A Note on A New Argument for the Proof of the Gevrey Regularity of Solutions of Non-linear Elliptic Equations

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Abstract. We give a new simplifying argument for the proof of the Gevrey regularity of solutions of non-linear elliptic equations. Our proof relies on a weak form of the Schauder estimates and therefore, we hope, it can be applied to treat other cases other than the elliptic one.

In this paper we present a new argument for the proof of the Gevrey regularity of solutions of a non-linear elliptic equation. This kind of results is well-known. We mention some works of Berstein, Gevrey, Hopf, Lewy, Giraud, Morey, Petrovskii, Friedman,.... who proved the analyticity of classical solutions of such equations or systems. Recently we treated the Gevrey regularity of classical solutions of some models of semilinear elliptic degenerate equations, see [1], [2], [3]. These, we hope, will shed some light for further research of the Gevrey regularity of solutions of more general classes of non-linear non elliptic equations (like the Laplacian does for general elliptic equations). But the method there is based on some geometric properties of explicit fundamental solutions for the principle linear part. So it is hard to be extended to treat more general situations. In this note we present a new method to deal with non-linear elliptic equations. We follow the scheme proposed by Friedman [4], but our proof here is based solely on the Schauder estimates. Therefore, we hope, it may be well applied to a general situation, ([5]). We will actually work in a more general space of functions than the space of Gevrey functions. Let L_k be a sequence of positive numbers, satisfying the monotonicity condition $\binom{k}{i} L_i L_{k-i} \le C_1 L_k (i = 1, 2...; k = 1, 2...)$, where C_1 is a positive constant. We note that if the sequence L_k satisfies the monotonicity condition then the sequence $C^k L_k$ also satisfies the same condition for an arbitrary positive constant C. A function $\mathcal{F}(x, v)$, defined for $x = (x_1, ..., x_n)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ and for $v = (v_1, ..., v_\mu) \in E \subset \mathbb{C}^\mu$, is said to belong to the class $C\{L_{k-a}; \Omega, E\}$ (a is an integer) if and only if $\mathcal{F}(x, v)$ is infinitely differentiable and to every pair of compact subsets $\Omega_0 \subset \Omega$ and $E_0 \subset E$ there corresponds a constant C_2 such that for $(x, v) \in \Omega_0 \times E_0$

$$\left| \frac{\partial^{j+k} \mathcal{F}(x, v)}{\partial x_1^{j_1} ... \partial x_n^{j_n} \partial v_1^{k_1} ... \partial v_\mu^{k_\mu}} \right| \le C_2^{j+k} L_{j-a} L_{k-a},$$

$$\Big(j_1 + \cdots + j_n = j, k_1 + \cdots + k_\mu = k; j, k = a, a+1, a+2... \Big).$$

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If $\mathcal{F}(x,v) = f(x)$, we simply write $f(x) \in C\{L_{n-a};\Omega\}$. Note that $C\{n!;\Omega\}$, $(C\{n!^s;\Omega\})$ is the space of all analytic functions (s-Gevrey functions), respectively, in Ω. For $\alpha = (\alpha_1,\dots,\alpha_n)$ we write $D^\alpha u$ for $\frac{\partial^{|\alpha|}u}{i^{|\alpha|}\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}$, where $|\alpha| = \alpha_1+\cdots+\alpha_n, i=\sqrt{-1}$.

We focus on the Gevrey regularity of $C^\infty-$solutions. The $C^\infty-$smoothness of a classical solution of a non-linear elliptic equation may be deduced from a well-known theory (see [6]). Since we deal with $C^\infty-$solutions, by differentiating the initial equation, we can always assume that our equation is quasi-linear (if necessary we can consider a system of equations as well), i. e. of the following form:

$$\sum_{|\alpha|=m} \mathcal{A}_\alpha(x,u,Du,...,D^\beta u)_{|\beta|\le m-1} D^\alpha u = \mathcal{B}(x,u,Du,...,D^\beta u)_{|\beta|\le m-1},$$

or in a short form

$$\sum_{|\alpha|=m} \mathcal{A}_\alpha D^\alpha u = \mathcal{B}. \tag{1}$$

For $x,y\in\Omega$ let us write $d_x = dist(x,\partial\Omega), d_{x,y} = \min(d_x,d_y)$. For $k=0,1,2,\dots,\gamma\in(0,1), u(x)\in C^\infty(\bar\Omega)$ set

$$[u]_{k,0;\Omega} = \sup_{\substack{|\beta|=k\\ x\in\Omega}} |D^\beta u(x)|, [u]^*_{k,0;\Omega} = \sup_{\substack{|\beta|=k\\ x\in\Omega}} d_x^k|D^\beta u(x)|, |u|^*_{k,0;\Omega} = \sum_{j=0}^k [u]^*_{j;\Omega},$$

$$[u]^*_{k,\gamma;\Omega} = \sup_{\substack{|\beta|=k\\ (x,y)\in\Omega^2; x\neq y}} d_{x,y}^{k+\gamma}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\gamma},$$

$$|u|^*_{k,\gamma;\Omega} = |u|^*_{k,0;\Omega} + [u]^*_{k,\gamma;\Omega}$$

$$|u|^{(k)}_{0,\gamma;\Omega} = \sup_{x\in\Omega} d_x^k|u(x)| + \sup_{(x,y)\in\Omega^2;x\neq y} d_{x,y}^{k+\gamma}\frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

From the theory of linear elliptic equations the following a priori estimate is well-known (see [6],[7]):

If v is a solution of a linear elliptic equation

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha v = f(x)$$

with, say, $C^\infty-$coefficients, then for $\gamma\in(0,1)$ the following weighted Schauder estimate holds

$$|v|^*_{m,\gamma;\Omega} \le C_3\Big([v]_{0,0;\Omega} + |f|^{(m)}_{0,\gamma;\Omega}\Big) \tag{2}$$

where the constant C_3 depends only on a finite number of derivatives of the coefficients $a_\alpha(x)$, the diameter of Ω. Suppose that we are given a number d. At every point $x \in \Omega$ we define a ball $B_d(x)$ with center at x and radius d. Put $\Omega_d = \cup_{x\in\Omega} B_d(x)$. For every point $x \in \Omega$ applying (2) in $B_d(x)$ we arrive at the following estimate

$$[v]_{m,0;\Omega} \le C_3\Big(\frac{[v]_{0,0;\Omega_d}}{d^m} + |f|_{0,0;\Omega_d} + d^\gamma [f]_{0,\gamma;\Omega_d}\Big). \tag{3}$$

By using an interpolation inequality we will use (3) in the following form that for every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that

$$[v]_{m,0;\Omega} \le \varepsilon d[f]_{1,0;\Omega_d} + C(\varepsilon)[f]_{0,0;\Omega_d} + \frac{C_3[v]_{0,0;\Omega_d}}{d^m}. \tag{4}$$

Denote by μ the (complex) dimension of the variables $(u, Du, ..., D^\beta u)_{|\beta|\le m-1}$. The next lemmas are essential in [4].

Lemma 1. *There exist a constant C_4 such that if $g(\delta)$ be a non-negative monotone decreasing function defined in the interval $0 < \delta \le 1$ and satisfying*

$$g(\delta) \le \frac{1}{10} g\Big(\delta\Big(1 - \frac{1}{N}\Big)\Big) + \frac{C}{\delta^{N-1}} \quad (N \ge 3),$$

where C is an arbitrary constant, then $g(\delta) \le \frac{CC_4}{\delta^{N-1}}$.

Lemma 2. *Assume that $\bar\Omega \subset \Omega_1$ and $\mathcal{F}(x_1, ..., x_n, u, Du, ..., D^\beta u)_{|\beta|\le m-1} \in C\{L_{k-2}; \Omega_1, \mathbb{C}^\mu\}$. Then there exist constants C_5, C_6 such that for every $H_0, H_1 \ge 1, H_1 \ge C_5 H_0^2$ if*

$$[u]_{k,0;\Omega} \le H_0, \quad 0 \le k \le m;$$
$$[u]_{k,0;\Omega} \le H_0 H_1^{k-m-1} L_{k-m-1}, \quad m+1 \le k \le N+m, 2 \le N;$$

then

$$\sup_{x\in\Omega} \big| D^\alpha \mathcal{F}(x_1, ..., x_n, u, Du, ..., D^\beta u)_{|\beta|\le m-1}\big| \le C_6 H_0 H_1^{N-1} L_{N-1}$$

for every α such that $|\alpha| = N + 1$.

For the sake of completion we reproduce the proof of this lemma.

Proof. To avoid unnecessary complications all constants C_i which appear in the proof will be chosen such that they are greater than 1. We will write $(w_1, w_2, ..., w_\mu)$ for $(u, Du, ..., D^\beta u)_{|\beta|\le m-1}$. From the Faa di Bruno we see that $D^\alpha \mathcal{F}$ is a linear combination terms of the form

$$\frac{\partial^{j+k}\mathcal{F}}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}\partial w_1^{k_1}\cdots\partial w_\mu^{k_\mu}} \prod_{l=1}^{\mu}\prod_{\alpha_j} \left(D^{\alpha_l} w_l\right)^{\zeta(\alpha_l)},$$

where $k+j=j_1+\cdots+j_n+k_1+...+k_\mu \le N+1$ and

$$\sum_l \sum_{\alpha_l} \alpha_l \cdot \zeta(\alpha_l) = \alpha - (j_1,\ldots,j_n).$$

Since $\mathcal{F} \in C\{L_{k-2};\Omega_1,\mathbb{C}^\mu\}$, there exist constants C_7 such that

$$\Big|\frac{D^{j+k}\mathcal{F}}{\partial x_1^{j_1}\cdots\partial x_n^{j_n}\partial w_1^{k_1}...\partial w_\mu^{k_\mu}}\Big| \le C_7^{j+k} L_{j-2}L_{k-2}$$
$$\text{for } \ x\in\Omega, w\in E\subset \bar{E}\subset\mathbb{C}^\mu \ (j=j_1+\cdots+j_n, k=k_1+\cdots+k_\mu; j,k\ge 2).$$

Hence we can choose constants C_8 such that

$$\sup_{x\in\Omega}|D^\alpha\mathcal{F}| \le \frac{d^{N+1}}{d\xi^{N+1}}X(\xi)\Big|_{\xi=0},$$

where $X(\xi)=X_1(\xi)\cdot X_2(\xi)$ and

$$X_1(\xi)=X_1(v(\xi))=1+C_8v(\xi)+\sum_{i=2}^{N+1}\frac{C_8^iL_{i-2}v^i(\xi)}{i!}, X_2(\xi)=1+C_8\xi+\sum_{i=2}^{N+1}\frac{C_8^iL_{i-2}\xi^i}{i!},$$

$$v(\xi)=H_0\Big(\xi+\sum_{j=2}^{N+1}\frac{H_1^{j-2}L_{j-2}\xi^j}{j!}\Big).$$

We introduce the following notation: for two infinitely differentiable functions $v(\xi),h(\xi)$ with non-negative derivatives, we say $v(\xi)\ll h(\xi)$ if and only if $v^{(j)}(0)\le h^{(j)}(0)$ for $0\le j\le N+1$. We note that if C is an arbitrary constant and

$$v_1(\xi)\ll h_1(\xi), v_2(\xi)\ll h_2(\xi)$$

then

$$Cv_1(\xi)\ll Ch_1(\xi), v_1(\xi)+v_2(\xi)\ll h_1(\xi)+h_2(\xi), v_1(\xi)v_2(\xi)\ll h_1(\xi)h_2(\xi).$$

We would like to estimate $v^2(\xi)$. We claim that, there exists a constant C_9 (independent of N) such that

$$v^2(\xi)\ll C_9H_0^2\Big(\xi^2+\sum_{j=3}^{N+1}\frac{H_1^{j-3}L_{j-3}\xi^j}{(j-1)!}\Big). \tag{5}$$

Indeed, to estimate the coefficient of ξ^i in $v^2(\xi)$, we consider the following cases
I) The coefficient of ξ^2,ξ^3 are $H_0^2, H_0^2L_0$.

II) The coefficient of $\xi^j (j \geq 4)$ is

$$H_0^2\Big(\frac{2H_1^{j-3}L_{j-3}}{(j-1)!}+\sum_{\lambda=2}^{j-2}\frac{H_1^{j-4}L_{\lambda-2}L_{j-\lambda-2}}{\lambda!(j-\lambda)!}\Big). \tag{6}$$

The second sum in (6) can be estimated in the following way

$$\sum_{\lambda=2}^{j-2}\frac{L_{\lambda-2}}{\lambda!}\frac{L_{j-\lambda-2}}{(j-\lambda)!}H_1^{j-4} \leq H_1^{j-4}\max_{\lambda}\frac{L_{\lambda-2}L_{j-\lambda-2}}{(\lambda-2)!(j-\lambda-2)!}\sum_{\lambda=2}^{j-2}\Big(\frac{1}{(\lambda-1)(j-\lambda-1)}\Big)^2 \leq$$
$$\frac{C_{10}H_1^{j-4}L_{j-4}}{(j-4)!j^2}\sum_{\Lambda=1}^{j-3}\Big(\frac{1}{\Lambda}+\frac{1}{j-\Lambda-2}\Big)^2 \leq \frac{C_{11}H_1^{j-4}L_{j-3}}{(j-1)!}.$$

Therefore we have (5). Now by induction we can easily deduce that

$$v^i(\xi) \ll C_9^{i-1}H_0^i\Big(\xi^i+\sum_{j=i+1}^{N+1}\frac{H_1^{j-i-1}L_{j-i-1}\xi^j}{(j-i+1)!}\Big), (2 \leq i \leq N)$$

and finally

$$v^{N+1}(\xi) \ll C_9^N H_0^{N+1}\xi^{N+1}.$$

Next, it is easy to verify that $X_1(0)=1, X_2(0)=1, X_1'(0)=C_8H_0, X_2'(0)=C_8, X_1^{(2)}(0)=C_8H_0L_0+C_8^2C_9H_0^2L_0 \leq 2C_8H_0H_1L_0$ if we take $H_1 \geq C_8C_9H_0$ and $X_2^{(j)}(0)=C_8^jL_{j-2}$ for $j \geq 2$.
We now compute $X_1^{(j)}(0)$ when $3 \leq j \leq N+1$. Since

$$X_1(v) \ll 1+C_8H_0\xi+\Big(C_8H_0L_0+C_8^2C_9H_0^2L_0\Big)\frac{\xi^2}{2}+$$
$$\sum_{j=3}^{N+1}\Big(\frac{C_8H_0H_1^{j-2}L_{j-2}}{j!}+\frac{C_8^jC_9^{j-1}H_0^jL_{j-2}}{j!}+\sum_{i=2}^{j-1}\frac{C_8^iC_9^{i-1}H_0^iH_1^{j-i-1}L_{i-2}L_{j-i-1}}{i!(j-i+1)!}\Big)\xi^j$$

it follows that

$$X_1^{(j)}(0) \leq C_8H_0H_1^{j-2}L_{j-2}+C_8^jC_9^{j-1}H_0^jL_{j-2}+$$
$$+\sum_{i=2}^{j-1}\frac{C_8^iC_9^{i-1}H_0^iH_1^{j-i-1}L_{i-2}L_{j-i-1}j!}{i!(j-i+1)!} \text{ (for } j=2$$
$$\leq C_{12}H_0H_1^{j-2}L_{j-2}+\frac{C_{13}H_0H_1^{j-2}j!L_{j-3}}{(j-3)!}\sum_{i=2}^{j-1}\frac{1}{i(i-1)(j-i+1)(j-i)} \leq$$
$$C_{14}H_0H_1^{j-2}L_{j-2},$$

by taking $H_1 \geq (C_8 C_9 H_0)^2$.
Therefore by taking $H_1 \geq (C_8 C_9 H_0)^2 = C_5 H_0^2$ we obtain

$$\frac{d^{N+1} X(\xi)}{d\xi^{N+1}}\Big|_{\xi=0} = \sum_{j=0}^{N+1} \binom{j}{N+1} X_1^{(j)}(0) X_2^{(N+1-j)}(0) \leq$$

$$C_8 C_{14} N H_0 H_1^{N-2} L_{N-2} + C_{14} H_0 H_1^{N-1} L_{N-1} + C_8^{N+1} L_{N-1} + (N+1) C_8^{N+1} H_0 L_{N-2} +$$

$$2(N+1) N C_8^N H_0 H_1 L_0 L_{N-3} + \sum_{j=3}^{N-1} \binom{j}{N+1} C_{14} H_0 H_1^{j-2} L_{j-2} C_8^{N+1-j} L_{N-j-1} \leq$$

$$\text{(the final sum is absent if } N \leq 3) \;\; C_{15} H_0 H_1^{N-1} L_{N-1}.$$

Hence

$$\sup_{x\in\Omega} \left| D^\alpha \mathcal{F} \right| \leq C_{15} H_0 H_1^{N-1} L_{N-1} =: C_6 H_0 H_1^{N-1} L_{N-1}. \square$$

Lemma 3. *Under the same hypotheses of Lemma 2 with $k \leq N + m$ replaced by $k \leq N + m - 1$ then*

$$\sup_{x\in\Omega} \left| D^\alpha \mathcal{F} \right| \leq C_{16} [u]_{N+m,0;\Omega} + C_6 H_0 H_1^{k-m-1} L_{k-m-1}$$

for every α such that $|\alpha| = N + 1$.

Proof. Indeed, as in the proof of Lemma 2 all the terms in $D^\alpha \mathcal{F}$ can be estimated by known bounds for $[u]_{k,0;\Omega}$ $(0 \leq k \leq N+m-1)$ except terms of the form $\left(\frac{\partial \mathcal{F}}{\partial (D^\beta u)}\right)_{|\beta|=m-1} D^{\beta+\alpha} u$. There are no more than n^{m-1} such terms and each term is bounded by

$$\sup_{x\in\Omega} \left| \left(\frac{\partial \mathcal{F}}{\partial (D^\beta u)}\right)_{|\beta|=m-1} \right| [u]_{N+m,0;\Omega}.$$

Therefore the conclusion of Lemma 3 follows.$\square$

The well-known result that we are to prove is:

Theorem. *Suppose that $A_\alpha, \mathcal{B} \in C\{L_{k-2}; \Omega, \mathbb{C}^\mu\}$. If u is a $C^\infty-$solution of (1) which in turn is elliptic at u, i. e.*

$$\sum_{|\alpha|=m} \mathcal{A}_\alpha (x, u, Du, ..., D^\beta u)_{|\beta| \leq m-1} \xi^\alpha \neq 0$$

for every $(x, \xi) \in \Omega \times \mathbb{R}^n \backslash 0$. Then $u \in C\{L_{k-m-1}; \Omega\}$. In particular, if $\mathcal{A}_\alpha, \mathcal{B}$ are analytic (s-Gevrey) functions then so is u.

Proof. Since the theorem is purely local it suffices to prove that for every point $x_0 \in \Omega$, there exists a neighborhood $O(x_0)$ such that $u \in C\{L_{k-m-1}; O(x_0)\}$. Denote by $B_\rho(x_0)$

the ball with center at x_0 and radius ρ. Without loss of generality we can assume that for $\rho \leq 2$ the closed ball $\bar{B}_\rho(x_0)$ belongs to Ω. We will prove by induction that there exist two constants $H_0, H_1 \geq 1$ such that

$$
\begin{aligned}
&[u]_{k,0;B_\rho} \leq H_0 \quad \text{for} \quad 0 \leq k \leq m, \\
&[u]_{k,0;B_\rho} \leq H_0 \left(\frac{H_1}{2-\rho} \right)^{k-m-1} L_{k-m-1} \quad \text{for} \quad m+1 \leq k, \ 1 \leq \rho < 2.
\end{aligned} \tag{7}
$$

Hence the desired conclusion follows. Since $u \in C^\infty(\Omega)$, we can always find constants H_0, H_1 big enough such that (7) satisfies for $0 \leq k \leq 2m+3$. Assume that (7) holds for $k = N+m-1, N \geq m+4$. We shall prove (7) for $k = N+m$. Now, for $0 < \delta \leq 1$ let us write B, B' respectively for $B_{2-\delta}(x_0), B_{2-\frac{\delta(N-1)}{N}}(x_0)$. Put $g_N(\delta) = [u]_{N+m,0;B}$.

Since both sides are smooth, by $D^{\alpha'}$–differentiating (with $|\alpha'| = N$) the equation (1) we obtain:

$$
\begin{aligned}
\sum_{|\alpha|=m} \mathcal{A}_\alpha D^\alpha (D^{\alpha'} u) = - \sum_{|\alpha|=m} D^{\alpha'} \mathcal{A}_\alpha D^\alpha u + D^{\alpha'} \mathcal{B} - \\
\sum_{|\alpha|=m} \sum_{0<\alpha''<\alpha'} \binom{\alpha''}{\alpha'} D^{\alpha'-\alpha''} \mathcal{A}_\alpha D^{\alpha+\alpha''} u =: \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 \quad \text{in} \quad \Omega.
\end{aligned}
$$

Applying (4) for $\Omega = B, \Omega_d = B', d = \frac{\delta}{N}$ we have

$$
\begin{aligned}
[D^{\alpha'} u]_{m,0;B} \leq \frac{\varepsilon\delta}{N}([\mathcal{F}_1]_{1,0;B'} + [\mathcal{F}_2]_{1,0;B'} + [\mathcal{F}_3]_{1,0;B'}) + \frac{C_{17} N^m [u]_{N,0;B'}}{\delta^m} + \\
C(\varepsilon)([\mathcal{F}_1]_{0,0;B'} + [\mathcal{F}_2]_{0,0;B'} + [\mathcal{F}_3]_{0,0;B'}).
\end{aligned} \tag{8}
$$

By Lemma 2, from the inductive assumptions we deduce that

$$
\max\{[\mathcal{F}_1]_{0,0;B'}, [\mathcal{F}_2]_{0,0;B'}\} \leq C_{18} H_0 \Big(\frac{N H_1}{(N-1)\delta} \Big)^{N-2} L_{N-2} \leq C_{19} H_0 \Big(\frac{H_1}{\delta} \Big)^{N-2} L_{N-2}, \tag{9}
$$

$$
\begin{aligned}
[\mathcal{F}_3]_{0,0;B'} \le & \sum_{|\alpha|=m} \sum_{0<\alpha''<\alpha'} \binom{\alpha''}{\alpha'} [D^{\alpha'-\alpha''}\mathcal{A}_\alpha]_{0,0;B'} [D^{\alpha+\alpha''}u]_{0,0;B'} \le \\
& C_{20} N H_0 L_{N-2} \Big(\frac{NH_1}{(N-1)\delta}\Big)^{N-2} + \\
& \sum_{|\alpha|=m} \sum_{1\le|\alpha''|\le N-2} C_{21} H_0^2 \binom{\alpha''}{\alpha'} \Big(\frac{NH_1}{(N-1)\delta}\Big)^{N-3} L_{N-2-|\alpha''|} L_{|\alpha''|-1} \le \\
& C_{22} H_0 L_{N-1} \Big(\frac{H_1}{\delta}\Big)^{N-2} + C_{23} H_0^2 L_{N-3} \Big(\frac{H_1}{\delta}\Big)^{N-3} \sum_{|\alpha''|=1}^{N-2} \frac{\binom{|\alpha''|}{N}}{\binom{|\alpha''|-1}{N-3}} \le \\
& C_{22} H_0 L_{N-1} \Big(\frac{H_1}{\delta}\Big)^{N-2} + \\
& C_{24} H_0^2 L_{N-1} \Big(\frac{H_1}{\delta}\Big)^{N-3} \sum_{|\alpha''|=1}^{N-2} \frac{N}{|\alpha''|(N-|\alpha''|)(N-|\alpha''|-1)} \le \\
(10) \qquad & C_{25} H_0 \Big(\frac{H_1}{\delta}\Big)^{N-2} L_{N-1}
\end{aligned}
$$

if we take $H_1 \ge H_0$.

By Lemma 3, from the inductive assumptions we see that

$$
\begin{aligned}
(11) \quad \max\{[\mathcal{F}_1]_{1,0;B'}, [\mathcal{F}_2]_{1,0;B'}\} \le C_{26}[u]_{N+m,0;B'} + C_{27} H_0 \Big(\frac{NH_1}{(N-1)\delta}\Big)^{N-1} L_{N-1} \le \\
C_{26}[u]_{N+m,0;B'} + C_{28} H_0 \Big(\frac{H_1}{\delta}\Big)^{N-1} L_{N-1},
\end{aligned}
$$

$$
\begin{aligned}
[\mathcal{F}_3]_{1,0;B'} \le & \sum_{|\alpha|=m} \sum_{0<\alpha''<\alpha'} \binom{\alpha''}{\alpha'} [D^{\alpha'-\alpha''}\mathcal{A}_\alpha]_{1,0;B'} [D^{\alpha+\alpha''}u]_{0,0;B'} + \\
& \sum_{|\alpha|=m} \sum_{0<\alpha''<\alpha'} \binom{\alpha''}{\alpha'} [D^{\alpha'-\alpha''}\mathcal{A}_\alpha]_{0,0;B'} [D^{\alpha+\alpha''}u]_{1,0;B'} \le \\
& C_{29} N [u]_{N+m,0;B'} + \\
& \sum_{|\alpha|=m} \sum_{1\le|\alpha''|\le N-1} C_{30} H_0^2 \binom{\alpha''}{\alpha'} \Big(\frac{NH_1}{(N-1)\delta}\Big)^{N-2} L_{N-1-|\alpha''|} L_{|\alpha''|-1} + \\
& \sum_{|\alpha|=m} \sum_{1\le|\alpha''|\le N-2} C_{31} H_0^2 \binom{\alpha''}{\alpha'} \Big(\frac{NH_1}{(N-1)\delta}\Big)^{N-2} L_{N-2-|\alpha''|} L_{|\alpha''|} \le
\end{aligned}
$$

$$C_{29}N[u]_{N+m,0;B'} + C_{32}H_0^2 L_{N-2}\Big(\frac{H_1}{\delta}\Big)^{N-2}\left(\sum_{|\alpha''|=1}^{N-1}\frac{\binom{|\alpha''|}{N}}{\binom{|\alpha''|-1}{N-2}} + \sum_{|\alpha''|=1}^{N-2}\frac{\binom{|\alpha''|}{N}}{\binom{|\alpha''|}{N-2}}\right) \leq$$

$$C_{29}N[u]_{N+m,0;B'} +$$

$$C_{33}NH_0^2 L_{N-1}\Big(\frac{H_1}{\delta}\Big)^{N-2}\left(\sum_{|\alpha''|=1}^{N-1}\frac{1}{|\alpha''|(N-|\alpha''|)} + \sum_{|\alpha''|=1}^{N-2}\frac{1}{(N-|\alpha''|)(N-|\alpha''|-1)}\right) \leq$$

$$C_{29}N[u]_{N+m,0;B'} + C_{34}NH_0^2\Big(\frac{H_1}{\delta}\Big)^{N-2} L_{N-1} \leq$$

(12)

$$C_{29}N[u]_{N+m,0;B'} + C_{35}NH_0\Big(\frac{H_1}{\delta}\Big)^{N-1} L_{N-1}$$

if we take $H_1 \geq H_0$.
Therefore combining (8)-(12) we obtain

$$\begin{aligned}[u]_{N+m,0;B} \leq \frac{\varepsilon\delta}{N}\Big(C_{36}N[u]_{N+m,0;B'} + C_{37}NH_0\Big(\frac{H_1}{\delta}\Big)^{N-1} L_{N-1}\Big) + \\ \frac{C_{38}H_0H_1^{N-m-1}L_{N-m-1}N^m}{\delta^{N-1}} + C(\varepsilon)C_{39}H_0\Big(\frac{H_1}{\delta}\Big)^{N-2} L_{N-1} \leq \\ C_{36}\varepsilon\delta[u]_{N+m,0;B'} + C_{37}\varepsilon\delta H_0\Big(\frac{H_1}{\delta}\Big)^{N-1} L_{N-1} + \\ \frac{C_{40}H_0H_1^{N-m-1}L_{N-1}}{\delta^{N-1}} + C(\varepsilon)C_{39}H_0\Big(\frac{H_1}{\delta}\Big)^{N-2} L_{N-1}.\end{aligned}$$

Now choose ε such that $\varepsilon \leq \min\{\frac{1}{10C_{36}}, \frac{1}{2C_{37}C_4}\}$ we arrive at

$$g_N(\delta) \leq \frac{1}{10}g_N\Big(\delta\Big(1-\frac{1}{N}\Big)\Big) + \frac{H_0H_1^{N-1}L_{N-1}}{\delta^{N-1}}\Big(\frac{C_{40}}{H_1^m} + \frac{C(\varepsilon)C_{39}}{H_1} + \frac{1}{2C_4}\Big).$$

Hence by Lemma 1 we have

$$g_N(\delta) \leq \frac{C_4H_0H_1^{N-1}L_{N-1}\Big(\frac{C_{40}}{H_1^m} + \frac{C(\varepsilon)C_{39}}{H_1} + \frac{1}{2C_4}\Big)}{\delta^{N-1}}.$$

If we choose H_1 big enough such that $\frac{C_{40}}{H_1^m} + \frac{C(\varepsilon)C_{39}}{H_1} \leq \frac{1}{2C_4}$ we arrive at

$$[u]_{N+m,0;B} = g_N(\delta) \leq H_0\left(\frac{H_1}{\delta}\right)^{N-1} L_{N-1}.$$

That is the same as (7) for $k = N+m, \delta = 2-\rho$. The proof of the theorem is therefore completed. □

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