

# ZEROS OF JACOBI-SOBOLEV ORTHOGONAL POLYNOMIALS

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ABSTRACT. We investigate zeros of Jacobi-Sobolev orthogonal polynomials with respect to

$$\phi(f, g) = \int_{-1}^1 f(x)g(x)(1-x)^\alpha(1+x)^\beta dx + \gamma \int_{-1}^1 f'(x)g'(x)(1-x)^{\alpha+1}(1+x)^\beta dx$$

where  $\alpha > -1$ ,  $-1 < \beta \leq 0$  and  $\gamma > 0$ .

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## 1. INTRODUCTION

Consider a Sobolev inner product on the space  $\mathbb{P}$  of real polynomials given by

$$(1.1) \quad \phi(p, q) := \langle \sigma, pq \rangle + \gamma \langle \tau, p'q' \rangle$$

where  $\sigma$  and  $\tau$  are positive-definite moment functionals and  $\gamma > 0$ . Let  $\{P_n(x)\}_{n=0}^\infty$ ,  $\{Q_n(x)\}_{n=0}^\infty$ , and  $\{S_n^{(\gamma)}(x)\}_{n=0}^\infty$  be the sequences of monic polynomials orthogonal with respect to  $\sigma$ ,  $\tau$ , and  $\phi(\cdot, \cdot)$  respectively. Set

$$\langle \sigma, P_n^2(x) \rangle = u_n, \quad \langle \tau, Q_n^2(x) \rangle = v_n, \quad \phi(S_n^{(\gamma)}, S_n^{(\gamma)}) = s_n(\gamma), \quad n \geq 0.$$

Then it is well known([5]) that both  $P_n(x)$  and  $Q_n(x)$  have  $n$  real simple zeros. There also have been many works on zeros of Sobolev orthogonal polynomials  $S_n^{(\gamma)}$  for various choices of  $\sigma$  and  $\tau$  ([2, 3, 6, 16]). Recently, Marcellán, Pérez, and Piñar showed that  $S_n^{(\gamma)}$  has  $n$  real simple zeros, which interlace with zeros of  $P_n(x)$  when  $\sigma = \tau$  is the Laguerre moment functional([13]) and the Gegenbauer moment functional([11]). These results not only extend the previous works by Althammer[2] and Cohen[6] but also motivate the works by M. G. de Bruin and H. G. Meijer[4, 15]. In [15], they presented an exhaustive overview about the location of zeros of  $S_n^{(\gamma)}(x)$  when  $\{\sigma, \tau\}$  is a coherent pair.

Here, we are interested in the location of zeros of Jacobi-Sobolev orthogonal polynomials  $\{S_n^{(\gamma)}(x)\}_{n=0}^\infty$  when

$$\sigma = (1-x)^\alpha(1+x)^\beta dx \quad \text{and} \quad \tau = (1-x)^{\alpha+1}(1+x)^\beta dx$$

on  $[-1, 1]$ . In this case,  $\{\sigma, \tau\}$  is a coherent pair of type C if  $-1 < \beta < 0$ , type B if  $\beta = 0$ , and type A and C if  $\beta > 0$  according to the classification in [15]. So we can deduce from [15, Theorem 4.2] that for  $-1 < \beta < 0$ ,  $S_n^{(\gamma)}(x)$  has  $n$  real simple zeros, all of which lie in  $(-1, 1)$  except possibly the smallest zero. Furthermore, they showed ([15, Theorem 5.2]) that the smallest zero must be greater than

$$\frac{\alpha - \beta}{\alpha + \beta + 2} - \frac{5}{2}.$$

Note that the above lower bound for zeros of  $S_n^{(\gamma)}(x)$  is always less than  $-1$  for  $\alpha > -1$  and  $-1 < \beta < 0$ .

In this work, we give more precise location for the smallest zero with respect to the point  $-1$ .

## 2. MAIN RESULTS

Assume that  $\{\sigma, \tau\}$  is a coherent pair ([4, 8, 10]), that is, there are non-zero constants  $a_n$  such that

$$P'_n(x) + a_{n-1}P'_{n-1}(x) = nQ_{n-1}(x), \quad n \geq 2.$$

Expanding  $P_n(x) + a_{n-1}P_{n-1}(x)$  in terms of  $\{S_k^{(\gamma)}(x)\}_{k=0}^n$ , we obtain

$$(2.1) \quad P_n(x) + a_{n-1}P_{n-1}(x) = S_n^{(\gamma)}(x) + d_{n-1}(\gamma)S_{n-1}^{(\gamma)}(x), \quad n \geq 2;$$

$$d_{n-1}(\gamma) = \frac{a_{n-1}u_{n-1}}{s_{n-1}(\gamma)}, \quad n \geq 2.$$

Set  $\phi_{ij} := \phi(x^i, x^j)$  and  $\Delta_n(\phi) := \det[\phi_{ij}]_{i,j=0}^n$ . Then  $s_n(\gamma) = \frac{\Delta_n(\phi)}{\Delta_{n-1}(\phi)}$  ( $\Delta_{-1}(\phi) = 1$ ),  $n \geq 0$  so that

$$d_n(\gamma) = \frac{a_n u_n \Delta_{n-1}(\phi)}{\Delta_n(\phi)}, \quad n \geq 1.$$

Since  $\Delta_n(\phi)$  is a polynomial in  $\gamma$  of degree  $n$ ,  $\lim_{\gamma \rightarrow \infty} d_n(\gamma) = 0$  for  $n \geq 1$ .

It is easy to see from the orthogonality that  $S_n^{(\infty)}(x) := \lim_{\gamma \rightarrow \infty} S_n^{(\gamma)}(x)$  exists for  $n \geq 0$ . Since  $\lim_{\gamma \rightarrow \infty} d_n(\gamma) = 0$ , by (2.1),

$$(2.2) \quad S_n^{(\infty)}(x) = P_n(x) + a_{n-1}P_{n-1}(x), \quad n \geq 2.$$

Hence,  $S_n^{(\infty)}(x)$  is quasi-orthogonal of order  $n$  with respect to  $\sigma$  so that  $S_n^{(\infty)}(x)$  ( $n \geq 2$ ) has  $n$  real simple zeros  $\{y_{nk}(\infty)\}_{k=1}^n$  satisfying

$$(2.3) \quad y_{n1}(\infty) < x_{n1} < y_{n2}(\infty) < x_{n2} < \cdots < y_{nn}(\infty) < x_{nn}.$$

If we write  $S_n^{(\gamma)}(x) = x^n + \sum_{k=0}^{n-1} C_k^{(n)}(\gamma)x^k$ ,  $n \geq 1$ , then we can easily obtain from the orthogonality of  $\{S_n^{(\gamma)}(x)\}_{n=0}^{\infty}$

$$C_k^{(n)}(\gamma) = -\Delta_{n-1}^{(k)}(\phi)/\Delta_{n-1}(\phi), \quad 0 \leq k \leq n-1,$$

where  $\Delta_{n-1}^{(k)}(\phi)$  is the determinant of  $[\phi_{ij}]_{i,j=0}^{n-1}$  whose  $k$ -th column is replaced by  $[\phi(x^n, x^j)]_{j=0}^{n-1}$ . Note that  $\Delta_{n-1}(\phi)$  and  $\Delta_{n-1}^{(k)}(\phi)$  ( $0 \leq k \leq n-1$ ) are polynomials in  $\gamma$  of degree at most  $n-1$ . Since  $\Delta_{n-1}(\phi) \neq 0$ , zeros of  $S_n^{(\gamma)}(x)$  ( $n \geq 1$ ) are continuous functions in  $\gamma$  for  $\gamma \geq 0$ .

Consider now the Jacobi differential equation for  $\alpha + \beta \neq -1, -2, \dots$

$$(2.4) \quad (1-x^2)y''(x) + \{(\beta-\alpha) - (\alpha+\beta+2)x\}y'(x) + n(\alpha+\beta+n+1)y(x) = 0,$$

which is admissible ([9]) so that it has for each  $n \geq 0$  a unique monic polynomial solution of degree  $n$ , i.e., Jacobi polynomial

$$P_n^{(\alpha, \beta)}(x) = \binom{2n+\alpha+\beta}{n}^{-1} \sum_{k=0}^n \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k} (x+1)^k.$$

For  $\alpha + \beta \neq -1, -2, \dots$  (see [1])

$$(2.5) \quad P_n^{(\alpha, \beta)}(x) + a_{n-1}P_{n-1}^{(\alpha, \beta)}(x) = P_n^{(\alpha, \beta-1)}(x), \quad n \geq 1$$

where

$$a_n = \frac{2(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0$$

and

$$(2.6) \quad P_n^{(\alpha, \beta-1)}(x)' = nP_{n-1}^{(\alpha+1, \beta)}(x), \quad n \geq 0.$$

For  $\alpha, \beta > -1$  and  $\gamma > 0$ , let

$$\phi(p, q) := \left\langle \sigma_J^{(\alpha, \beta)}, pq \right\rangle + \gamma \left\langle \sigma_J^{(\alpha+1, \beta)}, p'q' \right\rangle$$

and  $\{S_n^{(\gamma)}(x; \alpha, \beta)\}_{n=0}^{\infty}$  the monic Jacobi-Sobolev orthogonal polynomials with respect to  $\phi(\cdot, \cdot)$ , where  $\sigma_J^{(\alpha, \beta)}$  is the positive-definite Jacobi moment functional defined by

$$\left\langle \sigma_J^{(\alpha, \beta)}, p(x) \right\rangle := \int_{-1}^1 p(x)(1-x)^\alpha(1+x)^\beta dx, \quad p \in \mathbb{P}.$$

Then by the relations (2.5) and (2.6),  $\{\sigma_J^{(\alpha, \beta)}, \sigma_J^{(\alpha+1, \beta)}\}$  is a coherent pair so that

$$(2.7) \quad S_n^{(\gamma)}(x; \alpha, \beta) + d_{n-1}(\gamma)S_{n-1}^{(\gamma)}(x; \alpha, \beta) = P_n^{(\alpha, \beta)}(x) + a_{n-1}P_{n-1}^{(\alpha, \beta)}(x), \quad n \geq 1$$

for some constants  $d_{n-1}(\gamma)$ , which are positive since  $a_n > 0$  for  $\alpha, \beta > -1$ . We also have

$$S_n^{(\infty)}(x; \alpha, \beta) := \lim_{\gamma \rightarrow \infty} S_n^{(\gamma)}(x; \alpha, \beta) = P_n^{(\alpha, \beta-1)}(x), \quad n \geq 0.$$

According to the classification in [15],  $\{\sigma_J^{(\alpha, \beta)}, \sigma_J^{(\alpha+1, \beta)}\}$  is of type C so that by Theorem 4.1 and Theorem 5.2 in [15],  $S_n^{(\gamma)}(x; \alpha, \beta)$  has  $n$  real simple zeros  $y_{nk} = y_{nk}(\gamma)$  ( $1 \leq k \leq n$ ) such that

$$\frac{\alpha - \beta}{\alpha + \beta + 2} - \frac{5}{2} < y_{n1} < y_{n2} < \cdots < y_{nn} < 1 \quad \text{and} \quad y_{n2} > -1.$$

Moreover, if  $\beta > 0$ , then  $\{\sigma_J^{(\alpha, \beta)}, \sigma_J^{(\alpha+1, \beta)}\}$  is also of type A. Hence by Theorem 4.2 in [15],  $\{y_{nk}\}_{k=1}^n$  interlace with the zeros  $\{x_{nk}\}_{k=1}^n$  of  $S_n^{(\infty)}(x; \alpha, \beta) = P_n^{(\alpha, \beta-1)}(x)$  as

$$x_{n1} < y_{n1} < x_{n2} < y_{n2} < \cdots < x_{nn} < y_{nn}.$$

In particular,  $y_{n1} > -1$  if  $\beta > 0$ .

We are now concerned with the location of the smallest zero  $y_{n1}$  of  $S_n^{(\gamma)}(x; \alpha, \beta)$  with respect to the point  $-1$  for  $\alpha > -1$  and  $-1 < \beta \leq 0$ .

**Theorem 2.1.** *If  $\alpha > -1$  and  $\gamma > 0$ , then  $S_n^{(\gamma)}(x; \alpha, 0)$  ( $n \geq 2$ ) has  $n$  real simple zeros  $\{y_{nk}\}_{k=1}^n$  with*

$$(2.8) \quad -1 < y_{n1} < x_{n1} < y_{n2} < x_{n2} < \cdots < y_{nn} < x_{nn} < 1$$

where  $\{x_{nk}\}_{k=1}^n$  are zeros of  $P_n^{(\alpha, 0)}(x)$ .

We will prove Theorem 2.1 taking into account ideas used in ([11, 13]). Set

$$F[\cdot] = (1-x^2)I - \gamma(1-x)[(-(\alpha+\beta+1)x - (\alpha-\beta+1))D + (1-x^2)D^2]$$

where  $D = \frac{d}{dx}$ . Then it is shown in [12] that  $F[\cdot]$  is a symmetric operator for  $\phi(\cdot, \cdot)$  in the sense that  $\phi(F[p], q) = \phi(p, F[q])$  for any  $p(x)$  and  $q(x)$  in  $\mathbb{P}$ .

**Lemma 2.2.** *Let  $\beta = 0$ . Then for any polynomial  $p(x)$  of degree  $k$  ( $\geq 0$ ), there exists a unique polynomial  $p_1(x)$  of degree  $k$  such that*

$$F[p_1(x)] = (1 - x^2)p(x).$$

**Proof.** When  $\beta = 0$ ,  $F[\cdot] = (1 - x^2)[I + \gamma(\alpha + 1)D - \gamma(1 - x)D^2]$ . Let

$$p_1(x) = \sum_{i=0}^k b_i(1 + x)^i \text{ and } p(x) = \sum_{i=0}^k c_i(1 + x)^i.$$

Then we obtain the following linear system of equations

$$(2.9) \quad \begin{cases} b_k = c_k, \\ b_{k-1} + \gamma k(\alpha + k)b_k = c_{k-1} \\ b_i + \gamma(i + 1)(\alpha + i + 1)b_{i+1} - 2\gamma(i + 1)(i + 2)b_{i+2} = c_i, \quad 0 \leq i \leq k - 2 \end{cases}$$

from which  $\{b_i\}_{i=0}^k$  can be obtained recursively.  $\square$

**Lemma 2.3.**  $\text{sgn}S_n^{(\gamma)}(-1; \alpha, 0) = (-1)^n$ ,  $n \geq 0$ .

**Proof.** From (2.5) and (2.7),

$$S_n^{(\gamma)}(-1; \alpha, 0) + d_{n-1}(\gamma)S_{n-1}^{(\gamma)}(-1; \alpha, 0) = P_n^{(\alpha, -1)}(-1) = 0, \quad n \geq 1.$$

Since  $d_n(\gamma) > 0$ ,  $n \geq 0$  and  $S_0(x; \alpha, 0) = 1$ ,  $\text{sgn}S_n^{(\gamma)}(-1; \alpha, 0) = (-1)^n$ ,  $n \geq 0$ .  $\square$

**Proof of Theorem 2.1.** Fix any  $n \geq 1$  and set

$$w_i(x) = \frac{P_n^{(\alpha, 0)}(x)}{x - x_i}, \quad 1 \leq i \leq n, \text{ where } x_i = x_{ni} \quad (1 \leq i \leq n) \text{ are zeros of } P_n^{(\alpha, 0)}(x).$$

By Lemma 2.2, there exists a unique polynomial  $p_i(x)$  of degree  $n - 1$  such that

$$F[p_i(x)] = (1 - x^2)w_i(x), \quad 1 \leq i \leq n.$$

Hence,

$$\begin{aligned} & \phi(S_n^{(\gamma)}(x; \alpha, 0), p_i(x)) \\ &= \int_{-1}^1 S_n^{(\gamma)}(x; \alpha, 0)p_i(x)(1 - x)^\alpha dx + \gamma \int_{-1}^1 S_n^{(\gamma)}(x; \alpha, 0)'p_i'(x)(1 - x)^{\alpha+1} dx \\ &= \int_{-1}^1 S_n^{(\gamma)}(x; \alpha, 0)[p_i(x) + \gamma(\alpha + 1)p_i'(x) - \gamma(1 - x)p_i''(x)](1 - x)^\alpha dx \\ & \quad - 2^{\alpha+1}\gamma S_n^{(\gamma)}(-1; \alpha, 0)p_i'(-1) \\ &= \int_{-1}^1 S_n^{(\gamma)}(x; \alpha, 0)w_i(x)(1 - x)^\alpha dx - 2^{\alpha+1}\gamma S_n^{(\gamma)}(-1; \alpha, 0)p_i'(-1) \\ &= \lambda_i S_n^{(\gamma)}(x_i; \alpha, 0)w_i(x_i) - 2^{\alpha+1}\gamma S_n^{(\gamma)}(-1; \alpha, 0)p_i'(-1) \end{aligned}$$

where  $\lambda_i$ 's are the Christoffel numbers for the Jacobi polynomial  $P_n^{(\alpha, 0)}(x)$ .

Since  $\text{sgn}w_i(x_i) = \text{sgn}P_n^{(\alpha, 0)}(x_i)' = (-1)^{n-i}$ , then

$$\text{sgn}S_n^{(\gamma)}(x_i; \alpha, 0) = \text{sgn}(w_i(x_i)S_n^{(\gamma)}(-1; \alpha, 0)p_i'(-1)) = (-1)^i \text{sgn}p_i'(-1).$$

Because  $c_{n-1} = 1$  and  $w_i(x)$  has  $n-1$  simple zeros in  $(-1, 1)$ , we obtain by the Cardano-Vieta formula  $(-1)^{n-i-1}c_i > 0$ ,  $0 \leq i \leq n-1$ . Then we have from (2.9)

$$\operatorname{sgn}b_i = (-1)^{n-1-i} \quad \text{and} \quad \operatorname{sgn}p'_i(-1) = \operatorname{sgn}b_1 = (-1)^n, \quad 0 \leq i \leq n-1.$$

Hence,  $\operatorname{sgn}S_n^{(\gamma)}(x_i; \alpha, 0) = (-1)^{n-i}$ ,  $1 \leq i \leq n$ . Since  $\operatorname{sgn}S_n^{(\gamma)}(-1; \alpha, 0) = (-1)^n$ ,  $S_n^{(\gamma)}(x; \alpha, 0)$  has  $n$  real simple zeros  $\{y_{nk}\}_{k=1}^n$  with

$$-1 < y_{n1} < x_1 < y_{n2} < x_2 < \cdots < y_{nn} < x_n < 1. \quad \square$$

Note that Theorem 2.1 also follows Theorem 4.1 in [15] and the subsequent remark, where different arguments are used.

For  $\alpha > -1$  and  $-1 < \beta < 0$ , we have the following which is the Jacobi version of Theorem 5.1 in [15].

**Theorem 2.4.** [cf. Theorem 5.1 in [15]] Let  $y_{n1}(\gamma)$  denote the smallest zero of  $S_n^{(\gamma)}(x; \alpha, \beta)$  and  $y_{n,1}(\infty)$  the smallest zero of  $S_n^{(\infty)}(x; \alpha, \beta)$ . For  $\alpha > -1$  and  $-1 < \beta < 0$ , we have

- (i)  $y_{n1}(\infty) < -1$  if  $n \geq 2$ ;
- (ii)  $y_{21}(\infty)$  is a lower bound for the zeros of  $S_n^{(\gamma)}(x; \alpha, \beta)$  for all  $n \geq 1$  and all  $\gamma > 0$ ;
- (iii) if  $n \geq 3$ , then for  $\gamma$  large

$$y_{n-1,1}(\gamma) < y_{n,1}(\gamma) < y_{n,1}(\infty)$$

and for  $\gamma$  small

$$y_{n,1}(\infty) < -1 < y_{n,1}(\gamma) < y_{n-1,1}(\gamma).$$

*Proof.* From the relation in (2.5)

$$P_n^{(\alpha, \beta-1)}(x) = P_n^{(\alpha, \beta)}(x) + \frac{2n(n+\alpha)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)} P_{n-1}^{(\alpha, \beta)}(x), \quad n \geq 1$$

and the three-term recurrence relation ([5, (2.29), page 153]) for monic Jacobi polynomials

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) = & \left( x - \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)} \right) P_{n-1}^{(\alpha, \beta)}(x) \\ & - \frac{4(n-1)(n+\alpha-1)(n+\beta-1)(n+\alpha+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)^2(2n+\alpha+\beta-3)} P_{n-2}^{(\alpha, \beta)}(x), \quad n \geq 1 \end{aligned}$$

we get the relations

$$\frac{(\alpha+\beta+2)(\alpha+\beta+3)}{(\alpha+1)(\alpha+2)} S_2^{(\infty)}(x; \alpha, \beta)(x) = (x+1)^2 + 2\left(\frac{\beta+1}{\alpha+1}\right)(x^2-1) + \frac{\beta(\beta+1)}{(\alpha+2)(\alpha+1)}(x-1)^2$$

and for  $n \geq 3$

$$(2.10) \quad S_n^{(\infty)}(x; \alpha, \beta) = (x+1)P_{n-1}^{(\alpha, \beta)}(x) - \frac{2(n+\beta-1)(n+\alpha+\beta-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)} S_{n-1}^{(\infty)}(x; \alpha, \beta).$$

Then by the same arguments used to prove Theorem 5.1 in [15], we have the theorem.  $\square$

We see from Theorem 2.4 that if  $\alpha > -1$  and  $-1 < \beta < 0$ , then for  $n \geq 2$ , the smallest zero  $y_{n1}(\gamma)$  of  $S_n^{(\gamma)}(x; \alpha, \beta)$  can be less than or equal to or greater than  $-1$  depending on  $\gamma$ .

**Theorem 2.5.** Let  $n \geq 2$ . Then for the zeros  $\{y_{nk}(\gamma)\}_{k=1}^n$  of  $S_n^{(\gamma)}(x; \alpha, \beta)$  ( $\gamma > 0$ ), we have

- (i)  $y_{n-1,1}(\gamma) < y_{n,1}(\gamma) < y_{n1}(\infty)$  or  $y_{n-1,1}(\gamma) > y_{n,1}(\gamma) > y_{n1}(\infty)$  or  $y_{n-1,1}(\gamma) = y_{n,1}(\gamma) = y_{n1}(\infty)$ ;

- (ii)  $\{y_{n,1}(\infty)\}_{n=2}^{\infty}$  is strictly increasing with an upper bound  $-1$ ;
- (iii) if there is an  $n \geq 3$  such that  $y_{n-1,1}(\gamma) \leq y_{n,1}(\gamma)$  then  $y_{m-1,1}(\gamma) < y_{m,1}(\gamma)$ ,  $m > n$ , so that  $\{y_{n,1}(\gamma)\}_{n=2}^{\infty}$  is strictly decreasing or  $\{y_{n,1}(\gamma)\}_{n=m}^{\infty}$  is strictly increasing for some  $m \geq 2$ ;
- (iv) for  $0 \leq \gamma \leq \infty$ ,  $\lim_{n \rightarrow \infty} y_{n,1}(\gamma) = -1$ .

*Proof.* (i) From (i) of Theorem 2.4 and (ii) of Theorem 4.1 in [15], we have

$$(2.11) \quad y_{n1}(\infty) < -1 < y_{n2}(\gamma) \quad \text{and} \quad y_{n1}(\gamma) < x_{n1} < y_{n2}(\gamma), \quad n \geq 3,$$

where  $x_{n1}$  is the smallest zero of  $P_n^{(\alpha, \beta)}(x)$ . We deduce from (2.2) that if  $y_{n1}(\gamma) = y_{n-1,1}(\gamma)$ , then  $y_{n1}(\gamma) = y_{n-1,1}(\gamma) = y_{n1}(\infty)$ . If  $y_{n1}(\gamma) < y_{n-1,1}(\gamma)$ , then  $y_{n1}(\infty) < y_{n1}(\gamma) < y_{n-1,1}(\gamma)$  since

$$\text{sgn}S_n^{(\infty)}(y_{n1}(\gamma)) = \text{sgn}S_{n-1}^{(\gamma)}(y_{n1}(\gamma)) = (-1)^{n-1}.$$

If  $y_{n1}(\gamma) > y_{n-1,1}(\gamma)$ , then  $y_{n1}(\infty) > y_{n1}(\gamma) > y_{n-1,1}(\gamma)$  since

$$\text{sgn}S_n^{(\infty)}(y_{n1}(\gamma)) = \text{sgn}S_{n-1}^{(\gamma)}(y_{n1}(\gamma)) = (-1)^n.$$

(ii) From (2.10), we obtain

$$\text{sgn}(S_n^{(\infty)}(y_{n-1,1}(\infty))) = \text{sgn}(y_{n-1,1}(\infty) + 1)\text{sgn}P_{n-1}^{(\alpha, \beta)}(y_{n-1,1}(\infty)) = (-1)^n.$$

Hence for  $n \geq 2$ ,  $y_{n,1}(\infty) < y_{n+1,1}(\infty) < -1$ . Thus  $\{y_{n,1}(\infty)\}_{n=2}^{\infty}$  is a strictly increasing sequence with an upper bound  $-1$ .

(iii) Assume that there exists an  $n \geq 3$  such that  $y_{n-1,1}(\gamma) \leq y_{n,1}(\gamma)$  but  $y_{n1}(\gamma) \geq y_{n+1,1}(\gamma)$ . Then from (i), we have

$$y_{n-1,1}(\gamma) \leq y_{n1}(\gamma) \leq y_{n1}(\infty) \quad \text{and} \quad y_{n+1,1}(\infty) \leq y_{n+1,1}(\gamma) \leq y_{n,1}(\gamma).$$

It implies that  $y_{n+1,1}(\infty) \leq y_{n1}(\infty)$ , which is a contradiction to (ii). Thus  $\{y_{n,1}(\gamma)\}_{n=2}^{\infty}$  is a strictly decreasing sequence or  $\{y_{n,1}(\gamma)\}_{n=m}^{\infty}$  is a strictly increasing sequence for some  $m \geq 2$ .

(iv) For  $\gamma = 0$ ,  $y_{n1}(0) = x_{n1}$ . Thus  $\{y_{n1}(0)\}_{n=2}^{\infty}$  is a decreasing sequence and  $\lim_{n \rightarrow \infty} y_{n,1}(0) = -1$ . Let  $\gamma \in (0, \infty)$ . Then (iii) shows that  $\{y_{n,1}(\gamma)\}_{n=2}^{\infty}$  converges :

$$\lim_{n \rightarrow \infty} y_{n,1}(\gamma) := y(\gamma).$$

On the other hand, from Theorem 2 and Corollary 1 in [7], we have that the limit  $y(\gamma)$  lies in  $[-1, 1]$ , which shows that  $y(\gamma) = -1$ . For  $\gamma = \infty$ , (ii) implies that

$$\lim_{n \rightarrow \infty} y_{n,1}(\infty) := y(\infty) \leq -1.$$

Finally, by Theorem 2.4 (iii), there is a  $\gamma$  with  $0 < \gamma < \infty$  and

$$y_{2,1}(\gamma) < y_{3,1}(\gamma) < y_{3,1}(\infty)$$

so that we have from (i) and (iii) that

$$y_{n-1,1}(\gamma) < y_{n,1}(\gamma) < y_{n,1}(\infty), \quad n \geq 3,$$

which shows that  $y(\infty) = -1$ . □

Since  $S_n^{(\infty)}(x; \alpha, \beta) = P_n^{(\alpha, \beta-1)}(x)$ ,  $n \geq 0$ , we have the following.

**Corollary 2.6.** For  $\alpha > -1$  and  $-2 < \beta < -1$ , the smallest zeros of  $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$  converge to  $-1$ .

The following table gives numerical computations of the smallest zeros of  $S_n^{(\gamma)}(x; -\frac{1}{2}, -\frac{1}{2})$ .

	$\gamma = 0$	$\gamma = 1$	$\gamma = 100$	$\gamma = 100000$	$\gamma = \infty$
$n = 2$	-0.7071067	-1.1150692	-1.3621040	-1.3660250	-1.3660254
$n = 3$	-0.8660254	-1.1202612	-1.1209032	-1.1206532	-1.1206532
$n = 4$	-0.9238795	-1.0627289	-1.0601785	-1.0601489	-1.0601489
$n = 5$	-0.9510565	-1.0367560	-1.0360529	-1.0360461	-1.0360461
$n = 6$	-0.9659258	-1.0242415	-1.0240185	-1.0240162	-1.0240162
$n = 7$	-0.9749279	-1.0172377	-1.0171494	-1.0171485	-1.0171485
$n = 8$	-0.9807852	-1.0128997	-1.0128590	-1.0128586	-1.0128586
$n = 9$	-0.9848077	-1.0100208	-1.0099999	-1.0099997	-1.0099997
$n = 10$	-0.987688363	-1.008010659	-1.007999075	-1.007998969	-1.007998963

**Table of the smallest zeros of  $S_n^{(\gamma)}(x; -\frac{1}{2}, -\frac{1}{2})$**

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