{m_\Psi})$ which vanishes on c_0. Letting

$$P_1(x) = \frac{x(1)^2}{4} \quad \text{and} \quad P_2(x) = \frac{x(1)^2}{4} + \frac{x(2)}{2}\varphi(x),$$

we obtain two distinct norm-preserving extensions of P from c_0 to ℓ_∞.

So if Ψ is not strictly increasing we cannot, in general, obtain Lemma 6.1 and its consequences. Note also that m_Ψ is a symmetric space not satisfying the assumption (3.1) of Theorem 3.2.

Example 6.7. Let $\Psi(0) = 0$, $\Psi(n) = \max\{\sqrt{n}, 2\}$ for $n \in \mathbb{N}$. Then m^0_Ψ is an M-ideal of its bidual m_Ψ (see Theorem 5.5) with norm

$$\|x\| = \|x\|_\Psi = \max\left\{\max_{k\in\{1,2,3,4\}}\frac{\sum_{i=1}^k x^*(i)}{2},\ \sup_{k\ge 5}\frac{\sum_{i=1}^k x^*(i)}{\sqrt{k}}\right\}.$$

Then exactly the same P, Q, P_i, $i = 1,2$, as in the previous example, can be used to show that P_i, $i = 1,2$, are two distinct norm-preserving extensions of P from m_Ψ^0 to m_Ψ. Notice also that Q is norm-attaining on m_Ψ^0 but is not finite.

So even though m_Ψ^0 is an M-ideal of m_Ψ, we cannot obtain the result similar to Corollary 6.4 without the assumption (3.1) of Theorem 3.2.

We can see that the preceding examples are parts of the general situation.

Theorem 6.8. *Let $\lim_{n\to\infty} \Psi(n) = \infty$ and m_Ψ, m_Ψ^0 be complex spaces. The following conditions are equivalent.*

(1) *Ψ is strictly increasing.*
(2) *For each $x \in B_{m_\Psi^0}$, there are $n \in \mathbb{N}$ and $\epsilon > 0$ such that for every $y = (0,\cdots,0,y(n+1),\cdots) \in B_{m_\Psi}$ and for every $|\lambda| < \epsilon$, $\|x+\lambda y\| \le 1$.*
(3) *No element in $S_{m_\Psi^0}$ is a complex extreme point of $B_{m_\Psi^0}$.*
(4) *No element in $S_{m_\Psi^0}$ is a complex extreme point of B_{m_Ψ}.*
(5) *Every norm-attaining 2-homogeneous polynomial on m_Ψ^0 is finite.*
(6) *Every norm-attaining 2-homogeneous polynomial on m_Ψ^0 has a unique norm-preserving extension to m_Ψ.*
(7) *Every norm-attaining bounded linear functional on m_Ψ^0 is finite.*

Proof. By Lemma 6.1, (1) $\Rightarrow$ (2) holds and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear by definition.

Suppose for the rest of the proof that Ψ is not strictly increasing. Then there is $n \in \mathbb{N}$ such that $\Psi(n) = \Psi(n+1)$. Set

$$x_0 = \sum_{i=1}^{n} \frac{\Psi(n)}{n} e_i.$$

We know that $\frac{\Psi(n)}{n} \le \frac{\Psi(k)}{k}$ for each k, $1 \le k \le n$. This yields

$$\sup_{k\ge 1} \frac{\sum_{i=1}^k x_0^*(i)}{\Psi(k)} = \sup_{k\ge 1} \frac{k\Psi(n)}{n\Psi(k)} = 1.$$

So $x_0 \in S_{m_\Psi^0}$. We shall show that x_0 is a complex extreme point of B_{m_Ψ}. Suppose that there is $y \in m_\Psi$ such that $\|x_0 + \zeta y\| \le 1$ for all $|\zeta| \le 1$. Then

$$\frac{1}{\Psi(n)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| \le \sum_{i=1}^{n} \frac{(x_0+\zeta y)^*(i)}{\Psi(n)} \le 1, \quad \text{for all} \quad |\zeta| \le 1.$$

Consider the analytic function $f : B_{\mathbb{C}} \to \ell_1$, defined by

$$f(\zeta) = \frac{1}{\Psi(n)} \sum_{i=1}^{n} \left(\frac{\Psi(n)}{n} + \zeta y(i) \right) e_i.$$

Then $\|f(\zeta)\|_1$ has maximum 1 at $\zeta = 0$. Since S_{ℓ_1} consists entirely of complex extreme points, the strong form of the Maximum Modulus Theorem holds true (cf. Theorem 3.1 in [15]), and thus f is constant. Therefore $y(i) = 0$ for $1 \le i \le n$. For each $y(k)$, $k > n$,

$$\frac{1}{\Psi(n+1)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| + \frac{|\zeta y(k)|}{\Psi(n+1)} \le \sum_{i=1}^{n+1} \frac{(x_0+\zeta y)^*(i)}{\Psi(n+1)} \le 1, \quad \text{for all} \quad |\zeta| \le 1.$$

This implies that

$$\begin{aligned}\frac{1}{\Psi(n+1)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| + \frac{|\zeta y(k)|}{\Psi(n+1)} &= \frac{1}{\Psi(n)} \sum_{i=1}^{n} \left| \frac{\Psi(n)}{n} + \zeta y(i) \right| + \frac{|\zeta y(k)|}{\Psi(n)} \\ &= 1 + \frac{|\zeta y(k)|}{\Psi(n)} \le 1, \quad \text{for all} \quad |\zeta| \le 1.\end{aligned}$$

So we obtain $y(k) = 0$ for any $k > n$. Therefore $y = 0$ and x_0 is a complex extreme point of B_{m_Ψ}. Thus we showed the equivalence of (1), (2), (3) and (4). Now, let's take 2-homogeneous polynomials on m_Ψ

$$P(x) = \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2},$$

$$Q(x) = \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2} + \frac{x(n+1)}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x(k+n+1)}{\Psi(1)2^k}.$$

Observe that $P(x_0) = Q(x_0) = 1$. So P is a norm-attaining 2-homogeneous polynomial. We can see that Q is also norm-attaining. Indeed, for each $\|x\| \le 1$,

$$\begin{aligned}
|Q(x)| &\le \left(\frac{|x(1)| + \cdots + |x(n)|}{\Psi(n)}\right)^2 + \frac{|x(n+1)|}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x^*(1)}{2^k\Psi(1)} \\
&\le \frac{|x(1)| + \cdots + |x(n)|}{\Psi(n)} + \frac{|x(n+1)|}{\Psi(n)} \\
&\le \frac{x^*(1) + \cdots + x^*(n+1)}{\Psi(n+1)} \le 1,
\end{aligned}$$

in view of the assumption that $\Psi(n) = \Psi(n+1)$. Hence, we get a norm-attaining 2-homogeneous polynomial on m_Ψ^0 which is not finite. So (5) $\Rightarrow$ (1) is proved. Choose further a norm one linear functional ϕ on m_Ψ which vanishes on m_Ψ^0. Letting for $x \in m_\Psi$,

$$P_1(x) = \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2},$$

$$P_2(x) = \frac{(x(1) + \cdots + x(n))^2}{\Psi(n)^2} + \frac{x(n+1)}{\Psi(n+1)}\phi(x),$$

we can easily see that they are two distinct norm-preserving extensions of P to m_Ψ. This proves (6) $\Rightarrow$ (1). Finally, we will construct a norm-attaining bounded linear functional which is not finite. Define a linear functional φ on m_Ψ^0 as follows

$$\varphi(x) = \frac{x(1) + \cdots + x(n)}{\Psi(n)} + \frac{1}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x(n+k)}{2^k}.$$

Then $\varphi(x_0) = 1$, $\|\varphi\| = 1$, and φ is not finite. Indeed, for each $\|x\| \le 1$, by the Hardy-Littlewood inequality [4],

$$\begin{aligned}
|\varphi(x)| &\le \frac{x^*(1) + \cdots + x^*(n)}{\Psi(n)} + \frac{1}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x^*(n+k)}{2^k} \\
&\le \frac{x^*(1) + \cdots + x^*(n)}{\Psi(n)} + \frac{1}{\Psi(n)} \sum_{k=1}^{\infty} \frac{x^*(n+1)}{2^k} \\
&\le \frac{x^*(1) + \cdots + x^*(n) + x^*(n+1)}{\Psi(n+1)} \le 1.
\end{aligned}$$

This proves (7) $\Rightarrow$ (1).

In order to complete the proof we observe that (2) $\Rightarrow$ (5), (6) by Corollary 6.5 and that (2) $\Rightarrow$ (7) by Proposition 3.7. $\square$

Corollary 6.9. *Let* $\lim_{n\to\infty} \Psi(n) = \infty$ *and* m_Ψ, m_Ψ^0 *be real or complex spaces. Then both* m_Ψ^0 *and* m_Ψ *are not rotund.*

Proof. If Ψ is strictly increasing then the hypothesis is an immediate corollary of Lemma 6.1, which is valid for both real and complex spaces.

Suppose now that Ψ is not strictly increasing. Then there is $n \in \mathbb{N}$ such that $\Psi(n) = \Psi(n+1)$. Let

$$x = \sum_{i=1}^{n-1} ae_i + ae_n + be_{n+1},$$

$$y = \sum_{i=1}^{n-1} ae_i + be_n + ae_{n+1},$$

where $a > b > 0$. Then $x^* = y^* = x$, and

$$\|x\| = \|y\| = \max\left\{\frac{a}{\Psi(1)}, \frac{2a}{\Psi(2a)}, \cdots, \frac{an}{\Psi(n)}, \frac{an+b}{\Psi(n+1)}\right\} = \frac{an+b}{\Psi(n+1)} = \frac{an+b}{\Psi(n)},$$

since $\{\frac{n}{\Psi(n)}\}$ is increasing. Moreover,

$$\frac{x+y}{2} = \left(\frac{x+y}{2}\right)^* = \sum_{i=1}^{n-1} ae_i + \frac{a+b}{2}e_n + \frac{a+b}{2}e_{n+1}.$$

So

$$\begin{aligned}\left\|\frac{x+y}{2}\right\| &= \max\left\{\frac{(n-1)a}{\Psi(n-1)}, \frac{(n-1)a+(a+b)/2}{\Psi(n)}, \frac{(n-1)a+a+b}{\Psi(n+1)}\right\} \\ &= \max\left\{\frac{(n-1)a}{\Psi(n-1)}, \frac{an+b}{\Psi(n)}\right\} = \frac{an+b}{\Psi(n)} = \|x\| = \|y\|.\end{aligned}$$

Thus a sphere of the space m_Ψ^0 has a line segment, and so the space is not rotund. □

Suppose now that X is a complex r.i. sequence space with the Fatou property. We will apply Theorem 6.8 to X. Let Φ and Ψ be the *norm fundamental functions* of X and X' respectively, which are defined by $\Phi(0) = 0 = \Psi(0)$ and for each $n \in \mathbb{N}$,

$$\Phi(n) = \|e_1 + \cdots + e_n\|_X, \quad \text{and} \quad \Psi(n) = \|e_1 + \cdots + e_n\|_{X'}.$$

It is well known [4] that Φ and Ψ are quasi-concave and for each $n \in \mathbb{N} \cup \{0\}$,

$$\Phi(n)\Psi(n) = n.$$

Given X with the norm fundamental function Φ, define the Marcinkiewicz sequence space m_Ψ with the following norm

$$\|x\|_{m_\Psi} = \sup_{n\in\mathbb{N}}\left\{\frac{\sum_{k=1}^n x^*(k)}{\Psi(n)}\right\} = \sup_{n\in\mathbb{N}}\left\{\frac{\Phi(n)}{n}\sum_{k=1}^n x^*(k)\right\}.$$

Then the norm fundamental function of m_Ψ is Φ, and $\|x\|_{m_\Psi} \le \|x\|_X$ for all $x \in X$ ([4]). This implies that if $x \in S_X$ is a complex extreme point of B_{m_Ψ}, then x is a complex extreme point of B_X.

In the proof of Theorem 6.8, we showed that if Ψ is not strictly increasing then there is an $n \in \mathbb{N}$ such that

$$x_0 = \sum_{i=1}^n \frac{\Psi(n)}{n} e_i$$

is a complex extreme point of B_{m_Ψ}. Note that

$$\|x_0\|_X = \frac{\Psi(n)}{n}\|e_1 + \cdots + e_n\|_X = \frac{\Psi(n)\Phi(n)}{n} = 1.$$

Hence if Ψ is not strictly increasing, then x_0 is a complex extreme point of B_X. Note also that if Ψ is not strictly increasing, then we can take Q and φ as in the proof of Theorem 6.8. Since $\|x\|_{m_\Psi} \le \|x\|_X$, Q is 2-homogeneous norm-attaining polynomial on X and φ is norm-attaining bounded linear functional on X. Moreover they are not finite. Thus we proved the following proposition.

Proposition 6.10. *Suppose a complex r.i. sequence space X with the Fatou property has a norm fundamental function Φ such that $\{\frac{\Phi(n)}{n}\}$ is not strictly decreasing. Then B_X has a complex extreme point. Moreover, there is a norm-attaining 2-homogeneous polynomial on X which is not finite, and there is a norm-attaining bounded linear functional on X which is not finite.*

Corollary 6.11. *Let X be a complex r.i. sequence space with the Fatou property. Assume no point of S_X is a complex extreme point of B_X. Then the norm fundamental function of its associate space X' is strictly increasing.*

Now, we present a simple but useful fact about complex extreme points of unit ball for r.i. sequence spaces.

Proposition 6.12. *Suppose X is a complex r.i. sequence space and suppose that $x_0 \in X$ is an order continuous element of X. Then $x_0 \in S_X$ is a complex extreme point of B_X if and only if x_0^* is a complex extreme point of B_X.*

Proof. Observe that if $T : X \to X$ is an isometric isomorphism, then T preserves the complex extreme points of B_X.

Let $x_0 \in S_X$ and x_0 be an order continuous element. Then $\lim_{n\to\infty} x_0^*(n) = 0$. So there is a permutation σ of $\mathbb{N}$ such that $|x_0(\sigma(n))| = x_0^*(n)$ for each $n \in \mathbb{N}$. Let $\lambda_n = \text{sign}(x_0(\sigma(n)))$ for $n \in \mathbb{N}$. Define an isometric isomorphism T on X as follows

$$Tx = \{\lambda_n x(\sigma(n))\}, \quad x \in X,$$

Then $Tx_0 = x_0^*$, and so x_0 is a complex extreme point of B_X if and only if x_0^* is a complex extreme point of B_X. □

Example 6.13. We shall show that the converse of Corollary 6.11 does not hold in general, even though X is an order continuous symmetric sequence space. Let X be the set of all complex sequences x such that

$$\|x\| = \sum_{k=1}^{\infty}(\sqrt{n} - \sqrt{n-1})x^*(n) < \infty.$$

Since the sequence $\{\sqrt{n} - \sqrt{n-1}\}$ is decreasing, $(X, \|\ \|)$ is a Lorentz space and it is order continuous [11, 12]. The norm fundamental functions Φ and Ψ of X and X', respectively, are equal and $\Phi(n) = \sqrt{n} = \Psi(n)$ for all $n \in \mathbb{N}$.

We shall show that every point of S_X is a complex extreme point of B_X. By Proposition 6.12, we have only to show that every point $x^* \in S_X$ is a complex extreme point of B_X. Let $x^* \in S_X$ and $y \in X$ be such that $\|x^* + \zeta y\| \le 1$ for all $|\zeta| < 1$. Then by the Hardy-Littlewood inequality [4],

$$\sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n-1})\,|x^*(n) + \zeta y(n)| \le \sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n-1})(x^* + \zeta y)^*(n) \le 1.$$

The function $f : B_{\mathbb{C}} \to \ell_1$ defined by

$$f(\zeta) = \sum_{n=1}^{\infty}(\sqrt{n} - \sqrt{n-1})\,(x^*(n) + \zeta y(n))\,e_n,$$

is analytic and $\|f(\zeta)\|_1$ attains its maximum at $\zeta = 0$. By the Maximum Modulus Theorem (Theorem 3.1 in [15]), f is constant. Hence $y = 0$, and x^* is a complex extreme point of B_X.

Note that even though both Φ and Ψ are strictly increasing concave functions and X is order continuous, we cannot obtain the converse of Corollary 6.11.

Note also that although m_Ψ^0 is order continuous and it has the same norm fundamental function as X, no point of $S_{m_\Psi^0}$ is a complex extreme point of B_{m_Ψ} since Ψ is strictly increasing. Therefore we cannot completely determine the extreme point of r.i. space X by its norm fundamental function.

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