A PRECONDITIONER FOR THE FETI-DP FORMULATION OF THE STOKES PROBLEM WITH MORTAR METHODS

HYEA HYUN KIM AND CHANG-OCK LEE

ABSTRACT. We consider a FETI-DP formulation for the Stokes problem on nonmatching grids in $2D$. The FETI-DP method is a domain decomposition method that uses Lagrange multipliers to match the solutions continuously across the subdomain boundaries in the sense of dual-primal variables. We use the mortar matching condition as the continuity constraints for the FETI-DP formulation. Moreover, to satisfy the compatibility condition of the local Stokes problem and to solve the Stokes problem efficiently, redundant continuity constraints are introduced. Lagrange multipliers corresponding to the redundant constraints are treated as primal variables in the FETI-DP formulation. We propose a preconditioner for the FETI-DP operator, which is derived from a dual norm on the Lagrange multiplier space. The dual norm is obtained from a duality pairing between the Lagrange multiplier space and the velocity function space restricted on the nonmortar sides. Then, we show that the condition number of the preconditioned FETI-DP operator is bounded by $C \max_{i=1,\cdots,N} \left\{(1+\log(H_i/h_i))^2\right\}$, where H_i and h_i are the subdomain size and the mesh size, respectively, and C is a constant independent of H_i's and h_i's.

1. INTRODUCTION

In this paper, an iterative substructuring method with Lagrange multipliers is studied for the Stokes problem under nonconforming discretizations. Nonconforming discretizations are important for multiphysics simulations, contact-impact problems, the generation of meshes and partitions aligned with jumps in diffusion coefficients, hp-adaptive methods, and special discretizations in the neighborhood of singularities. Of the many methods for nonmatching discretizations, including [6] and [17], we consider the mortar methods ([1, 3, 20, 21]). With the mortar matching condition as the continuity constraints, the FETI-DP equation is formulated.

Dual-primal FETI(FETI-DP) methods were introduced by Farhat *et al.*[9] as a generation of FETI method[10]. The idea is to use primal variables at corner points and Lagrange multipliers on edges to match solutions continuously across subdomain boundaries. They also showed numerically that the FETI-DP method is scalable with respect to the mesh size, the subdomain size and the number of elements per subdomain for second and fourth order elliptic problems both. Mandel and Tezaur[15] analyzed that the condition number of the FETI-DP method is

2000 *Mathematics Subject Classification.* 65N30; 65N55; 76D07.
Key words and phrases. FETI-DP, mortar methods, Stokes problem, preconditioner.
This work was partially supported by KRF-2002-070-C00004 and KOSEF R01-2000-00008.

bounded by $C\,(1+\log\,(H/h))^2$ for both second and fourth order elliptic problems in $2D$, where H and h denote the subdomain size and mesh size. For $3D$ elliptic problems with heterogeneous coefficients, Klawonn *et al.*[13] obtained the same bound of the condition number.

Li[14] developed a FETI-DP method for the Stokes problem by adding redundant continuity constraints. The Lagrange multipliers to the redundant constraints are treated as the primal variables. Hence, the enlarged coarse problem accelerates the convergence of the method. Moreover, the compatibility condition of the local Stokes problem is satisfied at the FETI-DP iterations. It was also shown that the Dirichlet preconditioner gives a condition number bound $C(1+\log(H/h))^2$.

Recently, FETI(-DP) methods, which were originally developed for conforming discretizations, have been applied to nonmatching discretizations([7, 8, 11, 16, 18, 19]). For elliptic problems in $2D$, Dryja and Widlund[7] proposed the Dirichlet preconditioner which gives the condition number bound $C(1+\log(H/h))^2$ with the Neumann-Dirichlet ordering of substructures. In general cases, that is, without considering ordered substructures, they obtained $C(1+\log(H/h))^4$ for the condition number bound. Moreover, in [8], they proposed a different preconditioner which is similar to the one in [12], and proved the condition number bound $C(1+\log(H/h))^2$. However, in their analysis, they imposed a restriction that the mesh sizes on the nonmortar side and the mortar side are comparable. This restriction is impractical when the coefficients of elliptic problems are highly discontinuous between subdomains (See Wohlmuth[21]).

For the same problem, Kim and Lee [11] formulated a FETI-DP operator in a different way from Dryja and Widlund[7, 8] and proposed a Neumann-Dirichlet preconditioner, which gives the condition number bound $C(1+\log(H/h))^2$ without the restriction on mesh size between neighboring subdomains. The proposed preconditioner is different from the early developed FETI-DP preconditioners. In this preconditioner, the connectivity matrix is multiplied by the function values only on the nonmortar sides of interfaces not on the both sides of interfaces. For the elliptic problems with heterogeneous coefficients, the authors chose the slave and master sides according to the magnitude of coefficients. Then they obtained the same condition number bound which does not depend on the coefficients.

We extend the result in [11] to the Stokes problem. In doing this, we use the inf-sup stable $P_1(h)-P_0(2h)$ finite elements in each subdomain. For the optimality of the approximation under nonmatching discretizations, we impose mortar matching conditions on the velocity functions using the standard Lagrange multiplier space introduced in [3].

For the Stokes problem, Belgacem[2] showed the optimality of approximation with mortar methods. The inf-sup constant for the mortar finite element function space is crucial in the analysis of the approximation order. If the constant is independent of mesh size and subdomain size, then the optimal order of approximation follows independently of the number of subdomains and mesh size as in the case of elliptic problems. In [2], it was shown that the inf-sup constant is independent of mesh size but not shown for the subdomain size. For the $P_1(h)-P_0(2h)$ mortar finite elements, we compute the inf-sup constant numerically by increasing the number of subdomains and can see that the constant seems to be independent of the number of subdomains.

We follow the FETI-DP formulation in [14]. With the same reason as mentioned before, we introduce the redundant constraints to solve the Stokes problem efficiently and correctly. Moreover, we propose a Neumann-Dirichlet preconditioner and analyze the condition number bound. In the analysis of the condition number bound, the continuity of the mortar projection in $H_{00}^{1/2}$-norm is used. Hence, our results can be extended to the Lagrange multiplier spaces satisfying this property. A few of such Lagrange multiplier spaces are developed by Wohlmuth[20, 21].

This paper is organized as follows. Section 2 contains a brief introduction to Sobolev spaces and finite elements. In Section 3, we derive a FETI-DP operator for the Stokes problem and propose a preconditioner for the FETI-DP operator. Section 4 is devoted to analyze the condition number bound of the preconditioned FETI-DP operator. Numerical results are included in Section 5.

2. SOBOLEV SPACES AND FINITE ELEMENTS

2.1. **Model problem.** Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and $L^2(\Omega)$ be the space of square integrable functions defined in Ω equipped with the norm $\|\cdot\|_{0,\Omega}$:

$$\|v\|_{0,\Omega}^2 := \int_\Omega v^2\,dx.$$

$L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ satisfying $\int_\Omega v\,dx = 0$. $H^1(\Omega)$ is the space of functions, which are square integrable up to the first weak derivatives, and the norm is given by

$$\|v\|_{1,\Omega} := \left(\int \nabla v \cdot \nabla v\,dx + \frac{1}{d_\Omega^2} \int v^2\,dx \right)^{1/2},$$

where d_Ω means the diameter of Ω. The space $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with zero trace on the boundary of Ω.

In this paper, we consider the following Stokes problem: For $\mathbf{f} \in [L^2(\Omega)]^2$, find $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ satisfying

$$\begin{aligned} -\triangle \mathbf{u} + \nabla p &= \mathbf{f} \ \text{ in } \Omega, \\ -\nabla \cdot \mathbf{u} &= 0 \ \text{ in } \Omega, \\ \mathbf{u} &= \mathbf{0} \ \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

We assume that Ω is partitioned into nonoverlapping bounded polygonal subdomains $\{\Omega_i\}_{i=1}^N$ and the partition is geometrically conforming. That is, a subdomain intersects with neighboring subdomains on the whole of an edge or at a vertex. For each subdomain, the following function spaces are introduced:

$$H_D^1(\Omega_i) := \left\{ v \in H^1(\Omega_i) \ : \ v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega \right\},$$
$$L_0^2(\Omega_i) := \left\{ q \in L^2(\Omega_i) \ : \ \int_{\Omega_i} q\, dx = 0 \right\},$$
$$\Pi^0 := \left\{ q^0 \in L_0^2(\Omega) \ : \ q^0|_{\Omega_i} \text{ is a constant for each } i \right\}.$$

Then, in a variational form, the problem (2.1) becomes:

Find $\left(\mathbf{u}, p_I, p^0\right) \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2 \times \prod_{i=1}^N L_0^2(\Omega_i) \times \Pi^0$ such that

(2.2)
$$\sum_{i=1}^N (\nabla\mathbf{u}, \nabla\mathbf{v})_{\Omega_i} - \sum_{i=1}^N (p_I + p^0, \nabla\cdot\mathbf{v})_{\Omega_i} = \sum_{i=1}^N (\mathbf{f}, \mathbf{v})_{\Omega_i} \quad \forall\, \mathbf{v} \in \prod_{i=1}^N \left[H_D^1(\Omega_i)\right]^2,$$
$$-\sum_{i=1}^N (\nabla\cdot\mathbf{u}, q_I)_{\Omega_i} = 0 \quad \forall\, q_I \in \prod_{i=1}^N L_0^2(\Omega_i),$$
$$-\sum_{i=1}^N (\nabla\cdot\mathbf{u}, q^0)_{\Omega_i} = 0 \quad \forall\, q^0 \in \Pi^0,$$

and the velocity $\mathbf{u}$ is continuous across the subdomain interfaces $\Gamma = \bigcup_{i,j=1}^N (\partial\Omega_i \cap \partial\Omega_j)$. Here, $(\cdot,\cdot)_{\Omega_i}$ denotes the inner product in $[L^2(\Omega_i)]^n$ for $n = 1, 2$.

We triangulate each subdomain Ω_i. Then $\Omega_i^{2h_i}$ denotes the quasi-uniform triangulation with the maximum diameter $2h_i$ of the triangles. After bisecting each edge of triangles in $\Omega_i^{2h_i}$, we obtain a finer triangulation $\Omega_i^{h_i}$ from $\Omega_i^{2h_i}$. Note that these triangulations need not match across the subdomain interfaces. From these triangulations, we consider the inf-sup stable $P_1(h_i) - P_0(2h_i)$ finite elements in each subdomain Ω_i. Let

$$X_i := \left\{ \mathbf{v}_i \in \left[H_D^1(\Omega_i) \cap C^0(\Omega_i)\right]^2 \ : \ \mathbf{v}_i|_\tau \in [P_1(\tau)]^2 \ \ \forall\, \tau \in \Omega_i^{h_i} \right\},$$
$$Q_i := \left\{ q_i \in L^2(\Omega_i) \ : \ q_i|_\tau \in P_0(\tau) \ \ \forall\, \tau \in \Omega_i^{2h_i} \right\},$$
$$Q_i^0 := Q_i \cap L_0^2(\Omega_i),$$

where $P_l(\tau)$ is a set of polynomials of degree less then or equal to l in τ.

To do FETI-DP formulation, we define the following spaces:

$$X := \left\{ \mathbf{v} \in \prod_{i=1}^{N} X_i \ : \ \mathbf{v} \text{ is continuous at subdomain corners} \right\},$$
$$Q := \prod_{i=1}^{N} Q_i^0,$$
$$W_i := X_i|_{\partial\Omega_i} \quad \text{for } i = 1, \cdots, N,$$
$$W = \left\{ \mathbf{w} \in \prod_{i=1}^{N} W_i \ : \ \mathbf{w} \text{ is continuous at subdomain corners} \right\}.$$

In this paper, we will use the same notation for a finite element function and the vector of nodal values of that function, that is, $\mathbf{v}_i$ is used to denote a finite element function or the corresponding vector of nodal values. The same applies to the notations W_i, X, W, etc.

For $\mathbf{v} = (\mathbf{v}_1^t, \cdots, \mathbf{v}_N^t)^t \in X$, we write

$$\mathbf{v}_i = \begin{pmatrix} \mathbf{v}_I^i \\ \mathbf{v}_\Delta^i \\ \mathbf{v}_c^i \end{pmatrix},$$

where the symbol I, Δ and c represent the d.o.f.(degrees of freedom) corresponding to the interior nodes, nodes on edges and nodes at corners, respectively. Since $\mathbf{v}$ is continuous at subdomain corners, there exists a vector $\mathbf{v}_c$ satisfying $\mathbf{v}_c^i = L_c^i \mathbf{v}_c$ for all $i = 1, \cdots, N$, where L_c^i is a map that restricts $\mathbf{v}_c$ at the corners of the subdomain Ω_i. The vector $\mathbf{v}_c$ has the d.o.f. corresponding to the union of subdomain corners. Let

$$\mathbf{v}_I = \begin{pmatrix} \mathbf{v}_I^1 \\ \vdots \\ \mathbf{v}_I^N \end{pmatrix}, \mathbf{v}_\Delta = \begin{pmatrix} \mathbf{v}_\Delta^1 \\ \vdots \\ \mathbf{v}_\Delta^N \end{pmatrix}.$$

We define the spaces X_I, W_Δ and W_c which consist of vectors $\mathbf{v}_I$, $\mathbf{v}_\Delta$ and $\mathbf{v}_c$, respectively. Similarly, for $\mathbf{w} = (\mathbf{w}_1^t, \cdots, \mathbf{w}_N^t)^t \in W$, we consider $\mathbf{w}_i$ to be ordered into

$$\mathbf{w}_i = \begin{pmatrix} \mathbf{w}_\Delta^i \\ \mathbf{w}_c^i \end{pmatrix}.$$

Then, for $\mathbf{w} \in W$, we define $\mathbf{w}_\Delta \in W_\Delta$ and $\mathbf{w}_c \in W_c$ as $\mathbf{w}_\Delta|_{\Omega_i} = \mathbf{w}_\Delta^i$ and $L_c^i \mathbf{w}_c = \mathbf{w}_c^i$ for $i = 1, \cdots, N$.

Now, we introduce Sobolev spaces defined on the boundaries of subdomains. For $w_i \in L^2(\partial\Omega_i)$, define

$$|w_i|_{1/2,\partial\Omega_i}^2 := \int_{\partial\Omega_i} \int_{\partial\Omega_i} \frac{|w_i(x) - w_i(y)|^2}{|x-y|^2} \, ds(x) \, ds(y).$$

Then $H^{1/2}(\partial\Omega_i)$ is the trace space of $H^1(\Omega_i)$ normed by

$$\|w_i\|_{1/2,\partial\Omega_i}^2 := |w_i|_{1/2,\partial\Omega_i}^2 + \frac{1}{d_{\partial\Omega_i}} \|w_i\|_{0,\partial\Omega_i}^2,$$

where $d_{\partial\Omega_i}$ is the diameter of $\partial\Omega_i$. For any $\Gamma_{ij} \subset \partial\Omega_i$, $H_{00}^{1/2}(\Gamma_{ij})$ is the set of functions in $L^2(\Gamma_{ij})$ such that the zero extension of the function into $\partial\Omega_i$ is contained in $H^{1/2}(\partial\Omega_i)$. Let

$$|v|^2_{H_{00}^{1/2}(\Gamma_{ij})} := \left(|v|^2_{H^{1/2}(\Gamma_{ij})} + \int_{\Gamma_{ij}} \frac{v(x)^2}{\text{dist}(x, \partial\Gamma_{ij})}\, ds \right)$$

and the norm for $v \in H_{00}^{1/2}(\Gamma_{ij})$ is given by

$$\|v\|_{H_{00}^{1/2}(\Gamma_{ij})} := \left(|v|^2_{H_{00}^{1/2}(\Gamma_{ij})} + \frac{1}{d_{\Gamma_{ij}}} \|v\|^2_{0,\Gamma_{ij}} \right)^{1/2},$$

where $d_{\Gamma_{ij}}$ denotes the diameter of Γ_{ij}. From Section 4.1 in [22], for $v \in H_{00}^{1/2}(\Gamma_{ij})$ we have the following relation:

$$C_1 \|\widetilde{v}\|_{1/2,\partial\Omega_i} \le \|v\|_{H_{00}^{1/2}(\Gamma_{ij})} \le C_2 \|\widetilde{v}\|_{1/2,\partial\Omega_i}, \tag{2.3}$$

where C_1 and C_2 are constants independent of $d_{\Gamma_{ij}}$ and $\widetilde{v}$ is the zero extension of v into $\partial\Omega_i$. For the product spaces $[H^{1/2}(\partial\Omega_i)]^2$ and $[H_{00}^{1/2}(\Gamma_{ij})]^2$, those norms are defined using the product norms and the inequalities in (2.3) also hold.

2.2. **Mortar methods.** Note that the space X is not contained in $[H_0^1(\Omega)]^2$. To approximate the solution of the problem (2.1) in the space X, we impose the mortar matching condition on the velocity functions. Let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Since the triangulations are different across Γ_{ij}, we distinguish them by choosing one as a mortar side and the other as a nonmortar side. Hence, for each subdomain Ω_i, we define

$$m_i := \left\{ j \ : \ \Omega_i^h|_{\Gamma_{ij}} \text{ is the nonmortar side of } \Gamma_{ij} \right\},$$
$$s_i := \left\{ j \ : \ \Omega_i^h|_{\Gamma_{ij}} \text{ is the mortar side of } \Gamma_{ij} \right\}.$$

Then, we can write

$$\overline{\partial\Omega_i \setminus \partial\Omega} = \Big(\bigcup_{j\in m_i} \Gamma_{ij}\Big) \bigcup \Big(\bigcup_{j\in s_i} \Gamma_{ij}\Big).$$

Now, we define the following spaces from the finite elements on the nonmortar sides of interfaces:

$$W_{ij} := \begin{cases} W_i|_{\Gamma_{ij}} & \text{if } j \in m_i, \\ W_j|_{\Gamma_{ij}} & \text{if } j \in s_i, \end{cases}$$
$$W_{ij}^0 := \left\{ \mathbf{w}_{ij} \in W_{ij} \ : \ \mathbf{w}_{ij} \text{ vanishes at the end points of } \Gamma_{ij} \right\},$$
$$W^0 := \prod_{i=1}^{N} \prod_{j\in m_i} W_{ij}^0$$

and consider the standard Lagrange multiplier space M_{ij} introduced by Bernardi *et al.*[3]. On Γ_{ij}, let us denote the triangulation of the nonmortar side by T_{ij}, then the Lagrange multiplier space M_{ij} corresponding to Γ_{ij} is defined as

$$M_{ij} := \{\boldsymbol{\psi} \in \left[C^0(\Gamma_{ij})\right]^2 \ : \ \boldsymbol{\psi}|_\tau \in [P_l(\tau)]^2, \text{ if } \tau \cap \partial\Gamma_{ij} = \emptyset, \ l = 1, \text{ otherwise } l = 0, \forall \tau \in T_{ij}\}.$$

Then we take the Lagrange multiplier space

$$M := \prod_{i=1}^{N} \prod_{j \in m_i} M_{ij}$$

and impose the following mortar matching condition on the velocity functions: For $\mathbf{v} = (\mathbf{v}_1, \cdots, \mathbf{v}_N) \in X$, $\mathbf{v}$ satisfies that

$$\int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds = 0 \quad \forall \boldsymbol{\lambda}_{ij} \in M_{ij}, \forall\, i = 1, \cdots, N, \forall j \in m_i. \tag{2.4}$$

Let us define the spaces

$$V := \{\mathbf{v} \in X \ : \ \mathbf{v} \text{ satisfies (2.4)}\},$$
$$P := \left\{q \in L_0^2(\Omega) \ : \ q|_{\Omega_i} \in Q_i \ \ \forall\, i = 1, \cdots, N\right\}$$

for the velocity and pressure, respectively. The space P is written into a direct sum of the L^2-orthogonal spaces Q and Π^0, that is,

$$P = Q \oplus \Pi^0.$$

When Hood-Taylor finite elements $P_2(h) - P_1(h)$ are used for each subdomain, it was shown in [2] that the best approximation property holds for the approximation space $V \times P$. For $P_2(h) - P_1(h)$ finite elements, the spaces M, V and P are defined similarly to the $P_1(h) - P_0(2h)$ finite elements. The inf-sup constant of the space $V \times P$ is crucial in the analysis of the approximation order. If the inf-sup constant is independent of mesh size and subdomain size then the best approximation property holds. It was also shown that the inf-sup constant is independent of the mesh size. However, it was not proved for the subdomain size. Following the similar idea to Belgacem [2], we can see that the inf-sup constant of the space $V \times P$ is independent of the mesh size for $P_1(h) - P_0(2h)$ finite elements. For the subdomain size H, we compute the inf-sup constant numerically and observe that the constant seems to be independent of H(see Section 5).

Now, we will rewrite (2.4) into a matrix form. Let B_i^{ij} be a matrix with entries

$$(B_i^{ij})_{lk} = \pm \int_{\Gamma_{ij}} \boldsymbol{\psi}_l \cdot \boldsymbol{\phi}_k \, ds \quad \forall l = 1, \cdots, L, \forall k = 1, \cdots, K, \tag{2.5}$$

where $\{\boldsymbol{\psi}_l\}_{l=1}^L$ is basis for M_{ij} and $\{\boldsymbol{\phi}_k\}_{k=1}^K$ is nodal basis for $W_i|_{\Gamma_{ij}}$. Here, $W_i|_{\Gamma_{ij}}$ means the restriction of functions in W_i on Γ_{ij}. In (2.5), the $+$sign is chosen if $\Omega_i|_{\Gamma_{ij}}$ is a nonmortar side, otherwise the $-$sign is chosen. Then we rewrite (2.4) as

$$B_i^{ij} \mathbf{v}_i|_{\Gamma_{ij}} + B_j^{ij} \mathbf{v}_j|_{\Gamma ij} = \mathbf{0} \quad \forall i = 1, \cdots, N, \forall j \in m_i. \tag{2.6}$$

Define $E_{ij} : M_{ij} \to M$ to be an extension operator by zero and $R^l_{ij} : W_l \to W_l|_{\Gamma_{ij}}$ for $l = i, j$ to be a restriction operator and let $B_i = \sum_{j \in m_i \cup s_i} E_{ij} B^{ij}_i R^i_{ij}$. Then (2.6) becomes

$$B\mathbf{w} = \mathbf{0}, \tag{2.7}$$

where

$$B = \begin{pmatrix} B_1 & \cdots & B_N \end{pmatrix},$$
$$\mathbf{w} = \begin{pmatrix} \mathbf{w}_1^t & \cdots & \mathbf{w}_N^t \end{pmatrix}^t \text{ with } \mathbf{w}_i = \mathbf{v}_i|_{\partial\Omega_i},\ \forall i = 1, \cdots, N.$$

We call B as the connectivity matrix borrowing the term from the FETI formulation with conforming discretizations. Let $B_{i,\Delta}$ and $B_{i,c}$ be matrices that consist of the columns of B_i corresponding to the d.o.f. on edges and corners, respectively. Then, using the notations introduced in Section 2, (2.7) is written into

$$B_\Delta \mathbf{w}_\Delta + B_c \mathbf{w}_c = \mathbf{0}, \tag{2.8}$$

where $B_\Delta = \begin{pmatrix} B_{1,\Delta} & \cdots & B_{N,\Delta} \end{pmatrix}$ and $B_c = \sum_{i=1}^N B_{i,c} L^i_c$.

3. FETI-DP FORMULATION

3.1. **FETI-DP operator.** In this section, we formulate a FETI-DP operator with the continuity constraints (2.8) obtained from the mortar matching condition (2.4). To solve the Stokes problem efficiently and correctly, we will add the redundant continuity constraints to the coarse problem:

$$\int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j)\, ds = \mathbf{0} \quad \forall i = 1, \cdots, N, \forall j \in m_i. \tag{3.1}$$

In the FETI-DP method, the mortar matching condition holds when the solution has converged. Hence, adding the redundant constraints to the coarse problem enhances the convergence of the FETI-DP method. When preconditioning the FETI-DP operator, we solve a Dirichlet problem, i.e. a local Stokes problem, in each subdomain. Furthermore, the compatibility condition of the local Stokes problem follows from the redundant constraints.

We rewrite (3.1) as

$$R^t(B_\Delta \mathbf{w}_\Delta + B_c \mathbf{w}_c) = \mathbf{0}, \tag{3.2}$$

where the matrix R has the number of columns corresponding to two times of the number of Γ_{ij}'s and rows corresponding to the d.o.f. on the space M and has entries 1 and 0. For $\boldsymbol{\lambda} \in M$, at each interior nodal point of Γ_{ij}, $\boldsymbol{\lambda}|_{\Gamma_{ij}}$ has two components corresponding to horizonal and vertical parts of velocity function. For $\boldsymbol{\lambda} \in M$, $R^t\boldsymbol{\lambda} = \mathbf{0}$ means that for all Γ_{ij}, the sums of $\boldsymbol{\lambda}|_{\Gamma_{ij}}$ corresponding to the horizonal and vertical parts of velocity function are zero.

Let N be the Lagrange multiplier space corresponding to the constraints (3.2) and for $\boldsymbol{\mu} \in N$, $\boldsymbol{\mu}|_{\Gamma_{ij}}$ has two components that correspond to the constraints for horizontal velocity and vertical velocity. Introducing Lagrange multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ to enforce the constraints (2.8) and (3.2), the followings are induced from the Galerkin approximation to (2.2):

Find $(\mathbf{u}_I, p_I, \mathbf{u}_\Delta, \mathbf{u}_c, p^0, \boldsymbol{\mu}, \boldsymbol{\lambda}) \in X_I \times Q \times W_\Delta \times W_c \times \Pi^0 \times N \times M$ such that

$$\begin{pmatrix} A_{II} & G_{II} & A_{I\Delta} & A_{Ic} & G_{I0} & \mathbf{0} & \mathbf{0} \\ G_{II}^t & \mathbf{0} & G_{\Delta I}^t & G_{cI}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{\Delta I} & G_{\Delta I} & A_{\Delta\Delta} & A_{\Delta c} & G_{\Delta 0} & B_\Delta^t R & B_\Delta^t \\ A_{cI} & G_{cI} & A_{c\Delta} & A_{cc} & G_{c0} & B_c^t R & B_c^t \\ G_{I0}^t & \mathbf{0} & G_{\Delta 0}^t & G_{c0}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R^t B_\Delta & R^t B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_\Delta & B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_c \\ p^0 \\ \boldsymbol{\mu} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{0} \\ \mathbf{f}_\Delta \\ \mathbf{f}_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \tag{3.3}$$

Here

$\begin{pmatrix} A_{II} & A_{I\Delta} & A_{Ic} \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Delta c} \\ A_{cI} & A_{c\Delta} & A_{cc} \end{pmatrix}$ is a stiffness matrix induced from $\sum_{i=1}^{N} (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i}$,

$\begin{pmatrix} G_{II} \\ G_{\Delta I} \\ G_{cI} \end{pmatrix}$ is a matrix induced from $\sum_{i=1}^{N} (-\nabla \cdot \mathbf{v}, p_I)_{\Omega_i}$,

$\begin{pmatrix} G_{I0} \\ G_{\Delta 0} \\ G_{c0} \end{pmatrix}$ is a matrix induced from $\sum_{i=1}^{N} (-\nabla \cdot \mathbf{v}, p^0)_{\Omega_i}$

and the subscripts I, Δ, and c denote the interior, edges and corners, respectively. Since $p^0|_{\Omega_i}$ is constant, we have $G_{I0} = \mathbf{0}$. Let

$$\mathbf{z}_\Delta = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \end{pmatrix}, \mathbf{z}_c = \begin{pmatrix} \mathbf{u}_c \\ p^0 \\ \boldsymbol{\mu} \end{pmatrix}.$$

We regard $\mathbf{z}_c$ as a primal variable. Then (3.3) can be written as

$$\begin{pmatrix} K_{\Delta\Delta} & K_{\Delta c} & \widetilde{B}_\Delta^t \\ K_{\Delta c}^t & K_{cc} & \widetilde{B}_c^t \\ \widetilde{B}_\Delta & \widetilde{B}_c & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_\Delta \\ \mathbf{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{f}}_\Delta \\ \widetilde{\mathbf{f}}_c \\ \mathbf{0} \end{pmatrix}.$$

After eliminating $\mathbf{z}_\Delta$, we obtain the following equation for $\mathbf{z}_c$ and $\boldsymbol{\lambda}$:

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{cl}^t & F_{ll} \end{pmatrix} \begin{pmatrix} \mathbf{z}_c \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}_c \\ \mathbf{d}_l \end{pmatrix}$$

where

$$\begin{aligned} F_{ll} &= \widetilde{B}_\Delta K_{\Delta\Delta}^{-1} \widetilde{B}_\Delta^t, \\ F_{cl} &= K_{\Delta c}^t K_{\Delta\Delta}^{-1} \widetilde{B}_\Delta^t - \widetilde{B}_c^t, \\ F_{cc} &= K_{cc} - K_{\Delta c}^t K_{\Delta\Delta}^{-1} K_{\Delta c}, \\ \mathbf{d}_l &= \widetilde{B}_\Delta K_{\Delta\Delta}^{-1} \widetilde{\mathbf{f}}_\Delta, \\ \mathbf{d}_c &= \widetilde{\mathbf{f}}_c - K_{\Delta c}^t K_{\Delta\Delta}^{-1} \widetilde{f}_\Delta. \end{aligned}$$

Note that $\begin{pmatrix} G_{\Delta 0} & B_\Delta^t R \\ G_{c0} & B_c^t R \end{pmatrix} \begin{pmatrix} p^0 \\ \mu \end{pmatrix} = \mathbf{0}$ implies that $\begin{pmatrix} p^0 \\ \mu \end{pmatrix} = \mathbf{0}$. Using this it can be shown easily that F_{cc} is invertible. Hence eliminating $\mathbf{z}_c$, we obtain the following equation for $\boldsymbol{\lambda}$:

$$(F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl})\boldsymbol{\lambda} = \mathbf{d}_l - F_{cl}^t F_{cc}^{-1}\mathbf{d}_c. \tag{3.4}$$

Let $F_{DP} = F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl}$ and call it the FETI-DP operator. Since we add the redundant constraints to the coarse problem, $\boldsymbol{\lambda}$ is not uniquely determined in M. Let us define

$$M_R = \left\{\boldsymbol{\lambda} \in M \ : \ R^t\boldsymbol{\lambda} = \mathbf{0}\right\}. \tag{3.5}$$

In Section 4, we will show that F_{DP} is symmetric and positive definite on M_R and $\boldsymbol{\lambda} \in M_R$ is uniquely determined. In the following section, we define several norms on the finite element function spaces and propose a preconditioner for the operator F_{DP}.

3.2. **Preconditioner.** For $\mathbf{w}_i \in W_i$, we define $S_i\mathbf{w}_i$ by

$$\begin{pmatrix} A_{II}^i & G_{II}^i & A_{I\Delta}^i & A_{Ic}^i \\ {G_{II}^i}^t & \mathbf{0} & {G_{\Delta I}^i}^t & {G_{cI}^i}^t \\ A_{\Delta I}^i & G_{\Delta I}^i & A_{\Delta\Delta}^i & A_{\Delta c}^i \\ A_{cI}^i & G_{cI}^i & A_{c\Delta}^i & A_{cc}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \\ \mathbf{w}_\Delta^i \\ \mathbf{w}_c^i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ S_i \begin{pmatrix} \mathbf{w}_\Delta^i \\ \mathbf{w}_c^i \end{pmatrix} \end{pmatrix},$$

where the superscript i for a matrix denotes the part of the matrix corresponding to subdomain Ω_i.

Let us define

$$S := \operatorname{diag}(S_1, \cdots, S_N)$$

and it can be seen easily that S is a symmetric and positive definite(s.p.d.) operator on W. Hence, we define

$$\|\mathbf{w}\|_W := \left(\sum_{i=1}^N < S_i\mathbf{w}_i, \mathbf{w}_i >\right)^{1/2} \tag{3.6}$$

as a norm for $\mathbf{w} \in W$. Here, $< \cdot, \cdot >$ denotes the l^2-inner product of vectors. For a function $\mathbf{w}_{ij} \in W_{ij}^0$ with $j \in m_i$, let $\widetilde{\mathbf{w}}_{ij}$ be the zero extension of $\mathbf{w}_{ij}$ into W_i. Using this, for $\mathbf{w} \in W^0$ we define an extension $\widetilde{\mathbf{w}} \in W$ by

$$\widetilde{\mathbf{w}} = (\widetilde{\mathbf{w}}_1, \cdots, \widetilde{\mathbf{w}}_N) \text{ with } \widetilde{\mathbf{w}}_i = \sum_{j\in m_i} \widetilde{\mathbf{w}}_{ij} \ \forall i = 1, \cdots, N,$$

and define a norm on W^0 by

$$\|\mathbf{w}\|_{W^0} := \|\widetilde{\mathbf{w}}\|_W. \tag{3.7}$$

We introduce the following subspaces with the norms induced from the spaces W and W^0:

$$\begin{aligned} W_R &:= \left\{ \mathbf{w} \in W \ : \ R^t(B_\Delta \mathbf{w}_\Delta + B_c \mathbf{w}_c) = \mathbf{0} \right\}, \\ W_{R,G} &:= \left\{ \mathbf{w} \in W_R \ : \ G^t_{\Delta 0} \mathbf{w}_\Delta + G^t_{c0} \mathbf{w}_c = \mathbf{0} \right\} \\ W^0_R &:= \left\{ \mathbf{w} \in W^0 \ : \ \widetilde{\mathbf{w}} \in W_R \right\}. \end{aligned}$$

Recall the definition of M_R in (3.5) and let $< \cdot, \cdot >_m$ be a duality pairing between M_R and W^0_R defined as

$$< \boldsymbol{\lambda}, \mathbf{w} >_m = \sum_{i=1}^N \sum_{j \in m_i} \int_{\Gamma_{ij}} \boldsymbol{\lambda}_{ij} \cdot \mathbf{w}_{ij} \, ds.$$

Then we define a dual norm for $\boldsymbol{\lambda} \in M_R$ by

$$\|\boldsymbol{\lambda}\|^2_{M_R} := \max_{\mathbf{w} \in W^0_R \setminus \{0\}} \frac{< \boldsymbol{\lambda}, \mathbf{w} >^2_m}{\|\mathbf{w}\|^2_{W^0}}. \tag{3.8}$$

Now, we will find an operator $\widehat{F}_{DP}$ which gives

$$< \widehat{F}_{DP} \boldsymbol{\lambda}, \boldsymbol{\lambda} > = \|\boldsymbol{\lambda}\|^2_{M_R} \tag{3.9}$$

and propose $\widehat{F}^{-1}_{DP}$ as a preconditioner for the FETI-DP operator in (3.4). Define $R_{ij} : W^0 \to W^0_{ij}$ as a restriction operator and $E^i_{ij} : W^0_{ij} \to W_i$ as an extension operator by zero. Then for $\mathbf{w} \in W^0_R$,

$$\begin{aligned} \|\mathbf{w}\|^2_{W^0} &= \|\widetilde{\mathbf{w}}\|^2_W \\ &= \sum_{i=1}^N < S_i \widetilde{\mathbf{w}}_i, \widetilde{\mathbf{w}}_i > \\ &= \sum_{i=1}^N < S_i (\sum_{j \in m_i} E^i_{ij} R_{ij} \mathbf{w}), \sum_{j \in m_i} E^i_{ij} R_{ij} \mathbf{w} > . \end{aligned}$$

Let $\widehat{S} = \sum_{i=1}^N (\sum_{j \in m_i} E^i_{ij} R_{ij})^t S_i (\sum_{j \in m_i} E^i_{ij} R_{ij})$. Moreover, we have

$$< \boldsymbol{\lambda}, \mathbf{w} >_m = < \widehat{B} \mathbf{w}, \boldsymbol{\lambda} > \tag{3.10}$$

where $\widehat{B} = \text{diag}_{i=1,\cdots,N} \left(\text{diag}_{j \in m_i} \widehat{B}^{ij}_i \right)$ and $\widehat{B}^{ij}_i$ is a matrix obtained from B^{ij}_i after deleting the columns corresponding to the d.o.f. at the end points of Γ_{ij}. Note that $\widehat{B}^{ij}_i$ is invertible. Since, we restrict $\boldsymbol{\lambda} \in M_R$ and $\mathbf{w} \in W^0_R$, to find $\widehat{F}_{DP}$ in a matrix form we need the following l^2-orthogonal projections:

$$\begin{aligned} P_{W^0_R} &: W^0 \to W^0_R, \\ P_{M_R} &: M \to M_R. \end{aligned}$$

For $\boldsymbol{\lambda} \in M_R$ and $\mathbf{w} \in W^0_R$, we may write

$$< \boldsymbol{\lambda}, \mathbf{w} >_m = < \widehat{B}_p \mathbf{w}, \boldsymbol{\lambda} >, \quad \|\mathbf{w}\|^2_{W^0} = < \widehat{S}_p \mathbf{w}, \mathbf{w} >, \tag{3.11}$$

where

$$\widehat{S}_p = P_{W_R^0} \widehat{S} P_{W_R^0}, \quad \widehat{B}_p = P_{M_R} \widehat{B} P_{W_R^0}.$$

Then it can be shown that the operators

$$\widehat{S}_p : W_R^0 \to W_R^0,$$
$$\widehat{B}_p : W_R^0 \to M_R$$

are invertible and $\widehat{S}_p$ is s.p.d. on W_R^0. Hence, using (3.11), the maximum in (3.8) occurs when $\mathbf{w} \in W_R^0$ satisfies $\widehat{S}_p \mathbf{w} = \widehat{B}_p^t \boldsymbol{\lambda}$. Therefore, we have

$$\|\boldsymbol{\lambda}\|_{M_R}^2 = < \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t \boldsymbol{\lambda}, \boldsymbol{\lambda} > .$$

Let

$$\widehat{F}_{DP}^{-1} = (\widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t)^{-1} = (\widehat{B}_p^t)^{-1} \widehat{S}_p \widehat{B}_p^{-1}.$$

and we call it a Neumann-Dirichlet preconditioner for the F_{DP} operator.

Define the l^2-orthogonal projections

$$P_{W_R^0}^{ij} : W^0|_{\Gamma_{ij}} \to W_R^0|_{\Gamma_{ij}},$$
$$P_{M_R}^{ij} : M|_{\Gamma_{ij}} \to M_R|_{\Gamma_{ij}}.$$

Then the projection operators $P_{W_R^0}$ and P_{M_R} are composed of diagonal blocks of $P_{W_R^0}^{ij}$'s and $P_{M_R}^{ij}$'s, respectively. Moreover, it can be shown easily that

$$P_{M_R}^{ij} \widehat{B}_i^{ij} P_{W_R^0}^{ij} : W_R^0|_{\Gamma_{ij}} \to M_R|_{\Gamma_{ij}}$$

is invertible. Hence, it follows that

$$\widehat{B}_p^{-1} = \text{diag}_{i=1,\cdots,N} \text{diag}_{j \in m_i} \left(\widehat{B}_{ij}^{-1} \right),$$

where $\widehat{B}_{ij} = P_{M_R}^{ij} \widehat{B}_i^{ij} P_{W_R^0}^{ij}$ and

$$\widehat{F}_{DP}^{-1} = \sum_{i=1}^{N} \left(\sum_{j \in m_i} E_{ij}^i \widehat{B}_{ij}^{-1} R_{ij} \right)^t S_i \left(\sum_{j \in m_i} E_{ij}^i \widehat{B}_{ij}^{-1} R_{ij} \right).$$

Therefore, the computation of $\widehat{F}_{DP}^{-1} \boldsymbol{\lambda}$ can be done parallely in each subdomain.

4. Condition number estimation

Lemma 4.1. *We have*

$$B(W_{R,G}) = B(W_R) = M_R.$$

Proof. Since $W_{R,G} \subset W_R$, $B(W_{R,G}) \subset B(W_R)$.

Now, we will show that $B(W_R) \subset B(W_{R,G})$. For $\mathbf{w} \in W^0$, we consider $\widetilde{\mathbf{w}} = (\widetilde{\mathbf{w}}_1, \cdots, \widetilde{\mathbf{w}}_N) \in W$, the zero extension of $\mathbf{w}$ into the space W. Since $\widetilde{\mathbf{w}}_j|_{\Gamma_{ij}} = 0$ for $j \in m_i$ and $\widetilde{\mathbf{w}}$ is zero at subdomain corners, we have

$$B\widetilde{\mathbf{w}} = \widehat{B}\mathbf{w}, \tag{4.1}$$

where $\widehat{B}$ is defined in (3.10). Since $\widehat{B}$ is a $1-1$ mapping from W^0 onto M, from the definition of W_R^0 and M_R, we get

$$\widehat{B}(W_R^0) = M_R. \tag{4.2}$$

For $\mathbf{w} \in W_R^0$, the zero extension $\widetilde{\mathbf{w}} = (\widetilde{\mathbf{w}}_1, \cdots, \widetilde{\mathbf{w}}_N)$ satisfies

$$\int_{\partial\Omega_i} \widetilde{\mathbf{w}}_i \, ds = \mathbf{0} \quad \forall\, i = 1, \cdots, N$$

and then applying divergence theorem

$$G_{\Delta 0}^t \widetilde{\mathbf{w}}_\Delta + G_{c0}^t \widetilde{\mathbf{w}}_c = \mathbf{0}$$

holds for $\widetilde{\mathbf{w}}$. Hence, for $\mathbf{w} \in W_R^0$, we have $\widetilde{\mathbf{w}} \in W_{R,G}$ and from (4.1) we obtain

$$\widehat{B}(W_R^0) \subset B(W_{R,G}). \tag{4.3}$$

From the definition of W_R and M_R,

$$B(W_R) = M_R. \tag{4.4}$$

Combining (4.4), (4.2) and (4.3), we have that $B(W_R) \subset B(W_{R,G})$. □

Remark 4.1. *For* $\mathbf{w} \in W_R^0$*, we have* $\widetilde{\mathbf{w}} \in W_{R,G}$*.*

Lemma 4.2. *For* $\boldsymbol{\lambda} \in M_R$*, we have*

$$< F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} >= \max_{\mathbf{w} \in W_{R,G} \setminus \{0\}} \frac{< B\mathbf{w}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|_W^2}.$$

Proof. The problem (3.3) is equivalent to solving the following min-max problem:

$$\max_{\boldsymbol{\lambda} \in B(W_{R,G})} \min_{\mathbf{w} \in W_{R,G}} \left\{ \sum_{i=1}^N \left(\frac{1}{2} < S_i \mathbf{w}_i, \mathbf{w}_i > - < \mathbf{d}_i, \mathbf{w}_i > \right) + < B\mathbf{w}, \boldsymbol{\lambda} > \right\}, \tag{4.5}$$

where $\mathbf{d}_i$ is the Schur complement forcing vector obtained from $\begin{pmatrix} \mathbf{f}_I^t & \mathbf{0}^t & \mathbf{f}_\Delta^t & \mathbf{f}_c^t \end{pmatrix}^t$ after solving Stokes problem in each subdomain Ω_i.

Define $P_{W_{R,G}}$ as the l^2-orthogonal projection from W onto $W_{R,G}$. Note that from Lemma 4.1, $B(W_{R,G}) = M_R$ and P_M is the projection operator from M onto M_R introduced in Section 3. From $(\mathbf{u}_\Delta, \mathbf{u}_c)$ in (3.3), let us define $\mathbf{w} = (\mathbf{w}_1, \cdots, \mathbf{w}_N) \in W_{R,G}$ such that $\mathbf{w}_i = \begin{pmatrix} \mathbf{u}_\Delta^i \\ L_c^i \mathbf{u}_c \end{pmatrix}$. Then taking Euler-Lagrangian in (4.5), we can see that the solution $(\mathbf{w}, \boldsymbol{\lambda}) \in W_{R,G} \times M_R$ of (3.3) satisfies

$$\begin{pmatrix} S_p & B_p^t \\ B_p & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} P_{W_{R,G}} \mathbf{d} \\ \mathbf{0} \end{pmatrix}, \tag{4.6}$$

where

$$S_p = P_{W_{R,G}} S P_{W_{R,G}}, \quad B_p = P_{M_R} B P_{W_{R,G}},$$
$$\mathbf{d} = \begin{pmatrix} \mathbf{d}_1^t & \cdots & \mathbf{d}_N^t \end{pmatrix}^t.$$

Since S is s.p.d. on $W_{R,G}$, the equation for $\boldsymbol{\lambda}$ follows by eliminating $\mathbf{w}$ in (4.6):

$$B_p S_p^{-1} B_p^t \boldsymbol{\lambda} = B_p S_p^{-1} \mathbf{d}, \tag{4.7}$$

which is the same as (3.4). Therefore we have

$$B_p S_p^{-1} B_p^t = F_{DP}. \tag{4.8}$$

For $\boldsymbol{\lambda} \in M_R$, consider

$$\max_{\mathbf{w} \in W_{R,G} \backslash \{0\}} \frac{< B\mathbf{w}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|_W^2}. \tag{4.9}$$

From (3.6), the definition of $\| \cdot \|_W$, we may write

$$\|\mathbf{w}\|_W^2 =< S\mathbf{w}, \mathbf{w} > .$$

Since S is s.p.d. on $W_{R,G}$, the maximum in (4.9) occurs when $\mathbf{w} \in W_{R,G}$ satisfies $S_p \mathbf{w} = B_p^t \boldsymbol{\lambda}$. Hence, we have

$$\max_{\mathbf{w} \in W_{R,G} \backslash \{0\}} \frac{< B\mathbf{w}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|_W^2} =< B_p S_p^{-1} B_p^t \boldsymbol{\lambda}, \boldsymbol{\lambda} > . \tag{4.10}$$

Combining (4.8) and (4.10), we complete the proof. □

Lemma 4.3. *For $\boldsymbol{\lambda} \in M_R$, we have*

$$\max_{\mathbf{w} \in W_{R,G} \backslash \{0\}} \frac{< B\mathbf{w}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|_W^2} \geq \|\boldsymbol{\lambda}\|_{M_R}^2.$$

Proof. For $\mathbf{w} \in W_R^0$, $\widetilde{\mathbf{w}} \in W_{R,G}$ and from (3.7), we obtain

$$\max_{\mathbf{w} \in W_{R,G} \backslash \{0\}} \frac{< B\mathbf{w}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|_W^2} \geq \max_{\mathbf{w} \in W_R^0 \backslash \{0\}} \frac{< B\widetilde{\mathbf{w}}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|_{W^0}^2}. \tag{4.11}$$

Since $\widetilde{\mathbf{w}}_j|_{\Gamma_{ij}} = 0$ for $j \in m_i$, we get

$$< B\widetilde{\mathbf{w}}, \boldsymbol{\lambda} >=< \widehat{B}\mathbf{w}, \boldsymbol{\lambda} >=< \boldsymbol{\lambda}, \mathbf{w} >_m . \tag{4.12}$$

Combining (4.11), (4.12) and (3.8), we complete the proof. □

Let us define a notation $| \cdot |_{S_i} :=< S_i \cdot, \cdot >^{1/2}$. Then the following lemma can be found in Bramble and Pasciak[5].

Lemma 4.4. *For $\mathbf{w}_i \in W_i$, we have*

$$C_1 \beta |\mathbf{w}_i|_{S_i} \leq |\mathbf{w}_i|_{1/2, \partial\Omega_i} \leq C_2 |\mathbf{w}_i|_{S_i},$$

where β is the inf-sup constant for the finite elements of subdomain Ω_i and the constants C_1 and C_2 are independent of h_i and H_i.

Since we have chosen inf-sup stable $P_1(h) - P_0(2h)$ finite elements for each subdomain, the constant β is independent of h_i and H_i. Therefore, we have

$$C_1 |\mathbf{w}_i|_{S_i} \leq |\mathbf{w}_i|_{1/2, \partial\Omega_i} \leq C_2 |\mathbf{w}_i|_{S_i}, \tag{4.13}$$

where C_1 and C_2 are constants independent of h_i and H_i.

We also have the following result which is derived from the Lemma 5.1 in [15].

Lemma 4.5. *For $\mathbf{w} \in W$, we have*

$$\|\mathbf{w}_i - \mathbf{w}_j\|^2_{H^{1/2}_{00}(\Gamma_{ij})} \leq C \max_{l \in \{i,j\}} \left\{ \left(1 + \log \frac{H_l}{h_l}\right)^2 \right\} \left(|\mathbf{w}_i|^2_{1/2,\partial\Omega_i} + |\mathbf{w}_j|^2_{1/2,\partial\Omega_j} \right),$$

where $\mathbf{w}_i$ is the restriction of $\mathbf{w}$ onto $\partial\Omega_i$ for $i = 1, \cdots, N$ and C is a constant independent of h_i's and H_i's.

Definition 4.1. *We define a projection $\pi_{ij} : [H^{1/2}_{00}(\Gamma_{ij})]^2 \to W^0_{ij}$ for $\mathbf{v} \in [H^{1/2}_{00}(\Gamma_{ij})]^2$ by*

$$\int_{\Gamma_{ij}} (\mathbf{v} - \pi_{ij}\mathbf{v}) \cdot \boldsymbol{\lambda}_{ij} \, ds \; = 0 \;\; \forall \boldsymbol{\lambda}_{ij} \in M_{ij}.$$

From Lemma 2.2 in [1], π_{ij} is a continuous operator on $H^{1/2}_{00}(\Gamma_{ij})$, i.e., there exists a constant C such that

$$\|\pi_{ij}\mathbf{v}\|_{H^{1/2}_{00}(\Gamma_{ij})} \leq C\|\mathbf{v}\|_{H^{1/2}_{00}(\Gamma_{ij})} \;\; \forall \mathbf{v} \in [H^{1/2}_{00}(\Gamma_{ij})]^2 \tag{4.14}$$

and the constant C is independent of H_i's and h_i's.

Lemma 4.6. *For $\boldsymbol{\lambda} \in M_R$, we have*

$$\max_{\mathbf{w} \in W_{R,G} \backslash \{0\}} \frac{< B\mathbf{w}, \boldsymbol{\lambda} >^2}{\|\mathbf{w}\|^2_W} \leq C \max_{i=1,\cdots,N} \left\{ \left(1 + \log \frac{H_i}{h_i}\right)^2 \right\} \|\boldsymbol{\lambda}\|^2_{M_R},$$

where C is a constant independent of h_i's and H_i's.

Proof. Note that

$$< B\mathbf{w}, \boldsymbol{\lambda} >= \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} (\mathbf{w}_i - \mathbf{w}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds.$$

Since $\mathbf{w}_i - \mathbf{w}_j \in [H^{1/2}_{00}(\Gamma_{ij})]^2$, from the definition of π_{ij}, we have

$$< B\mathbf{w}, \boldsymbol{\lambda} >= \sum_{i=1}^{N} \sum_{j \in m_i} \int_{\Gamma_{ij}} \pi_{ij}(\mathbf{w}_i - \mathbf{w}_j) \cdot \boldsymbol{\lambda}_{ij} \, ds. \tag{4.15}$$

Let $\mathbf{z}_{ij} = \pi_{ij}(\mathbf{w}_i - \mathbf{w}_j)$ and $\mathbf{z} \in W^0$ with $\mathbf{z}|_{\Gamma_{ij}} = \mathbf{z}_{ij}$. Since $\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} \in M_{ij}$ and $\mathbf{w} \in W_{R,G}$,

$$\int_{\Gamma_{ij}} \mathbf{z}_{ij} \, ds = \int_{\Gamma_{ij}} (\mathbf{w}_i - \mathbf{w}_j) \, ds = 0. \tag{4.16}$$

From (4.16), we can see that $R^t B\widetilde{\mathbf{z}} = 0$ with $\widetilde{\mathbf{z}}$ as the zero extension of $\mathbf{z}$. Hence, $\mathbf{z} \in W^0_R$ and (4.15) is the duality pairing between $\mathbf{z} \in W^0_R$ and $\boldsymbol{\lambda} \in M_R$. From (3.8), we get

$$< B\mathbf{w}, \boldsymbol{\lambda} >^2 = < \boldsymbol{\lambda}, \mathbf{z} >^2_m \leq \|\boldsymbol{\lambda}\|^2_{M_R} \|\mathbf{z}\|^2_{W^0}. \tag{4.17}$$

From (3.7), (3.6), (4.13), (2.3), (4.14) and Lemma 4.5, we obtain

$$
\begin{aligned}
\|\mathbf{z}\|_{W^0}^2 &= \|\widetilde{\mathbf{z}}\|_W^2 \\
&= \sum_{i=1}^N |\widetilde{\mathbf{z}}_i|_{S_i}^2 \\
&\le C \sum_{i=1}^N |\widetilde{\mathbf{z}}_i|_{1/2,\partial\Omega_i}^2 \\
&\le C \sum_{i=1}^N \sum_{j\in m_i} \|\mathbf{w}_i - \mathbf{w}_j\|_{H_{00}^{1/2}(\Gamma_{ij})}^2 \\
&\le C \max_{i=1,\cdots,N} \left\{ \left(1+\log\frac{H_i}{h_i}\right)^2 \right\} \sum_{i=1}^N |\mathbf{w}_i|_{1/2,\partial\Omega_i}^2 \\
&\le C \max_{i=1,\cdots,N} \left\{ \left(1+\log\frac{H_i}{h_i}\right)^2 \right\} \|\mathbf{w}\|_W^2 .
\end{aligned} \tag{4.18}
$$

Here, C is a generic constant which is independent of h_i's and H_i's.

Combining (4.17) and (4.18), we complete the proof. □

From Lemma 4.2, Lemma 4.3 and Lemma 4.6, we have

Theorem 4.1. *For $\boldsymbol{\lambda} \in M_R$,*

$$
\|\boldsymbol{\lambda}\|_{M_R}^2 \le < F_{DP}\boldsymbol{\lambda}, \boldsymbol{\lambda} > \le C \max_{i=1,\cdots,N} \left\{ \left(1+\log\frac{H_i}{h_i}\right)^2 \right\} \|\boldsymbol{\lambda}\|_{M_R}^2,
$$

where C is a constant independent of h_i's and H_i's.

Consequently, from (3.9) we obtain the following condition number estimate:

Corollary 4.1.

$$
\kappa(\widehat{F}_{DP}^{-1} F_{DP}) \le C \max_{i=1,\cdots,N} \left\{ \left(1+\log\frac{H_i}{h_i}\right)^2 \right\}.
$$

5. Numerical Results

In this section, we provide numerical tests for the FETI-DP formulation developed in this paper. Let $\Omega = [0,1]\times[0,1] \subset \mathbb{R}^2$ and consider the following problem:

$$
\begin{aligned}
-\triangle \mathbf{u} + \nabla p &= \mathbf{f} \ \text{ in } \Omega, \\
-\nabla\cdot\mathbf{u} &= 0 \ \text{ in } \Omega, \\
\mathbf{u} &= \mathbf{0} \ \text{ on } \partial\Omega,
\end{aligned} \tag{5.1}
$$

where $\mathbf{f}$ is chosen so that the exact solution of the problem becomes

$$
\mathbf{u} = \begin{pmatrix} \sin^3(\pi x)\sin^2(\pi y)\cos(\pi y) \\ -\sin^2(\pi x)\sin^3(\pi y)\cos(\pi x) \end{pmatrix} \ \text{ and } \ p = x^2 - y^2.
$$

TABLE 1. CG iterations(condition number) when $N = 4 \times 4$

n	Matching		Nonmatching	
	FETI-DP	PFETI-DP	FETI-DP	PFETI-DP
5	12(5.23)	9(2.62)	16(8.35)	12(3.75)
9	24(2.50e+1)	13(4.39)	50(1.15e+2)	15(5.79)
17	37(6.68e+1)	15(5.94)	86(5.01e+2)	17(7.93)
33	45(1.45e+2)	17(7.75)	119(1.31e+3)	20(9.88)
65	58(2.69e+2)	19(9.85)	153(3.29e+3)	22(1.20e+1)

Let N denote the number of subdomains. We only consider the uniform partition of Ω. The notation $N = 4 \times 4$ means that Ω is partitioned into 4×4 square subdomains. With this partition, we triangulate each subdomain in the following manner. For all subdomains, we take the same number of nodes n, including end points, in horizontal and vertical edges with $n = 4k + 1$ for some positive integer k. We solve (5.1) on matching and nonmatching grids both. For matching grids, we make uniform triangulations in each subdomain with $(n-1)/2+1$ nodes on horizontal and vertical edges of subdomain and denote it by $\Omega_i^{2h_i}$, a triangulation for the pressure. After bisecting each edge of triangles in $\Omega_i^{2h_i}$, we obtain $\Omega_i^{h_i}$, a triangulation for the velocity. For nonmatching grids, we take $(n-1)/2+1$ random quasi-uniform nodes on each horizontal and vertical edges of subdomain, and generate nonuniform structured triangulations. We denote it by $\Omega_i^{2h_i}$. The triangulation $\Omega_i^{h_i}$ is obtained from $\Omega_i^{2h_i}$ similarly to matching grids.

Now, we solve the FETI-DP operator with and without preconditioner varying N and n. Those cases are denoted by PFETI-DP and FETI-DP, respectively. The CG(Conjugate Gradient) iteration is stopped when the relative residual is less than 10^{-6}.

In Tables 1-3, the number of CG iterations and the corresponding condition number are shown varying N and n. In Table 1, $N = 4 \times 4$ and $n-1$ increases by double. On both matching and nonmatching grids, PFETI-DP performs well and the condition numbers seem to behave $\log^2$-growth as n increases. Especially on nonmatching grids, the CG iteration stops quickly in PFETI-DP compared with FETI-DP. In Tables 2 and 3, N increases with $n = 5$ and $n = 9$. For both cases of FETI-DP and PFETI-DP, the CG iteration becomes stable as N increases. Hence, we can see that the developed preconditioner gives the condition number bound as confirmed in theory.

Moreover, we have observed the convergent behaviors of the approximated solution. The H^1 and L^2-errors for velocity and pressure are examined. $\mathbf{u}^h$ and p^h denote the approximated solutions for the velocity and pressure and $\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$ means the square root of $\sum_{i=1}^N \|\mathbf{u}-\mathbf{u}^h\|_{1,\Omega_i}^2$. The errors and reduction factors are shown in Table 4 for various N and n with matching grids. Three cases are considered: when $n-1$ increases by double with $N = 4 \times 4$, when N increases by double in both edges of Ω with $n = 5$, and when N increases by double in both

TABLE 2. CG iterations(condition number) when $n = 5$

N	Matching		Nonmatching	
	FETI-DP	PFETI-DP	FETI-DP	PFETI-DP
4×4	12(5.23)	9(2.62)	16(8.35)	12(3.75)
8×8	12(5.42)	9(2.62)	16(9.18)	12(3.68)
16×16	10(5.54)	9(2.55)	16(9.57)	11(3.42)
32×32	10(5.61)	9(2.53)	16(10.88)	12(3.78)

TABLE 3. CG iterations(condition number) when $n = 9$

N	Matching		Nonmatching	
	FETI-DP	PFETI-DP	FETI-DP	PFETI-DP
4×4	24(2.50e+1)	13(4.39)	50(1.15e+2)	15(5.79)
8×8	25(2.60e+1)	13(4.35)	53(1.19e+2)	15(6.21)
16×16	24(2.62e+1)	12(4.27)	57(1.34e+2)	16(6.27)
32×32	23(2.62e+1)	12(4.27)	56(1.25e+2)	16(6.24)

TABLE 4. H^1 and L^2-errors(factor) on matching grids

$N =$ 4×4	$n = 5$	$n = 9$	$\|\mathbf{u} - \mathbf{u}^h\|_{1,*}$	$\|\mathbf{u} - \mathbf{u}^h\|_0$	$\|p - p^h\|_0$
n	N	N			
5	4×4		3.37e-1	3.75e-3	1.07e-1
9	8×8	4×4	1.72e-1 (0.510)	1.02e-3 (0.272)	5.99e-2 (0.559)
17	16×16	8×8	8.64e-2 (0.502)	2.64e-4 (0.258)	3.08e-2 (0.514)
33	32×32	16×16	4.32e-2 (0.500)	6.65e-5 (0.258)	1.55e-2 (0.503)
65		32×32	2.16e-2 (0.500)	1.66e-5 (0.249)	7.79e-3 (0.502)

edges of Ω with $n = 9$. For all cases, we can see that the H^1-error for velocity, $\|\mathbf{u} - \mathbf{u}^h\|_{1,*}$, and L^2-error for pressure, $\|p - p^h\|_0$, reduce by half and L^2-error for velocity, $\|\mathbf{u} - \mathbf{u}^h\|_0$, reduces by quarter. For the finite elements $P_1(h) - P_0(2h)$, these convergent behaviors are optimal.

For the case of nonmatching grids, the errors and reduction factors are shown in Tables 5-7 with various N and n. In Table 5, we observe that the error $\|\mathbf{u} - \mathbf{u}^h\|_{1,*}$ and $\|p - p^h\|_0$ reduce by half and the error $\|\mathbf{u} - \mathbf{u}^h\|_0$ reduces by quarter as $n - 1$ increases by double with $N = 4 \times 4$. When $n = 5$ and $n = 9$, as N increases, the errors also show the optimal convergent behaviors in Tables 6 and 7. These numerical results confirm that the stopping criterion for CG iteration in Tables 1-3 is sufficient.

As mentioned in Section 2, if the inf-sup constant for the space $V \times P$ is independent of N and n, then the optimality of approximation can be shown. Let β^* and β be the inf-sup constants for the space $V \times P$ and the $P_1(h) - P_0(2h)$

TABLE 5. H^1 and L^2-errors(factor) on nonmatching grids: $N = 4 \times 4$

n	$\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|p-p^h\|_0$
5	3.41e-1	3.79e-3	1.05e-1
9	1.78e-1 (0.521)	1.10e-3 (0.290)	6.08e-2 (0.579)
17	8.95e-2 (0.502)	2.85e-4 (0.259)	3.16e-2 (0.517)
33	4.48e-2 (0.500)	7.21e-5 (0.252)	1.58e-2 (0.500)
65	2.24e-2 (0.500)	1.81e-5 (0.251)	7.93e-3 (0.501)

TABLE 6. H^1 and L^2-errors(factor) on nonmatching grids: $n = 5$

N	$\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|p-p^h\|_0$
4×4	1.78e-1	1.10e-3	6.08e-2
8×8	8.95e-2 (0.502)	2.94e-4 (0.269)	3.28e-2 (0.539)
16×16	4.49e-2 (0.501)	7.33e-5 (0.249)	1.63e-2 (0.496)
32×32	2.25e-2 (0.501)	1.84e-5 (0.251)	8.18e-3 (0.501)

TABLE 7. H^1 and L^2-errors(factor) on nonmatching grids: $n = 9$

N	$\|\mathbf{u}-\mathbf{u}^h\|_{1,*}$	$\|\mathbf{u}-\mathbf{u}^h\|_0$	$\|p-p^h\|_0$
4×4	3.37e-1	3.75e-4	1.07e-1
8×8	1.72e-1 (0.510)	1.02e-3 (0.272)	5.99e-2 (0.559)
16×16	8.64e-2 (0.502)	2.64e-4 (0.258)	3.08e-2 (0.514)
33×32	4.32e-2 (0.500)	6.65e-5 (0.258)	1.55e-2 (0.503)

finite elements, respectively, and β_0 be the inf-sup constant for the space $V \times \Pi^0$. Then the constant β^* depends on β and β_0 from the trick conceived by Boland and Nicolaides [4]. Hence, if the constant β_0 is independent of n and N, then the same holds for β^*. In [2], for $V \times \Pi^0$ which is obtained from the Hood-Taylor finite elements, it was shown that the constant β_0 is independent of n, but not shown for N. Following the proofs in [2], we can obtain the same results for the space $V \times \Pi^0$ of the $P_1(h) - P_0(2h)$ finite elements. We have no proof that β_0 is independent of N. Instead, we compute the constant β_0 numerically as N increases. The results are given in Table 8 both for matching and nonmatching grids when $n = 5$ and $n = 9$. We observe that the constant β_0 becomes stable as N increases. Table 9 gives the constant β_0 as n increases with $N = 4 \times 4$. This confirms that the constant β_0 is independent of n.

REFERENCES

[1] F. B. BELGACEM, *The mortar finite element method with Lagrange multipliers*, Numer. Math., 84 (1999), pp. 173–197.

[2] ———, *The mixed mortar finite element method for the incompressible Stokes problem: convergence analysis*, SIAM J. Numer. Anal., 37 (2000), pp. 1085–1100.

TABLE 8. Inf-sup constant β_0 when $n = 5$ and $n = 9$

N	$n = 5$		$n = 9$	
	Nonmatching	Matching	Nonmatching	Matching
4×4	0.5780	0.5785	0.5921	0.5924
8×8	0.5293	0.5294	0.5352	0.5353
16×16	0.5008	0.5010	0.5041	0.5042
32×32	0.4827	0.4828	0.4854	0.4848

TABLE 9. Inf-sup constant β_0 when $N = 4 \times 4$

n	Nonmatching	Matching
5	0.5780	0.5785
9	0.5921	0.5294
17	0.5966	0.5967
33	0.5973	0.5979
65	0.5983	0.5983

[3] C. BERNARDI, Y. MADAY, AND A. T. PATERA, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991), vol. 299 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1994, pp. 13–51.

[4] J. M. BOLAND AND R. A. NICOLAIDES, *Stability of finite elements under divergence constraints*, SIAM J. Numer. Anal., 20 (1983), pp. 722–731.

[5] J. H. BRAMBLE AND J. E. PASCIAK, *A domain decomposition technique for Stokes problems*, Appl. Numer. Math., 6 (1990), pp. 251–261.

[6] B. COCKBURN AND C.-W. SHU, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440–2463.

[7] M. DRYJA AND O. B. WIDLUND, *A FETI-DP method for a mortar discretization of elliptic problems*, in Recent developments in domain decomposition methods (Zürich, 2001), vol. 23 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2002, pp. 41–52.

[8] ———, *A generalized FETI-DP method for a mortar discretization of elliptic problems*, in Domain decomposition methods in Science and Engineering (Cocoyoc, Mexico, 2002), UNAM, Mexico City, 2003, pp. 27–38.

[9] C. FARHAT, M. LESOINNE, AND K. PIERSON, *A scalable dual-primal domain decomposition method*, Numer. Linear Algebra Appl., 7 (2000), pp. 687–714.

[10] C. FARHAT AND F.-X. ROUX, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Int. J. Numer. Methods in Engrg., 32 (1991), pp. 1205–1227.

[11] H. H. KIM AND C.-O. LEE, *A FETI-DP preconditioner for elliptic problems on nonmatching grids*, in KAIST DAM Research Report 02-9, KAIST, 2002.

[12] A. KLAWONN AND O. B. WIDLUND, *FETI and Neumann-Neumann iterative substructuring methods: connections and new results*, Comm. Pure Appl. Math., 54 (2001), pp. 57–90.

[13] A. KLAWONN, O. B. WIDLUND, AND M. DRYJA, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal., 40 (2002), pp. 159–179.

[14] J. LI, *A dual-primal FETI method for incompressible stokes equations*, in Technical Report 816, Department of Computer Science, Courant Institute, New York Unversity, 2001.

[15] J. MANDEL AND R. TEZAUR, *On the convergence of a dual-primal substructuring method*, Numer. Math., 88 (2001), pp. 543–558.

[16] F. RAPETTI AND A. TOSELLI, *A FETI preconditioner for two-dimensional edge element approximations of Maxwell's equations on nonmatching grids*, SIAM J. Sci. Comput., 23 (2001), pp. 92–108.

[17] D. J. RIXEN, *Extended preconditioners for the FETI method applied to constrainted problems*, Internat. J. Numer. Methods Engrg., 54 (2002), pp. 1–26.

[18] D. STEFANICA, *A numerical study of FETI algorithms for mortar finite element methods*, SIAM J. Sci. Comput., 23 (2001), pp. 1135–1160.

[19] ———, *FETI and FETI-DP methods for spectral and mortar spectral elements: a performance comparison*, in Proceedings of the Fifth International Conference on Spectral and High Order Methods (ICOSAHOM-01) (Uppsala), vol. 17, 2002, pp. 629–638.

[20] B. I. WOHLMUTH, *A mortar finite element method using dual spaces for the Lagrange multiplier*, SIAM J. Numer. Anal., 38 (2000), pp. 989–1012.

[21] ———, *Discretization methods and iterative solvers based on domain decomposition*, vol. 17 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, Berlin, 2001.

[22] J. XU AND J. ZOU, *Some nonoverlapping domain decomposition methods*, SIAM Rev., 40 (1998), pp. 857–914.

DIVISION OF APPLIED MATHEMATICS, KAIST, DAEJEON 305-701, KOREA
E-mail address: mashy@amath.kaist.ac.kr, colee@amath.kaist.ac.kr