Combinatorics of lopsided sets

Hans-Jürgen Bandelt
Fachbereich Mathematik, Universität Hamburg,
Bundesstr. 55, D-20146 Hamburg, Germany
bandelt@math.uni-hamburg.de

Victor Chepoi[1]
Laboratoire d'Informatique Fondamentale, Faculté des Sciences de Luminy,
Université de la Méditerranée, F-13288 Marseille Cedex 9, France
Victor.Chepoi@lif.univ-mrs.fr

Andreas Dress
Forschungsschwerpunkt Mathematisierung,
Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany
dress@mathematik.uni-bielefeld.de

Jack Koolen
Division of Applied Mathematics, KAIST,
373-1 Kusongdong, Yusongku, Daejon 305 701 Korea
jhk@amath.kaist.ac.kr

[1] Corresponding author

Abstract

Lopsided sets (of sign vectors), as introduced by Lawrence in the context of uniform oriented matroids, can be regarded as those ("super")isometric subgraphs of hypercubes for which isometry is inherited by their associated graphs of "fibres", relative to any (Cartesian) factorization into two smaller hypercubes. There exists a plethora of equivalent characterizations, demonstrating that this concept is most natural and versatile in combinatorics. For example, antimatroids and median convexities are all particular instances of lopsided sets.

1 Introduction

In this paper, we investigate lopsided sets, which were introduced by Jim Lawrence [11] in 1983. Recently, they were rediscovered in the context of extremal combinatorics and named *ample sets*; see [7]. They constitute a certain class of subsets $\mathcal{S}$ of the elementary abelian 2-group

$$Sign\,(X) := \{-1, +1\}^X$$

of all *sign maps* defined on some finite set X, that is, the set of all maps from X into the two–element set $\{-1, +1\}$. Lopsided sets can be regarded as a common generalization of antimatroids (convex geometries) and median graphs (among which are trees, hypercubes, and covering graphs of distributive lattices). The primary motivation of Lawrence in his paper [11] was to investigate and generalize those subsets

$$\mathcal{S}(K) := \{s \in Sign\,(X) \mid \{t \in K \mid t(x)s(x) \geq 0 \text{ for all } x \in X\} \neq \emptyset\}$$

of $Sign\,(X)$ that arise from convex sets K of $\mathbb{R}^X$ defined to comprise exactly those sign maps from $Sign\,(X)$ that represent the closed orthants of $\mathbb{R}^X$ intersecting K. He presented examples of lopsided sets that cannot be realized in this way, and he also used lopsidedness to characterize uniform oriented matroids. One of the main results of [11] is the following strikingly elementary description of lopsidedness (via "total asymmetry"). First, viewing $Sign\,(X)$ as the vertex set of the "solid" hypercube $H(X) := [-1, +1]^X \subset \mathbb{R}^X$ of dimension $\#X$, one can speak of its faces

$$[s_1, s_2] := \{s \in H(X) \mid s(x) \in [s_1(x), s_2(x)] \text{ for all } x \in X\}$$

for $s_1, s_2 \in Sign\,(X)$. Two vertices s and t from a face $\mathcal{F}$ are said to be antipodes in $\mathcal{F}$ if $\mathcal{F} = [s, t]$. Now, according to [11, Theorems 3,4], a subset $\mathcal{S}$ of $Sign\,(X)$ is lopsided if and only if, whenever $\mathcal{F}$ is a face of $H(X)$ and $\mathcal{S} \cap \mathcal{F}$ is closed with respect to the antipodal mapping for $\mathcal{F}$ (i.e., if the antipode in $\mathcal{F}$ of any vertex from $\mathcal{S} \cap \mathcal{F}$ also belongs to $\mathcal{S}$), then $\mathcal{S} \cap \mathcal{F}$ is either empty or all of $Sign\,(X) \cap \mathcal{F}$. In particular, $\mathcal{S}$ is lopsided exactly when its complement in $Sign\,(X)$ is, so that one could speak of lopsided bipartitions of $Sign\,(X)$. Examples of bipartitions that are not lopsided are indicated in Figures 1 and 2: in each case, $Sign\,(X)$ is displayed as a graph, viz., the

$\#X$-dimensional cube, and either part of the bipartition is closed under the antipodal mapping of $Sign\,(X)$.

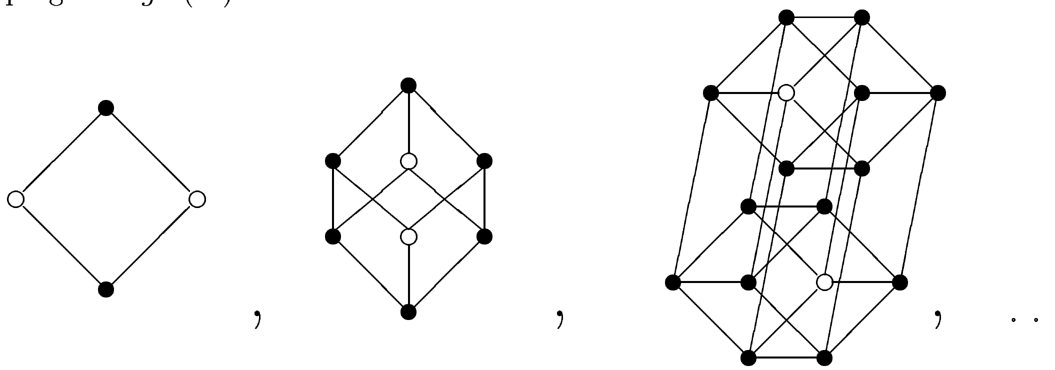

FIGURE 1. Obstructions to lopsidedness.

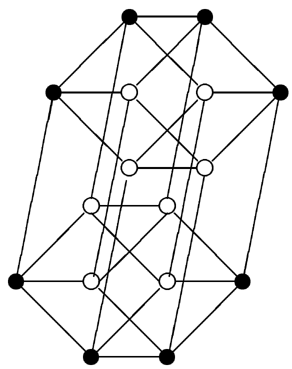

FIGURE 2. Two complementary isometric 8-cycles that are not lopsided.

In the present paper, we provide several combinatorial characterizations of lopsided sets, each emphasizing one or another feature of lopsidedness, as well as relationships with other properties of set systems.

Given any subset Y of X, one can always associate two subsets $\mathcal{S}_Y$ and $\mathcal{S}^Y$ of $Sign\,(X-Y)$ with an arbitrary set $\mathcal{S} \subseteq Sign\,(X)$ of sign maps:

$$\mathcal{S}_Y := \{t \in Sign\,(X-Y) \mid \text{ some extension } s \in Sign\,(X) \text{ of } t \text{ belongs to } \mathcal{S}\},$$

$$\mathcal{S}^Y := \{t \in Sign\,(X - Y) \mid \text{every extension } s \in Sign\,(X) \text{ of } t \text{ belongs to } \mathcal{S}\}.$$

These operations suggest two ways to derive a simplicial complex from $\mathcal{S}$:

$$\overline{\mathcal{X}}(\mathcal{S}) := \{Y \subseteq X \mid \mathcal{S}\,|_Y = \mathcal{S}_{X-Y} = Sign\,(Y)\},$$

$$\underline{\mathcal{X}}(\mathcal{S}) := \{Y \subseteq X \mid \mathcal{S}^Y \neq \emptyset\}.$$

Using this notation, the original definition of lopsidedness amounts to the condition that for each $A \subseteq X$,

$$\text{either} \quad A \in \underline{\mathcal{X}}(\mathcal{S}) \quad \text{or} \quad X - A \in \underline{\mathcal{X}}(Sign\,(X) - \mathcal{S}).$$

The starting point of our investigations was the simple, yet slightly surprising observation (cf. Section 4, Cor. 1, see also [7]) that

$$\#\underline{\mathcal{X}}(\mathcal{S}) \leq \#\mathcal{S} \leq \#\overline{\mathcal{X}}(\mathcal{S})$$

holds for whatever subset $\mathcal{S}$ of $Sign\,(X)$ one considers. So, it appeared to be natural to define a set $\mathcal{S}$ of sign maps to be *ample* if equality $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$ holds. Ampleness turned out to be preserved when passing to the sets $\mathcal{S}^Y$ and $\mathcal{S}_Y$, and to imply connectedness (and, even more, isometricity) of the subgraph induced by $\mathcal{S}$ in the hypercube $Sign\,(X)$. It followed that $\mathcal{S}_Y$ and $\mathcal{S}^Y$ had to be connected (isometric) subgraphs of $Sign\,(X - Y)$ for every ample subset $\mathcal{S}$ of $Sign\,(X)$. Conversely, connectivity (or isometricity) of $\mathcal{S}^Y$ for all $Y \subseteq X$ turned out to imply ampleness, suggesting to call such subsets *superconnected* or *superisometric*. Further investigation finally resulted in recognizing that our ample sets coincided exactly with Lawrence's lopsided sets and that an amazingly rich and multi–facetted theory regarding such subsets of $Sign\,(X)$ could be developed. Here is a list of some of the most remarkable properties of lopsided sets, each of which could be used to define them (altogether we establish 30 equivalent conditions):

superisometry: $\mathcal{S}^Y$ is isometric for all $Y \subseteq X$,

superconnectivity: $\mathcal{S}^Y$ is connected for all $Y \subseteq X$,

commutativity: $(\mathcal{S}^Y)_Z = (\mathcal{S}_Z)^Y$ holds for any disjoint $Y, Z \subseteq X$,

ampleness: $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$,

sparseness: $\#\mathcal{S} = \#\underline{\mathcal{X}}(\mathcal{S})$.

2 Sets of Sets and Sets of Maps

Throughout this paper, X denotes a finite set with $n := \#X$ elements, and $\mathcal{X}$ is any (set-theoretic) *simplicial complex* consisting of subsets of X, that is, we assume that $\mathcal{X} \subseteq \mathcal{P}(X)$ satisfies the condition

$$B \subseteq A \in \mathcal{X} \Rightarrow B \in \mathcal{X}.$$

It is not required that X or $\mathcal{X}$ be nonempty. A standard example of a simplicial complex $\mathcal{X}$ is given by the collection of independent sets of a matroid defined on X.

There are two natural notions of complementation for collections $\mathfrak{A}$ of subsets of X: one could consider either the complement $\mathcal{P}(X) - \mathfrak{A}$ of $\mathfrak{A}$ in $\mathcal{P}(X)$, or the set $\{X - A \mid A \in \mathfrak{A}\}$ of all complements of the sets in $\mathfrak{A}$. Remarkably, while neither $\mathcal{P}(X) - \mathcal{X}$ nor $\{X - A \mid A \in \mathcal{X}\}$ is a simplicial complex when $\mathcal{X}$ is a simplicial complex different from $\emptyset$ and $\mathcal{P}(X)$, the concatenation of the two complementation operators associates a simplicial complex

$$\mathcal{X}^* := \mathcal{P}(X) - \{X - A \mid A \in \mathcal{X}\} = \{X - A \mid A \in \mathcal{P}(X) - \mathcal{X}\} \tag{1}$$

to any given simplicial complex $\mathcal{X} \subseteq \mathcal{P}(X)$. Obviously, for all $A, B \subseteq X$ with $A \cup B = X$ and $A \cap B = \emptyset$, one has either $A \in \mathcal{X}$ or $B \in \mathcal{X}^*$, but not both. Further,

$$\mathcal{P}(X) = \{A \subseteq X \mid A \notin \mathcal{X}\} \,\dot\cup\, \{A \subseteq X \mid X - A \notin \mathcal{X}^*\},$$
$$\mathcal{X}^{**} = \mathcal{X},$$
$$\#\mathcal{X} + \#\mathcal{X}^* = 2^n.$$

Restriction to subsets of X lifts to an operation on complexes. As above, it is convenient to refer rather to the complement Y of the subset of X to which one wants to restrict, i.e. to define

$$\mathcal{X}_Y := \{A \cap (X - Y) \mid A \in \mathcal{X}\} = \{A \in \mathcal{X} \mid A \cap Y = \emptyset\} \tag{2}$$

for every subset Y of X. Regarding $\mathcal{X}_Y$ as a complex of subsets of $X - Y$, we have

$$\begin{aligned}(\mathcal{X}_Y)^* &= \{(X - Y) - A \mid A \in \mathcal{P}(X - Y) - \mathcal{X}_Y\} \\ &= \{(X - Y) - A \mid A \subseteq X - Y \text{ and } A \notin \mathcal{X}\} \\ &= \{B \subseteq X - Y \mid (X - Y) - B = X - (B \cup Y) \notin \mathcal{X}\} \\ &= \{B \subseteq X - Y \mid B \cup Y \in \mathcal{X}^*\}.\end{aligned} \tag{3}$$

Therefore, $((\mathcal{X}^*)_Y)^*$ coincides with

$$\mathcal{X}^Y := \{B \subseteq X - Y \mid B \cup Y \in \mathcal{X}\} = \{A - Y \mid Y \subseteq A \in \mathcal{X}\}, \tag{4}$$

implying that also

$$(\mathcal{X}_Y)^* = (\mathcal{X}^*)^Y \text{ and } (\mathcal{X}^Y)^* = (\mathcal{X}^*)_Y \tag{5}$$

must hold. We record the following elementary properties:

$$\begin{aligned} &\mathcal{X}^Y \subseteq \mathcal{X}_Y, \\ &\mathcal{X}^\emptyset = \mathcal{X} = \mathcal{X}_\emptyset, \\ &\mathcal{X}^X \neq \emptyset \iff \mathcal{X} = \mathcal{P}(X), \\ &\mathcal{X}_X \neq \emptyset \iff \mathcal{X} \neq \emptyset \iff \emptyset \in \mathcal{X}. \end{aligned} \tag{6}$$

Furthermore, for all $Y, Z \subseteq X$ with $Y \cap Z = \emptyset$, we have

$$\begin{aligned} (\mathcal{X}^Y)^Z &= \mathcal{X}^{Y \cup Z}, \\ (\mathcal{X}_Y)_Z &= \mathcal{X}_{Y \cup Z}, \\ (\mathcal{X}^Y)_Z &= \{A - Y \mid Y \subseteq A \in \mathcal{X}, A \cap Z = \emptyset\} \\ &= \{A - Y \mid Y \subseteq A \in \mathcal{X}_Z\} \\ &= (\mathcal{X}_Z)^Y. \end{aligned} \tag{7}$$

To motivate the next concept, recall that the "topes" of an oriented matroid defined on X are described by certain sign maps from $Sign\,(X)$. In particular, if $X \subseteq \mathbb{R}^n$ and if one assigns to every linear map $\lambda : \mathbb{R}^n \to \mathbb{R}$ with $X \cap \ker(\lambda) = \emptyset$ the sign vector s_λ from $Sign\,(X)$ defined by $s_\lambda(x) := \operatorname{sgn}(\lambda(x))$ for $x \in X$, then a subset of $Sign\,(X)$ is obtained, that is well known to encode a number of geometric properties of X.

In what follows, $\mathcal{S}$ is any subset of $Sign\,(X)$. By convention $Sign\,(\emptyset)$ consists of the empty map. The set-theoretic complement of $\mathcal{S}$ is denoted by $\mathcal{S}^*$:

$$\mathcal{S}^* := Sign\,(X) - \mathcal{S}. \tag{8}$$

As before, restriction of maps to a subset considered as an operation on subsets of $Sign\,(X)$ is referred to by the complement Y of that subset:

$$
\begin{aligned}
(9)\qquad \mathcal{S}_Y :&= \{s\,\big|_{X-Y}\big|\; s \in \mathcal{S}\}\\
&= \{t \in Sign\,(X-Y) \;\big|\; \text{ some extension } s \in Sign\,(X) \text{ of } t \text{ belongs to } \mathcal{S}\}.
\end{aligned}
$$

The set $((\mathcal{S}^*)_Y)^*$ then coincides with

$$
(10)\qquad \mathcal{S}^Y := \{t \in Sign\,(X-Y) \;\big|\; \text{ every extension } s \in Sign\,(X) \text{ of } t \text{ belongs to } \mathcal{S}\}.
$$

Therefore

$$
(11)\qquad (\mathcal{S}_Y)^* = (\mathcal{S}^*)^Y \text{ and } (\mathcal{S}^Y)^* = (\mathcal{S}^*)_Y.
$$

As above, we record some simple properties:

$$
\begin{aligned}
(12)\qquad &\mathcal{S}^Y \subseteq \mathcal{S}_Y,\\
&\mathcal{S}^\emptyset = \mathcal{S} = \mathcal{S}_\emptyset,\\
&\mathcal{S}^X \neq \emptyset \iff \mathcal{S} = Sign\,(X),\\
&\mathcal{S}_X \neq \emptyset \iff \mathcal{S} \neq \emptyset.
\end{aligned}
$$

Further, for $Y, Z \subseteq X$ with $Y \cap Z = \emptyset$, we have

$$
\begin{aligned}
(13)\qquad (\mathcal{S}^Y)^Z &= \mathcal{S}^{Y\cup Z},\\
(\mathcal{S}_Y)_Z &= \mathcal{S}_{Y\cup Z},\\
(\mathcal{S}^Y)_Z &= \{t\,\big|_{(X-Y)-Z}\big|\; t \in \mathcal{S}^Y\}\\
&= \{t\,\big|_{X-(Y\cup Z)}\big|\; t \in Sign\,(X-Y) \text{ and every}\\
&\qquad \text{extension of } t \text{ belongs to } \mathcal{S}\}\\
&\subseteq (\mathcal{S}_Z)^Y = \{t \in Sign\,(X-(Y\cup Z)) \;\big|\; \text{ every extension of}\\
&\qquad t \text{ to } X-Z \text{ can be extended to a map in } \mathcal{S}\}.
\end{aligned}
$$

If $Y = \{e\}$ is a singleton set, we omit set brackets in the corresponding sub- and superscripts for $\mathcal{X}$ and $\mathcal{S}$; then note that

$$
(14)\qquad \#\mathcal{X}_e + \#\mathcal{X}^e = \#\mathcal{X},
$$

$$\#\mathcal{S}_e + \#\mathcal{S}^e = \#\mathcal{S}. \tag{15}$$

There is, of course, a purpose for developing these concepts and notations in parallel: every simplicial complex $\mathcal{X}$ can be encoded by the set

$$\mathcal{S}(\mathcal{X}) := \{s_A \mid A \in \mathcal{X}\},$$

where

$$s_A(x) := \begin{cases} +1 & \text{if } x \in A, \\ -1 & \text{otherwise} \end{cases}$$

denotes the characteristic sign map of A (relative to X). Clearly, $\mathcal{X}$ coincides with

$$\{A \subseteq X \mid (\mathcal{S}(\mathcal{X}))_{X-A} = Sign\,(A)\}$$

as well as with

$$\{A \subseteq X \mid (\mathcal{S}(\mathcal{X}))^A \neq \emptyset\}.$$

In the same fashion, $\mathcal{X}$ is obtained from $(\mathcal{S}(\mathcal{X}))_{X-W}$ for any subset W of X that includes $\cup\mathcal{X}$. Thus, $\mathcal{X}_Y$ is obtained from $(\mathcal{S}(\mathcal{X}))_Y$ and so is $\mathcal{X}^Y$ from $(\mathcal{S}(\mathcal{X}))^Y$. The above equations for $\mathcal{X}$ suggest two ways to derive, quite generally, a simplicial complex from an arbitrary subset $\mathcal{S}$ of $Sign\,(X)$:

$$\begin{aligned} \overline{\mathcal{X}}(\mathcal{S}) &:= \{A \subseteq X \mid \mathcal{S}_{X-A} = \mathcal{S}\,\big|_A = Sign\,(A)\} = \{A \subseteq X \mid (\mathcal{S}_{X-A})^A \neq \emptyset\}, \\ \underline{\mathcal{X}}(\mathcal{S}) &:= \{A \subseteq X \mid \mathcal{S}^A \neq \emptyset\} = \{A \subseteq X \mid (\mathcal{S}^A)_{X-A} \neq \emptyset\}. \end{aligned} \tag{16}$$

To see that these complexes may be different, consider $X = \{1, 2\}$ and $\mathcal{S} = \{--, ++\}$ (where maps to $\{-1, +1\}$ are encoded as sign vectors). Then $\underline{\mathcal{X}}(\mathcal{S}) = \{\emptyset\}$, but $\overline{\mathcal{X}}(\mathcal{S}) = \{\emptyset, \{1\}, \{2\}\}$.

In general, $\underline{\mathcal{X}}(\mathcal{S}) \subseteq \overline{\mathcal{X}}(\mathcal{S})$ holds; and the two operators are related via complementation:

$$\overline{\mathcal{X}}(\mathcal{S}^*) = (\underline{\mathcal{X}}(\mathcal{S}))^* \text{ and } \underline{\mathcal{X}}(\mathcal{S}^*) = (\overline{\mathcal{X}}(\mathcal{S}))^*. \tag{17}$$

Hence

$$\{A \subseteq X \mid A \in \overline{\mathcal{X}}(\mathcal{S})\} \,\dot{\cup}\, \{A \subseteq X \mid X - A \in \underline{\mathcal{X}}(\mathcal{S}^*)\} = \mathcal{P}(X). \tag{18}$$

Moreover, for every subset Y of X, we have the inclusions

$$(\underline{\mathcal{X}}(\mathcal{S}))_Y \subseteq \underline{\mathcal{X}}(\mathcal{S}_Y) \subseteq \overline{\mathcal{X}}(\mathcal{S}_Y) = (\overline{\mathcal{X}}(\mathcal{S}))_Y, \tag{19}$$

$$(\underline{\mathcal{X}}(\mathcal{S}))^Y = \underline{\mathcal{X}}(\mathcal{S}^Y) \subseteq \overline{\mathcal{X}}(\mathcal{S}^Y) \subseteq (\overline{\mathcal{X}}(\mathcal{S}))^Y. \tag{20}$$

3 Conditional Antimatroids

As we have just seen, every simplicial complex $\mathcal{X}$ is trivially retrieved as

$$\mathcal{X} = \underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$$

from its set $\mathcal{S}$ of characteristic sign maps. To give a more general instance, first consider a subset $\mathcal{L}$ of $\mathcal{P}(X)$ satisfying

(i) $\emptyset \in \mathcal{L}$,

(ii) $K, L \in \mathcal{L}$ implies $K \cap L \in \mathcal{L}$.

Whenever $p \in K \in \mathcal{L}$ such that $K - \{p\} \in \mathcal{L}$, then p is called an *extreme* point; the set of all extreme points of K is denoted by $\mathrm{ex}(K)$. We say that $K \in \mathcal{L}$ is *generated* by $A \subseteq K$ if K is the smallest member of $\mathcal{L}$ containing A; this is expressed by the short-hand $[A] = K$. Note that A necessarily includes $\mathrm{ex}(K)$ whenever A generates K. Set systems $\mathcal{L}$ satisfying (i) and (ii) with the additional property that every member K of $\mathcal{L}$ is generated by its set of extreme points are called *conditional antimatroids* since such sets meeting the additional requirement $X \in \mathcal{L}$ are known as *antimatroids* or *convex geometries*; see Edelman and Jamison [9].

Proposition 1. *Let $\mathcal{S}$ be the set of characteristic sign maps encoding a set system $\mathcal{L} \subseteq \mathcal{P}(X)$ satisfying (i) and (ii). Then*

$$\begin{aligned}
\overline{\mathcal{X}}(\mathcal{S}) &= \{A \subseteq X \mid A \subseteq K \text{ for some } K \in \mathcal{L}, \text{ and } [A - a] \neq [A] \text{ for all } a \in A\} \\
&= \{A \subseteq X \mid A \text{ is a minimal generating set of some } K \in \mathcal{L}\}, \\
\underline{\mathcal{X}}(\mathcal{S}) &= \{A \subseteq X \mid \text{there exists some } C \subseteq X \text{ with } A \cap C = \emptyset \text{ such that} \\
&\qquad B \cup C \in \mathcal{L} \text{ for all } B \subseteq A\} \\
&= \{ex(K) \mid K \in \mathcal{L}\}.
\end{aligned}$$

In particular, $\underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$ holds exactly when $\mathcal{L}$ is a conditional antimatroid.

A natural example of a conditional antimatroid is given by the set $\mathcal{L}$ of all (strict) partial orders on a set M. We then regard each partial order as an asymmetric, transitive subset of the Cartesian square M^2 minus the diagonal, i.e., of $X = \{(u,v) \mid u,v \in M, u \neq v\}$. The extreme points of any member $<$ of $\mathcal{L}$ are exactly its "covering pairs" (u,v), that is, $u < v$ and there is no $w \in M$ with $u < w < v$. For the set $\mathcal{S}$ of characteristic sign maps associated to $\mathcal{L}$, we then have

$$\overline{\mathcal{X}}(\mathcal{S}) = \underline{\mathcal{X}}(\mathcal{S}) = \{H \subseteq X \mid H \text{ is the Hasse diagram of a partial order on } M\}.$$

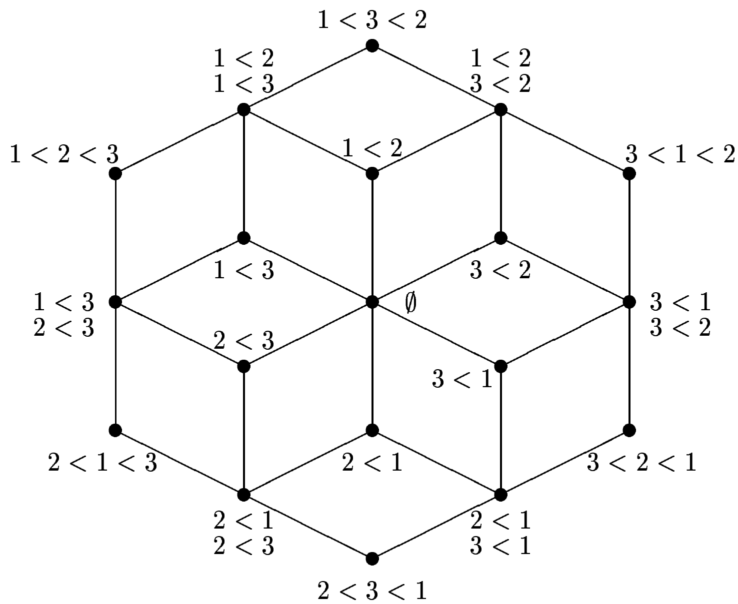

FIGURE 3. The conditional antimatroid of all partial orders on $\{1,2,3\}$ represented by their characteristic sign maps; cf. [5, Fig. 3].

A particular class of conditional antimatroids is given by set systems $\mathcal{L} \subseteq \mathcal{P}(X)$ which satisfies conditions (i) and (ii) and

(iii) for any $x \neq y$ in $\bigcup \mathcal{L}$, there exists some $K \in \mathcal{L}$ with $\#\{x,y\} \cap K = 1$,

(iv) $K, L, M \in \mathcal{L}$ with $K, L \subseteq M$ implies $K \cup L \in \mathcal{L}$.

For each $x \in \bigcup \mathcal{L}$ there exists a smallest member, $[x]$, of $\mathcal{L}$ containing x, by (ii), such that $[x] \neq [y]$ for $x, y \in \bigcup \mathcal{L}$ in view of (iii). Then every $p \in K$ with $p \notin [x]$ for all $x \in K - \{p\}$ is an extreme point of K (and vice versa) because

$$K - \{p\} = \bigcup \{[x] \mid x \in K \text{ but } p \notin [x]\} \in \mathcal{L}$$

by (iv). This shows that K is generated by its extreme points. Thus, conditions (i)–(iv) guarantee that $\mathcal{L}$ is a conditional antimatroid. Note that if in addition one imposes $X \in \mathcal{L}$, then $\mathcal{L}$ becomes an antimatroid as well as a distributive sublattice of $\mathcal{P}(X)$.

An important subclass of the former class is described by the requirements (i), (ii), (iii), and

(v) $K_i, M_i \in \mathcal{L}$ $(i = 1, 2, 3)$ with $K_i \cup K_j \subseteq M_k$ for $\{i, j, k\} = \{1, 2, 3\}$ implies $K_1 \cup K_2 \cup K_3 \in \mathcal{L}$,

which indeed implies (iv). We then call $\mathcal{L}$ a *median set system* since, by virtue of (ii) and (v), it is closed under the median operation m of $\mathcal{P}(X)$ defined by

$$m(L_1, L_2, L_3) := (L_1 \cap L_2) \cup (L_1 \cap L_3) \cup (L_2 \cap L_3).$$

Every abstract (finite) median algebra (for which the former set-theoretic ternary operation is axiomatized) can be represented by a median set system via the Sholander embedding into some power set $\mathcal{P}(X)$; minimality of the chosen set X then guarantees (iii); see [3, 12]. An inherent feature of median algebras is that they may be oriented so that any element can serve as the empty set in the associated set representation: a median set system $\mathcal{L}$ is mapped onto another one,

$$\mathcal{L} \vartriangle Z := \{K \vartriangle Z \mid K \in \mathcal{L}\},$$

by the automorphism of $\mathcal{P}(X)$ taking the symmetric difference with a fixed set $Z \in \mathcal{L}$ since

$$(K \vartriangle Z) \cap (L \vartriangle Z) = m(K, L, Z) \vartriangle Z \text{ for all } K, L \in \mathcal{L}.$$

Proposition 2. *A set system $\mathcal{L} \subseteq \mathcal{P}(X)$ is median if and only if $\mathcal{L} \vartriangle Z$ is a conditional antimatroid for each $Z \in \mathcal{L}$.*

This observation suggests to call a set $\mathcal{S} \subseteq Sign\,(X)$ a *median set* if for some $t \in \mathcal{S}$ the set $t\mathcal{S} := \{ts \mid s \in \mathcal{S}\}$ encodes a median set system, or equivalently, if $t\mathcal{S}$ encodes a conditional antimatroid for every $t \in \mathcal{S}$.

4 Linear Independence

Every sign map $s \in Sign\,(X)$ can be lifted to a sign map $\tau_s \in Sign\,(\mathcal{P}(X))$ by

$$\tau_s(A) := \prod_{a \in A} s(a) \qquad \text{for } A \subseteq X$$

(with $\tau_s(\emptyset) := +1$, by convention). Clearly, these maps form a basis of the vector space $\mathbb{R}^{\mathcal{P}(X)}$ of all maps from $\mathcal{P}(X)$ into $\mathbb{R}$. Then, restricting all τ_s to some subset $\mathcal{X}$ of $\mathcal{P}(X)$ will necessarily produce some linear dependence. However, simultaneously restricting the set $Sign\,(X)$ to some subset $\mathcal{S} \subseteq Sign\,(X)$, might restore linear independence. And indeed, one can show that every set $\mathcal{S} \subseteq Sign\,(X)$ lifts to a linearly independent set of maps defined on $\overline{\mathcal{X}}(\mathcal{S})$. This simple observation, which entails that $\overline{\mathcal{X}}(\mathcal{S})$ cannot be smaller in size than $\mathcal{S}$, is crucial for all that follows:

Theorem 1. *Assume $\mathcal{S} \subseteq Sign\,(X)$ and $\overline{\mathcal{X}}(\mathcal{S}) \subseteq \mathcal{X} \subseteq \mathcal{P}(X)$. Then the lifting $\{\tau_s|_{\mathcal{X}} \mid s \in \mathcal{S}\}$ of $\mathcal{S}$ constitutes a linearly independent subset of $\mathbb{F}^{\mathcal{X}}$ for any field $\mathbb{F}$ of characteristic different from 2.*

Proof. We proceed by induction on $n = \#X$. For $n = 0$, the assertion is trivial because the empty set is a linearly independent subset of every vector space (even if it has dimension 0, as with $\mathbb{F}^{\emptyset}$), and $\{+1\}$ is a linearly independent subset of $\mathbb{F} \cong \mathbb{F}^{\{\emptyset\}}$. So, assume that some linear combination of the restricted maps $\tau_s\,|_{\mathcal{X}}$ $(s \in \mathcal{S})$ gives the zero map, that is,

$$\sum_{s \in \mathcal{S}} \alpha_s \tau_s\,|_{\mathcal{X}} = \sum_{s \in Sign\,(X)} \alpha_s \tau_s\,|_{\mathcal{X}} \equiv 0$$

holds for some coefficients α_s $(s \in Sign\,(X))$ from $\mathbb{F}$ with $\alpha_s = 0$ for all $s \in \mathcal{S}^*$. For each $e \in X$, (19) implies

$$\overline{\mathcal{X}}(\mathcal{S}_e) = (\overline{\mathcal{X}}(\mathcal{S}))_e \subseteq \mathcal{X}_e \subseteq \mathcal{X}.$$

By virtue of the induction hypothesis, $\left\{\tau_s\,|_{\mathcal{X}_e} \,\middle|\, s \in \mathcal{S}_e\right\}$ is a linearly independent subset of $\mathbb{F}^{\mathcal{X}_e}$. For each map $s \in Sign\,(X)$, there exists a (unique) companion $s' \in Sign\,(X)$ with $s'\,|_{X-e} = s\,|_{X-e}$ but $s'(e) \neq s(e)$; so, the induction hypothesis implies that

$$\alpha_s + \alpha_{s'} = 0$$

must hold for all s, s' with $\#\{e \in X \mid s(e) \neq s'(e)\} = 1$, whether in $\mathcal{S}$ or not. A trivial induction on $\#\{e \in X \bigm| s(e) \neq t(e)\}$ then yields for any two maps $s, t \in Sign\,(X)$ that

$$\alpha_t = \begin{cases} \alpha_s & \text{if } \tau_s(X) = \tau_t(X), \\ -\alpha_s & \text{otherwise} \end{cases}$$

must hold. Consequently, we have

$$0 = \sum_{s \in Sign\,(X)} \alpha_s \tau_s(X) = 2^n \alpha_1,$$

where the subscript 1 refers to the constant sign map with value +1. Since $\mathrm{char}(\mathbb{F}) \neq 2$, we conclude that $\alpha_1 = 0$ must hold and, therefore, $\alpha_s = 0$ for all $s \in Sign\,(X)$ as required. □

Corollary 1. *For every set $\mathcal{S} \subseteq Sign\,(X)$, one has*

$$\#\underline{\mathcal{X}}(\mathcal{S}) \leq \#\mathcal{S} \leq \#\overline{\mathcal{X}}(\mathcal{S}). \tag{21}$$

Proof. The inequality $\#\mathcal{S} \leq \#\overline{\mathcal{X}}(\mathcal{S})$ is a trivial consequence of Theorem 1. Applying this inequality to $\mathcal{S}^*$ yields

$$2^n - \#\underline{\mathcal{X}}(\mathcal{S}) = \#(\underline{\mathcal{X}}(\mathcal{S}))^* = \#\overline{\mathcal{X}}(\mathcal{S}^*) \geq \#\mathcal{S}^* = 2^n - \#\mathcal{S},$$

which implies

$$\#\mathcal{S} \geq \#\underline{\mathcal{X}}(\mathcal{S}).$$

□

In thus setting the stage for the theory of lopsided sets, we closely follow a scheme that has been applied (if not invented) by Emil Artin in his treatment of Galois theory [1] and class field theory [2]. Using Dedekind's lemma (quite comparable with our Theorem 1) which states that a certain set A of maps, considered as vectors in a certain vector space V, is linearly independent, he derives the basic inequality $\#A \leq \dim V$ and then goes on to study in detail the situation(s) where equality holds. It is amazing to realize how often this simple idea (by far not exhausted by present day extremal combinatorics) has led to discovery or, at least, transparent organization of new insights in pure and applied mathematics.

5 Ampleness and Commutativity

As just pointed out, the preceding corollary suggests to study those systems $\mathcal{S}$ of sign maps for which equality $\underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$ holds. Clearly, the cardinality of this simplicial complex must coincide with that of $\mathcal{S}$ in this case. The next result lists a considerable number of equivalent properties. In particular, statement (v) served as the original definition of *ample* sets in [7], whereas (xvii) was the original definition of *lopsided* sets in [11].

Theorem 2. *For any subset $\mathcal{S} \subseteq Sign\,(X)$, the following assertions all are equivalent:*

(i) $\overline{\mathcal{X}}(\mathcal{S}) = \underline{\mathcal{X}}(\mathcal{S})$;

(ii) $\#\overline{\mathcal{X}}(\mathcal{S}) = \#\underline{\mathcal{X}}(\mathcal{S})$;

(iii) $\overline{\mathcal{X}}(\mathcal{S}^*) = \underline{\mathcal{X}}(\mathcal{S}^*)$;

(iv) $\#\overline{\mathcal{X}}(\mathcal{S}^*) = \#\underline{\mathcal{X}}(\mathcal{S}^*)$;

(v) $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$;

(vi) $\#\mathcal{S}^* = \#\overline{\mathcal{X}}(\mathcal{S}^*)$;

(vii) $\#\mathcal{S} = \#\underline{\mathcal{X}}(\mathcal{S})$;

(viii) $\#\mathcal{S}^* = \#\underline{\mathcal{X}}(\mathcal{S}^*)$;

(ix) $\#\mathcal{S}^e = \#(\overline{\mathcal{X}}(\mathcal{S}))^e$ *and* $\#\mathcal{S}_e = \#(\overline{\mathcal{X}}(\mathcal{S}))_e$ *for all* $e \in X$;

(x) $\#\mathcal{S}^e = \#(\underline{\mathcal{X}}(\mathcal{S}))^e$ *and* $\#\mathcal{S}_e = \#(\underline{\mathcal{X}}(\mathcal{S}))_e$ *for all* $e \in X$;

(xi) $\#(\mathcal{S}^Y)_Z = \#\overline{\mathcal{X}}((\mathcal{S}^Y)_Z) = \#((\overline{\mathcal{X}}(\mathcal{S}))^Y)_Z = \#((\overline{\mathcal{X}}(\mathcal{S}))_Z)^Y = \#\overline{\mathcal{X}}((\mathcal{S}_Z)^Y) = \#(\mathcal{S}_Z)^Y$ *for all* $Y, Z \subseteq X$ *with* $Y \cap Z = \emptyset$;

(xii) $\#(\mathcal{S}^Y)_Z = \#\underline{\mathcal{X}}((\mathcal{S}^Y)_Z) = \#((\underline{\mathcal{X}}(\mathcal{S}))^Y)_Z = \#((\underline{\mathcal{X}}(\mathcal{S}))_Z)^Y = \#\underline{\mathcal{X}}((\mathcal{S}_Z)^Y) = \#(\mathcal{S}_Z)^Y$ *for all* $Y, Z \subseteq X$ *with* $Y \cap Z = \emptyset$;

(xiii) $((\overline{\mathcal{X}}(\mathcal{S}))^Y)_Z = \overline{\mathcal{X}}((\mathcal{S}^Y)_Z) = \underline{\mathcal{X}}((\mathcal{S}^Y)_Z) = ((\underline{\mathcal{X}}(\mathcal{S}))^Y)_Z$ *for all* $Y, Z \subseteq X$ *with* $Y \cap Z = \emptyset$;

(xiv) $(\mathcal{S}^Y)_Z = (\mathcal{S}_Z)^Y$ *for all* $Y, Z \subseteq X$ *with* $Y \cap Z = \emptyset$*;*

(xv) $(\mathcal{S}^Y)_{X-Y} = (\mathcal{S}_{X-Y})^Y$ *for all* $Y \subseteq X$*;*

(xvi) $(\mathcal{S}^Y)_{X-Y} \neq \emptyset \iff (\mathcal{S}_{X-Y})^Y \neq \emptyset$ *for all* $Y \subseteq X$*;*

(xvii) *for all* $A, B \subseteq X$ *with* $A \cap B = \emptyset$ *and* $A \cup B = X$*, either* $A \in \underline{\mathcal{X}}(\mathcal{S})$ *or* $B \in \underline{\mathcal{X}}(\mathcal{S}^*)$*;*

(xviii) *for all* $A, B \subseteq X$ *with* $A \cap B = \emptyset$ *and* $A \cup B = X$*, either* $A \notin \overline{\mathcal{X}}(\mathcal{S})$ *or* $B \notin \overline{\mathcal{X}}(\mathcal{S}^*)$*.*

Proof. We proceed as indicated in Fig. 4: all implications and equivalences that are labelled in the figure are straightforward.

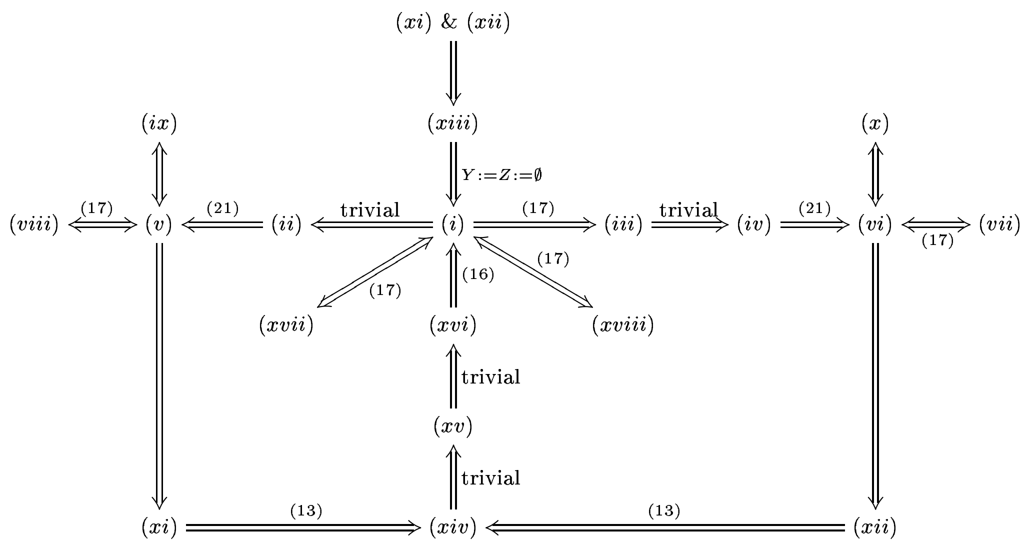

FIGURE 4. Schedule for the proof of Theorem 2.

The implication (xi) & (xii) $\Rightarrow$ (xiii) follows from (19) and (20) because these assertions imply that

$$\overline{\mathcal{X}}((\mathcal{S}^Y)_Z) = \overline{\mathcal{X}}(\mathcal{S}^Y)_Z \subseteq (\overline{\mathcal{X}}(\mathcal{S})^Y)_Z$$

and

$$\underline{\mathcal{X}}((\mathcal{S}^Y)_Z) \supseteq \underline{\mathcal{X}}(\mathcal{S}^Y)_Z = (\underline{\mathcal{X}}(\mathcal{S}^Y)_Z)$$

hold for every subset $\mathcal{S}$ of $Sign\,(X)$ and every pair of disjoint subsets Y, Z of X.

Using (14), (15), (19), and (20), the equivalence (v) $\iff$ (ix) follows from the inequality

$$\#\mathcal{S} = \#\mathcal{S}_e + \#\mathcal{S}^e \leq \#\overline{\mathcal{X}}(\mathcal{S}_e) + \#\overline{\mathcal{X}}(\mathcal{S}^e) \leq \#\overline{\mathcal{X}}(\mathcal{S})_e + \#\overline{\mathcal{X}}(\mathcal{S})^e = \#\overline{\mathcal{X}}(\mathcal{S}). \tag{22}$$

This inequality also shows that (v)$\Rightarrow$(xi) for $\#(Y \cup Z) \leq 1$. We now use induction on $\#(Y \cup Z)$ to establish (v)$\Rightarrow$(xi) for all $Y, Z \subseteq X$ with $Y \cap Z = \emptyset$. First assume $Z \neq \emptyset$. Pick $e \in Z$ and let $Z' := Z - \{e\}$. Then the induction hypothesis yields

$$\#(\mathcal{S}^Y)_{Z'} = \#\overline{\mathcal{X}}((\mathcal{S}^Y)_{Z'}) = \#(\overline{\mathcal{X}}(\mathcal{S})^Y)_{Z'} \quad = \#(\overline{\mathcal{X}}(\mathcal{S})_{Z'})^Y = \#\overline{\mathcal{X}}((\mathcal{S}_{Z'})^Y) = \#(\mathcal{S}_{Z'})^Y$$

and therefore

$$(\mathcal{S}^Y)_{Z'} = (\mathcal{S}_{Z'})^Y,$$
$$\overline{\mathcal{X}}((\mathcal{S}^Y)_{Z'}) = (\overline{\mathcal{X}}(\mathcal{S})^Y)_{Z'} = (\overline{\mathcal{X}}(\mathcal{S})_{Z'})^Y$$

holds. Hence, using (22) with $\mathcal{S}$ replaced by

$$\mathcal{S}' := (\mathcal{S}^Y)_{Z'} = (\mathcal{S}_{Z'})^Y$$

we get

$$\#\mathcal{S}'_e = \#\overline{\mathcal{X}}(\mathcal{S}'_e) = \#\overline{\mathcal{X}}(\mathcal{S}')_e,$$

that is,

$$\#(\mathcal{S}^Y)_Z = \#\overline{\mathcal{X}}((\mathcal{S}^Y)_Z) = \#(\overline{\mathcal{X}}(\mathcal{S})^Y)_Z = \#(\overline{\mathcal{X}}(\mathcal{S})_Z)^Y.$$

We may also apply our induction hypothesis to $\mathcal{S}_e$ because $\#\mathcal{S}_e = \#\overline{\mathcal{X}}(\mathcal{S}_e)$ is already established:

$$\#(\overline{\mathcal{X}}(\mathcal{S}_e)_{Z'})^Y = \#\overline{\mathcal{X}}(((\mathcal{S}_e)_{Z'})^Y) = \#((\mathcal{S}_e)_{Z'})^Y.$$

Similarly, in view of $\overline{\mathcal{X}}(\mathcal{S}_e) = \overline{\mathcal{X}}(\mathcal{S})_e$, we obtain

$$\#(\overline{\mathcal{X}}_Z)^Y = \#\overline{\mathcal{X}}((\mathcal{S}_Z)^Y) = \#(\mathcal{S}_Z)^Y,$$

completing the induction for the case $Z \neq \emptyset$.

If $Z=\emptyset$, a similar (yet simpler) argument works, picking some $e \in Y$.

The remaining implications (x) $\iff$ (vi) $\Rightarrow$ (xii) follow from their counterparts (ix) $\iff$ (v)$\Rightarrow$(xi) by complementation symmetry, that is, by exchanging the roles of $\mathcal{S}$ and $\mathcal{S}^*$, and applying the formulae $\underline{\mathcal{X}}((\mathcal{S}^Y)_Z)^* = \overline{\mathcal{X}}(((\mathcal{S}^*)_Y)^Z)$ and $((\underline{\mathcal{X}}(\mathcal{S})^Y)_Z)^* = ((\underline{\mathcal{X}}(\mathcal{S})^*)_Y)^Z = (\overline{\mathcal{X}}(\mathcal{S}^*)_Y)^Z$ for $Y, Z \subseteq X$ with $Y \cap Z = \emptyset$, which are derived from (5), (11), and (17). This completes the proof of Theorem 2. □

6 Superconnectivity and Superisometricity

The set $Sign\,(X)$, comprising the vertices of the solid hypercube $H(X) = [-1,+1]^X$, can be regarded as the graphic hypercube in which two sign maps s and t form an edge if and only if they differ at exactly one element $e \in X$. The shortest-path distance between s and t equals the *Hamming distance* $D(s,t)$, which is defined as the cardinality of the *difference set*

$$\Delta(s,t) := \{e \in X \mid s(e) \neq t(e)\}.$$

In particular, the pairs with Hamming distance 1 are the edges of $Sign\,(X)$. The set $\mathcal{S}$ is called *connected* if it induces a connected subgraph of $Sign\,(X)$, and it is called *isometric* if every pair of vertices s, s' of $\mathcal{S}$ can be connected in $\mathcal{S}$ by a path of length $D(s,s')$. Finally, $\mathcal{S}$ is said to be *weakly isometric* if every pair of vertices s, s' of $\mathcal{S}$ with $D(s,s') = 2$ has a common neighbour in $\mathcal{S}$. Using the shorthands

$$[s_1, s_2]_{\mathcal{S}} := [s_1, s_2] \cap \mathcal{S},$$

$$]s_1, s_2[_{\mathcal{S}} := [s_1, s_2]_{\mathcal{S}} - \{s_1, s_2\}$$

for $s_1, s_2 \in \mathcal{S}$, we may reformulate (weak) isometry as follows: $\mathcal{S}$ is weakly isometric if and only if $]s_1, s_2[_{\mathcal{S}} \neq \emptyset$ for all $s_1, s_2 \in \mathcal{S}$ with $D(s_1, s_2) = 2$; further, by a straightforward induction on $D(s_1, s_2)$, we infer that $\mathcal{S}$ is isometric if and only if $[s_1, s_2]_{\mathcal{S}}$ is connected for all $s_1, s_2 \in \mathcal{S}$, if and only if $]s_1, s_2[_{\mathcal{S}} \neq \emptyset$ holds for all $s_1, s_2 \in \mathcal{S}$ with $D(s_1, s_2) \geq 2$.

Lemma 1. *A set $\mathcal{S} \subseteq Sign\,(X)$ of sign maps is weakly isometric if and only if $(\mathcal{S}_e)^f = (\mathcal{S}^f)_e$ holds for all $e, f \in X$ with $e \neq f$. Moreover, $\mathcal{S}$ is isometric if and only if $\mathcal{S}_Y$ is weakly isometric for all $Y \subseteq X$ or, equivalently, exactly when $(\mathcal{S}_Z)^f = (\mathcal{S}^f)_Z$ holds for all $Z \subset X$ and $f \in X - Z$. In particular, lopsided sets are isometric.*

Proof. By (13) we have $(\mathcal{S}^f)_e \subseteq (\mathcal{S}_e)^f$. Every sign map $s \in (\mathcal{S}_e)^f$ has four extensions in $Sign\,(X)$, which together form the four vertices of a two-dimensional face $\mathcal{F}$ of $H(X)$, and at least two of those with distinct values at f must be contained in $\mathcal{S}$. Moreover $s \notin (\mathcal{S}^f)_e$ if and only if $\mathcal{S}$ intersects $\mathcal{F}$ in exactly two opposite vertices (antipodes) of $\mathcal{F}$. Hence, $\mathcal{S}$ is weakly isometric if and only if $(\mathcal{S}^f)_e = (\mathcal{S}_e)^f$ holds for all $e, f \in X$ with $e \neq f$.

If $[s,t]_{\mathcal{S}}$ is connected for $s, t \in \mathcal{S}$, then so is $[s\,|_{X-Y}, t\,|_{X-Y}]_{\mathcal{S}_Y}$ for all subsets Y of X. On the other hand, if $]s,t[_{\mathcal{S}}$ is empty for some $s, t \in \mathcal{S}$ with $D(s,t) \geq 2$, then $]s\,|_{X-Y}, t\,|_{X-Y}[_{\mathcal{S}_Y}$ is empty for any set $Y \subseteq \Delta(s,t)$ with $\#Y = D(s,t) - 2$. Therefore, isometry of $\mathcal{S}$ is equivalent to weak isometry of $\mathcal{S}_Y$ for all $Y \subseteq X$.

To prove the final equivalence, we employ the preceding characterizations. If $\mathcal{S}$ is isometric, proceed by induction on $\#Z$. Pick any $e \in Z$. Then

$$(\mathcal{S}_Z)^f = ((\mathcal{S}_{Z-\{e\}})_e)^f = ((\mathcal{S}_{Z-\{e\}})^f)_e = ((\mathcal{S}^f)_{Z-\{e\}})_e = (\mathcal{S}^f)_Z.$$

Conversely, from this equality we infer, for $Y \subset X$ and distinct $e, f \in X - Y$, that

$$((\mathcal{S}_Y)_e)^f = (\mathcal{S}_{Y\cup\{e\}})^f = (\mathcal{S}^f)_{Y\cup\{e\}} = ((\mathcal{S}^f)_Y)_e = ((\mathcal{S}_Y)^f)_e. \qquad \square$$

We can now establish several further characterizations of lopsidedness, all of which are based on Theorem 2. Conditions (iii) and (vi) below are referred to as *superisometry* and *superconnectivity,* respectively. That every lopsided set is isometric was observed by Lawrence [11] (by referring to the Djoković condition; see [6]).

Theorem 3. *For every $\mathcal{S} \subseteq Sign\,(X)$, the following assertions are equivalent:*

(i) *$\mathcal{S}$ is lopsided;*

(ii) *$(\mathcal{S}^Y)_Z$ is isometric for all disjoint subsets Y, Z of X;*

(iii) *$\mathcal{S}^Y$ is isometric for all $Y \subseteq X$;*

(iv) *$(\mathcal{S}^Y)_Z$ is weakly isometric for all disjoint subsets Y, Z of X;*

(v) *$(\mathcal{S}_Z)^Y$ is weakly isometric for all disjoint subsets $Y, Z \subseteq X$;*

(vi) *$\mathcal{S}^Y$ is connected for all $Y \subseteq X$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (v) follow from Theorem 2 and Lemma 1. The equivalence (iii)$\Leftrightarrow$(iv) is covered by Lemma 1. The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (vi) are trivial.

(v) $\Rightarrow$ (i): We show by induction on $\#Y + \#Z$ that $(\mathcal{S}^Y)_Z = (\mathcal{S}_Z)^Y$ holds for all disjoint subsets Y, Z of X. For $Y = \emptyset$ or $Z = \emptyset$ there is nothing to prove. If $Y = \{e\}$ for some $e \in X$, then $f \in Z$ implies

$$(\mathcal{S}_Z)^Y = ((\mathcal{S}_{Z-\{f\}})_f)^e = ((\mathcal{S}_{Z-\{f\}})^e)_f = ((\mathcal{S}^e)_{Z-\{f\}})_f = (\mathcal{S}^Y)_Z,$$

where the second equality follows from Lemma 1 and the assumption that $\mathcal{S}_{Z-\{f\}}$ is weakly isometric, and the third one follows from our induction hypothesis. In particular, if Y properly includes $\{e\}$ and is disjoint from $Z \subseteq X - \{e\}$, then

$$((\mathcal{S}^e)_Z)^{Y-\{e\}} = ((\mathcal{S}_Z)^e)^{Y-\{e\}} = (\mathcal{S}_Z)^Y$$

is weakly isometric by (v). Consequently, by applying the induction hypothesis to $\mathcal{S}^e$, $Y - \{e\}$, and Z, we get

$$(\mathcal{S}_Z)^Y = ((\mathcal{S}_Z)^e)^{Y-\{e\}} = ((\mathcal{S}^e)_Z)^{Y-\{e\}} = ((\mathcal{S}^e)^{Y-\{e\}})_Z = (\mathcal{S}^Y)_Z,$$

as asserted. We conclude that $\mathcal{S}$ is lopsided by Theorem 2.

(iv) $\Rightarrow$ (i): Then $((\mathcal{S}^e)^{Y-\{e\}})_Z = (\mathcal{S}^Y)_Z$ is weakly isometric for all $e \in Y \subseteq X$ and $Z \subseteq X - Y$. Hence we infer $(\mathcal{S}^Y)_Z = (\mathcal{S}_Z)^Y$ as above by induction, thus establishing lopsidedness.

(vi) $\Rightarrow$ (iii): By a straightforward induction, it suffices to show that $\mathcal{S}$ is isometric under the assumption that $\mathcal{S}$ is connected and all $\mathcal{S}^e$ $(e \in X)$ are isometric. Consider any shortest path $s_0, s_1, \ldots, s_k$ $(k \geq 2)$ in $\mathcal{S}$. Suppose by way of contradiction that

$$\Delta(s_{i-1}, s_i) = \Delta(s_{j-1}, s_j) =: \{e\} \text{ with } s_i(e) = s_{j-1}(e)$$

for some $1 \leq i < j \leq k$. Then in $\mathcal{S}^e$ there exists a path from $s_{i-1}\big|_{X-\{e\}} = s_i\big|_{X-\{e\}}$ to $s_{j-1}\big|_{X-\{e\}} = s_j\big|_{X-\{e\}}$ of length

$$\#\Delta(s_i\big|_{X-\{e\}}, s_{j-1}\big|_{X-\{e\}}) \leq D(s_i, s_{j-1}) \leq j - 1 - i$$

by isometry of $\mathcal{S}^e$. This entails a path of the same length from s_{i-1} to s_j in $\mathcal{S}$, contrary to the choice of $s_0, \ldots, s_k$ as a shortest path in $\mathcal{S}$. $\square$

Next, we have the following recursive characterizations:

Theorem 4. *For every set $\mathcal{S} \subseteq Sign\,(X)$, the following conditions are equivalent:*

(i) *$\mathcal{S}$ is lopsided;*

(ii) *$\mathcal{S}$ is isometric, and both $\mathcal{S}_e$ and $\mathcal{S}^e$ are lopsided for some $e \in X$;*

(iii) *$\mathcal{S}$ is weakly isometric, and both $\mathcal{S}_e$ and $\mathcal{S}^e$ are lopsided for some $e \in X$;*

(iv) *$\mathcal{S}$ is connected, and $\mathcal{S}^e$ is lopsided for every $e \in X$.*

Proof. From (iv), it follows immediately that S^Y is connected for every $Y \subseteq X$. In view of Theorem 3, this establishes (iv) $\Rightarrow$ (i). The implication (i) $\Rightarrow$ (ii) follows from Theorem 2 and Theorem 3, and (ii) $\Rightarrow$ (iii) is trivial.

To prove the remaining implication (iii) $\Rightarrow$ (iv), we will first show that $\mathcal{S}$ is connected. For $s, t \in \mathcal{S}$, select any path $u_0, u_1, \ldots, u_k$ in $\mathcal{S}_e$ joining $s\,\big|_{X-e} = u_0$ and $t\,\big|_{X-e} = u_k$. Each u_i extends to some $v_i \in \mathcal{S}$, and one necessarily has $1 \leq D(v_i, v_{i+1}) \leq 2$ for all $i = 0, \ldots, k-1$. Whenever $D(v_i, v_{i+1}) = 2$, we can adjoin a common neighbour $w_i \in \mathcal{S}$ of v_i and v_{i+1} by weak isometricity, and eventually obtain a path in $\mathcal{S}$ from s to t.

Next, we will prove that $\mathcal{S}^f$ is weakly isometric for every $f \in X - \{e\}$. Suppose by way of contradiction that $\mathcal{S}^f$ violates weak isometricity: then there are two sign maps s, t in $\mathcal{S}^f$ (at distance 2) having their two common neighbours u, v in $Sign\,(X - \{f\})$ outside $\mathcal{S}^f$. We denote the two extensions to $Sign\,(X)$ of each map s, t, u, v with indices $+$ and $-$ according to their value $+1$ or -1 at f. Then, by assumption, $\mathcal{S}$ includes $\{s_+, s_-\}$ and $\{t_+, t_-\}$, but neither $\{u_+, u_-\}$ nor $\{v_+, v_-\}$. On the other hand, $\mathcal{S}$ must intersect $\{u_+, v_+\}$ and $\{u_-, v_-\}$ because $\mathcal{S}$ is weakly isometric. Therefore, say, $u_-, v_+ \notin \mathcal{S}$, so that $\mathcal{S}$ contains the 6-cycle formed by $s_+, s_-, v_-, t_-, t_+, u_+$. Necessarily, all these maps have the same value at e, say -1, because $\mathcal{S}^e$ is lopsided and, hence, (weakly) isometric. For each $w \in Sign\,(X)$ with $w(e) = -1$, let w' denote its neighbour with $w'(e) = +1$. Since $\mathcal{S}_e$, being lopsided, cannot intersect a 3-dimensional face of $H(X - \{e\})$ in a 6-cycle, we infer that at least one of u'_-, v'_+ belongs to $\mathcal{S}$, say $v'_+ \in \mathcal{S}$. Now, as v_+ is a common neighbour of $s_+, t_+, v_-, v'_+ \in \mathcal{S}$ outside $\mathcal{S}$, the second common neighbour of v'_+ with each of s_+, t_+, v_- must lie in $\mathcal{S}$ because $\mathcal{S}$ is weakly isometric. Hence $\{s_+, s'_+\}, \{t_+, t'_+\}, \{v_-, v'_-\} \subseteq \mathcal{S}$, and consequently, by weak isometricity of $\mathcal{S}^e$, we also obtain $\{s_-, s'_-\}, \{u_+, u'_+\}, \{t_-, t'_-\} \subseteq \mathcal{S}$. This, however, implies that $\mathcal{S}^e$ intersects a face of $Sign\,(X - \{e\})$ in a 6-cycle, contradicting lopsidedness of $\mathcal{S}^e$.

To conclude the proof, we proceed by induction on $\#X$. We have just shown that $\mathcal{S}^f$ is weakly isometric for every $f \neq e$. Moreover, as $\mathcal{S}_e$ and $\mathcal{S}^e$ are lopsided, so are $(\mathcal{S}^f)_e = (\mathcal{S}_e)^f$ and $(\mathcal{S}^f)^e = (\mathcal{S}^e)^f$ by Theorem 3, Lemma 1, and (13). Therefore, by the induction hypothesis, $\mathcal{S}^f$ must satisfy condition (iv) and hence (i), that is, $\mathcal{S}^f$ is lopsided for every $f \in X$. Since $\mathcal{S}$ has already been shown to be connected, this establishes (iv). □

7 Push Downs and f-Vectors

For a set $\mathcal{S} \subseteq Sign\,(X)$, the *$f$-vector* $f(\mathcal{S})$ is the sequence $(f_0(\mathcal{S}), f_1(\mathcal{S}), \ldots, f_n(\mathcal{S}))$, where

$$f_i(\mathcal{S}) := \#\bigcup\{\mathcal{S}^Y \mid Y \subseteq X,\ \#Y = i\}$$

is the number of i-dimensional cubes in $\mathcal{S}$. For convenience, put $f_{-1}(\mathcal{S}) := 0$. Let us define the two facets of $H(X)$ corresponding to $e \in X$ by

$$H_e^+ := \{t \in H(X) \mid t(e) = +1\},$$

$$H_e^- := \{t \in H(X) \mid t(e) = -1\}.$$

There are straightforward relationships between the f–vector of a lopsided set $\mathcal{S}$ and the f–vectors of $\mathcal{S}_e, \mathcal{S}^e$, and of $\mathcal{S} \cap H_e^+, \mathcal{S} \cap H_e^-, \mathcal{S}^e$:

(23) $$f_i(\mathcal{S}) = f_i(\mathcal{S}_e) + f_i(\mathcal{S}^e) + f_{i-1}(\mathcal{S}^e),$$

for all $i = 1, 2, \ldots, n$ and

(24) $$f_i(\mathcal{S}) = f_i(\mathcal{S} \cap H_e^-) + f_i(\mathcal{S} \cap H_e^+) + f_{i-1}(\mathcal{S}^e),$$

for all $i = 0, 1, \ldots, n$. Recall from [11, Theorem 2] (or by arguing with superisometry) that lopsidedness is preserved under intersection with faces of $H(X)$. Then from (24) and lopsidedness of the sets $\mathcal{S} \cap H_e^+, \mathcal{S} \cap H_e^-, \mathcal{S}^e$ one infers by induction that the f-vector of a lopsided set satisfies the Euler's relation $\sum_{i \geq 0}(-1)^i f_i(\mathcal{S}) = 1$. Actually, one can easily turn this property into a new characterization of lopsidedness:

Corollary 2. *A nonempty set $\mathcal{S} \subseteq Sign\,(X)$ is lopsided if and only if*

$$\sum_{i \geq 0}(-1)^i f_i(\mathcal{S} \cap \mathcal{F}) = 1$$

holds for all faces $\mathcal{F}$ of $H(X)$.

The proof for the converse is by induction, showing that under the Euler's relation $\mathcal{S}$ is isometric and $\mathcal{S}^e$ is lopsided for every $e \in X$, which proves that $\mathcal{S}$ is lopsided by Theorem 4(iv).

We can now characterize lopsidedness of $\mathcal{S}$ in terms of the number $f_1(\mathcal{S})$ of edges of the graph of $\mathcal{S}$, in a way analogous to ampleness and sparseness (which involves the number $f_0(\mathcal{S}) = \#\mathcal{S}$ of vertices of this graph instead).

Theorem 5. *For every set $\mathcal{S} \subseteq Sign\,(X)$,*

$$\sum\{\#Y : Y \in \underline{\mathcal{X}}(\mathcal{S})\} \leq f_1(\mathcal{S}) \leq \sum\{\#Y : Y \in \overline{\mathcal{X}}(\mathcal{S})\}.$$

When $\mathcal{S}$ is connected, $f_1(\mathcal{S})$ attains the lower bound, or the upper bound, respectively, if and only if $\mathcal{S}$ is lopsided.

Proof. To establish the inequalities, we proceed by induction on $\#X$. First we consider the upper bound for $f_1(\mathcal{S})$. Put $\overline{\mathcal{X}}_k(\mathcal{S}) := \{Y \in \overline{\mathcal{X}}(\mathcal{S}) | \#Y = k\}$. The following inequalities are obvious for any $e \in X$:

$$f_1(\mathcal{S}) \leq f_1(\mathcal{S}_e) + f_1(\mathcal{S}^e) + \#\mathcal{S}^e,$$
$$\#\overline{\mathcal{X}}_k(\mathcal{S}) \geq \#\overline{\mathcal{X}}_k(\mathcal{S}_e) + \#\overline{\mathcal{X}}_{k-1}(\mathcal{S}^e) \text{ for } 1 \leq k \leq n.$$

Since $\#\mathcal{S}^e \leq \#\overline{\mathcal{X}}(\mathcal{S}^e)$, by the induction hypothesis we obtain

$$\begin{aligned} f_1(\mathcal{S}) &\leq f_1(\mathcal{S}_e) + f_1(\mathcal{S}^e) + \#\overline{\mathcal{X}}(\mathcal{S}^e) \\ &\leq \sum_{k=1}^{n} k \cdot \#\overline{\mathcal{X}}_k(\mathcal{S}_e) + \sum_{k=0}^{n-1} k \cdot \#\overline{\mathcal{X}}_k(\mathcal{S}^e) + \sum_{k=0}^{n-1} \#\overline{\mathcal{X}}_k(\mathcal{S}^e) \\ &= \sum_{k=1}^{n} k(\#\overline{\mathcal{X}}_k(\mathcal{S}_e) + \#\overline{\mathcal{X}}_{k-1}(\mathcal{S}^e)) \\ &\leq \sum_{k=1}^{n} k \cdot \#\overline{\mathcal{X}}_k(\mathcal{S}) \\ &= \sum\{\#Y | Y \in \overline{\mathcal{X}}(\mathcal{S})\}. \end{aligned}$$

If equality holds, then $f_1(\mathcal{S}^e) = \sum_{k=0}^{n-1} k \cdot \#\overline{\mathcal{X}}_k(\mathcal{S}^e)$, and therefore by the induction hypothesis, $\mathcal{S}^e$ is lopsided for all $e \in X$, whence $\mathcal{S}$ is lopsided by Theorem 4(iv).

To prove the first inequality, notice that

$$\sum\{\#Y|Y \in \underline{\mathcal{X}}(\mathcal{S})\} \leq \sum\{\#Y|Y \in \underline{\mathcal{X}}(\mathcal{S}\cap H_e^-)\}+\sum\{\#Y|Y \in \underline{\mathcal{X}}(\mathcal{S}\cap H_e^+)\}+\#\underline{\mathcal{X}}(\mathcal{S}^e).$$

Since $f_1(\mathcal{S}) = f_1(\mathcal{S} \cap H_e^-) + f_1(\mathcal{S} \cap H_e^+) + \#\mathcal{S}^e$ and $\#\mathcal{S}^e \geq \#\underline{\mathcal{X}}(\mathcal{S}^e)$, by the induction hypothesis we obtain the required inequality. If equality holds, then necessarily $\#\mathcal{S}^e = \#\underline{\mathcal{X}}(\mathcal{S}^e)$ for every $e \in X$, whence each $\mathcal{S}^e$ is lopsided. Again, by Theorem 4(iv) $\mathcal{S}$ is lopsided, concluding the proof. □

Recall that every simplicial complex $\mathcal{X}$ over X is retrieved from the lopsided set of its characteristic sign maps. There are typically many more sets $\mathcal{S} \subseteq Sign\,(X)$ giving rise to the same complex $\mathcal{X}$. For instance, every tree $\mathcal{T}$ with edges $1, 2, \ldots, m$ (comprising the set X) can be regarded as a lopsided set of sign maps yielding $\mathcal{X}(\mathcal{T}) = \{\emptyset, \{1\}, \ldots, \{m\}\}$. Namely, select an arbitrary vertex t of $\mathcal{T}$ as its root, which represents the constant sign map with value -1; to any vertex s of $\mathcal{T}$ one then associates the map that assigns $+1$ to the edges on the path from s to t, and -1 otherwise.

All lopsided sets with the same simplicial complex have the same f-vector. To see this, proceed by induction on the cardinality of X. Pick $e \in X$ and let $\mathcal{X}$ be the simplicial complex of the lopsided sets $\mathcal{S}$ and $\mathcal{T}$. Then $\mathcal{X}_e$ is the associated complex of the lopsided sets $\mathcal{S}_e$ and $\mathcal{T}_e$, while $\mathcal{X}^e$ is the complex of $\mathcal{S}^e$ and $\mathcal{T}^e$. By the induction hypothesis and (23) we immediately conclude that $f(\mathcal{S})$ and $f(\mathcal{T})$ coincide.

For a set system $\mathcal{L} \subseteq \mathcal{P}(X)$, the *push down operation* with respect to $e \in X$ replaces in $\mathcal{L}$ every set Y such that $Y - \{e\} \notin \mathcal{L}$ by $Y - \{e\}$; see [10]. The resulting set system is denoted by $\mathcal{L}[e\downarrow]$. Analogously, we define the push down operation of a set $\mathcal{S}$ of sign maps encoding $\mathcal{L}$ and denote the resulting set by $\mathcal{S}[e\downarrow]$: for each $s \in \mathcal{S}$ the value of s at e is changed from $+1$ to -1 provided that the resulting sign map with the flipped value was not yet in $\mathcal{S}$. When a sequence of push downs is executed with respect to (not necessarily distinct) elements $e_1, \ldots, e_k \in X$, write $\mathcal{L}[e_1, \ldots, e_k\downarrow] := \mathcal{L}[e_1\downarrow][e_2\downarrow]\ldots[e_k\downarrow]$ for the result of this serial push down. For an enumeration $e_1, \ldots, e_n$ of X, the system $\mathcal{L}[e_1, \ldots, e_n\downarrow]$, a *complete (serial) push down* of $\mathcal{L}$, is a simplicial complex, because $\mathcal{L}[e_1, \ldots, e_n, e\downarrow] = \mathcal{L}[e_1, \ldots, e_n\downarrow]$ holds for every $e \in X$.

For $\mathcal{S} \subseteq Sign\,(X)$, we have

$$\mathcal{S}[e\downarrow] \cap H_e^- \cong \mathcal{S}_e \text{ and } \mathcal{S}[e\downarrow] \cap H_e^+ \cong \mathcal{S}^e. \tag{25}$$

Thus the push down with respect to e allows to represent $\mathcal{S}_e$ and $\mathcal{S}^e$ internally within facets. More generally, for $Y \subseteq X$ with $Y = \{e_1, \ldots, e_k\} \neq \emptyset$, let

$$H_Y^+ := \{t \in H(X) \mid t(e) = +1 \text{ for all } e \in Y\},$$

$$H_Y^- := \{t \in H(X) \mid t(e) = -1 \text{ for all } e \in Y\}.$$

Then

$$\mathcal{S}[e_1, \ldots, e_k\downarrow] \cap H_Y^- \cong \mathcal{S}_Y,$$
$$\mathcal{S}[e_1, \ldots, e_k\downarrow] \cap H_Y^+ \cong \mathcal{S}^Y.$$

Hence, for $Z = \{f_1, \ldots, f_l\} \neq \emptyset$,

$$(\mathcal{S}_Z)^Y \cong \mathcal{S}[f_1, \ldots, f_l, e_1, \ldots, e_k\downarrow] \cap (H_Z^-)_Y^+, \tag{26}$$

$$(\mathcal{S}^Y)_Z \cong \mathcal{S}[e_1, \ldots, e_k, f_1, \ldots, f_l\downarrow] \cap (H_Y^+)_Z^-, \tag{27}$$

where $(H_Z^-)_Y^+ = (H_Y^+)_Z^-$ constitutes the same face of $H(X)$. Therefore, if the serial push downs commute, then $\mathcal{S}$ is lopsided by Theorem 2(xiv). From Theorem 2(xv) and the equalities (26) and (27) applied to $Z = X - Y$, one concludes that $\mathcal{S}$ is lopsided if all complete serial push downs yield the same simplicial complex: $\mathcal{S}[e_1, \ldots, e_n\downarrow] = \mathcal{S}[e_{\pi(1)}, \ldots, e_{\pi(n)}\downarrow]$ for all permutations π.

Proposition 3. *For a set $\mathcal{S} \subseteq Sign\,(X)$ and $e \in X$,*

$$\underline{\mathcal{X}}(\mathcal{S}) \subseteq \underline{\mathcal{X}}(\mathcal{S}[e\downarrow]) \subseteq \overline{\mathcal{X}}(\mathcal{S}[e\downarrow]) \subseteq \overline{\mathcal{X}}(\mathcal{S}).$$

In particular, if $\mathcal{S}$ is lopsided, then $\mathcal{S}[e\downarrow]$ is also lopsided such that $\mathcal{X}(\mathcal{S}[e\downarrow]) = \mathcal{X}(\mathcal{S})$ and

$$\mathcal{S}^Y[e\downarrow] = \mathcal{S}[e\downarrow]^Y \text{ for } Y \subseteq X - \{e\} \tag{28}$$

hold, whence $\mathcal{S}$ and $\mathcal{S}[e\downarrow]$ have the same f-vector.

Proof. The inclusion $\underline{\mathcal{X}}(\mathcal{S}) \subseteq \underline{\mathcal{X}}(\mathcal{S}[e\downarrow])$ follows from the definition of the push down operation, while $\overline{\mathcal{X}}(\mathcal{S}[e\downarrow]) \subseteq \overline{\mathcal{X}}(\mathcal{S})$ holds because

$$\overline{\mathcal{X}}(\mathcal{S}[e\downarrow]) = \overline{\mathcal{X}}(\mathcal{S}_e) \cup \{Y \cup \{e\} \mid Y \in \overline{\mathcal{X}}(\mathcal{S}^e)\} \subseteq \overline{\mathcal{X}}(\mathcal{S}). \qquad \square$$

Proposition 3 implies that all lopsided sets having the same simplicial complex can be obtained from each other by push down operations and inverse operations.

Let $\mathcal{S} \subseteq Sign\,(X)$ be lopsided. Then $\mathcal{S}[e\downarrow])$ is lopsided for each $e \in X$ by Proposition 3 such that $\mathcal{X}(\mathcal{S}[e\downarrow]) = \mathcal{X}(\mathcal{S})$. From this equality we conclude that all serial push downs yield the same simplicial complex $\mathcal{X}(\mathcal{S})$. Since intersections of lopsided sets with faces of $H(X)$ are always lopsided, it suffices to show $\mathcal{S}[e, f\downarrow]) = \mathcal{S}[f, e\downarrow])$ in order to establish commutativity. But as we may assume that $X = \{e, f\}$, this is now evident. Hence we have established the following characterization of lopsidedness.

Corollary 3. *For a subset $\mathcal{S} \subseteq Sign\,(X)$, the following assertions are equivalent:*

(i) *$\mathcal{S}$ is lopsided;*

(ii) *all serial push downs commute;*

(iii) *all complete serial push downs yield the same simplicial complex.*

A simplicial complex $\mathcal{X}$ is said to be *conformal* if any set of elements is included in a member of $\mathcal{X}$ whenever each pair of its elements is contained in a member of $\mathcal{X}$. Of course, a simplicial complex is conformal exactly when it is a median set system. On the other hand, median set systems can be characterized among lopsided sets by employing conformality:

Lemma 2. *If $\mathcal{L} \subseteq \mathcal{P}(X)$ is a conditional antimatroid or a median set system, respectively, then so is $\mathcal{L}[e\downarrow]$ for each $e \in X$.*

Proof. Let $M = m(L_1, L_2, L_3) \in \mathcal{L}$ for some $L_1, L_2, L_3 \in \mathcal{L}$. If $M - \{e\} \in \mathcal{L}$, then the push down with respect to e leaves the median in $\mathcal{L}[e\downarrow]$. So assume $M - \{e\} \notin \mathcal{L}$. If $\{i, j\} \subseteq \{1, 2, 3\}$ with $i \neq j$ and $L_i - \{e\}, L_j - \{e\} \in \mathcal{L}$, then

$$M - \{e\} = m(L_i - \{e\}, L_j - \{e\}, L_k) = M - \{e\} \in \mathcal{L} \text{ for } \{i, j, k\} = \{1, 2, 3\},$$

a contradiction. Therefore the push down applied to L_1, L_2, L_3 removes e in at least two instances, so that the median in $\mathcal{L}[e\downarrow]$ is $M - \{e\} \in \mathcal{L}[e\downarrow]$. $\square$

Proposition 4. *The following statements are equivalent for a lopsided set $\mathcal{S} \subseteq Sign(X)$:*

(i) *$\mathcal{S}$ is a median set;*

(ii) *$\#(\mathcal{S}^* \cap \mathcal{F}) \neq 1$ for every 3-dimensional face $\mathcal{F}$ of $H(X)$;*

(iii) *the complete push down of $\mathcal{S}$ is median;*

(iv) *$\mathcal{X}(\mathcal{S})$ is conformal.*

Proof. The implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) are trivial, while (i) $\Rightarrow$ (iii) follows from Lemma 2. To establish that (ii) $\Rightarrow$ (i) we proceed by way of contradiction. First observe that condition (ii) is preserved under push downs. Indeed, suppose by way of contradiction that $\mathcal{S}[e\downarrow]$ intersects a 3-dimensional face $\mathcal{F}$ of $H(X)$ along three 2-dimensional faces as indicated in Fig. 5. Consider the 3-dimensional face $\mathcal{F}' = \{s' : \Delta(s', s) = \{e\}$ for some $s \in \mathcal{F}\}$. Neither $\mathcal{F}$ nor $\mathcal{F}'$ is fully contained in $\mathcal{S}$. Since $\mathcal{S}$ fulfills the condition (ii), necessarily $\#(\mathcal{S} \cap \mathcal{F}) \leq 6$ and $\#(\mathcal{S} \cap \mathcal{F}') \leq 6$. Each 2-dimensional face of $\mathcal{S}[e\downarrow] \cap \mathcal{F}$ is the push down of a 2-dimensional face of either $\mathcal{S} \cap \mathcal{F}$ or $\mathcal{S} \cap \mathcal{F}'$. Hence we may assume without loss of generality that $\mathcal{S}$ shares with $\mathcal{F}$ two 2-dimensional faces and with $\mathcal{F}'$ one 2-dimensional face. This face share edges with two another 2-dimensional faces of $\mathcal{S}$ each intersecting both $\mathcal{F}$ and $\mathcal{F}'$. As a result, these three 2-dimensional faces will generate the forbidden configuration. This establishes our assertion.

Now, choose a lopsided set $\mathcal{S}$ satisfying (ii) such that $s_1, s_2, s_3 \in \mathcal{S}$ have their median t in $\mathcal{S}^*$ with the distance sum $k = D(t, s_1) + D(t, s_2) + D(t, s_3)$ being minimal. If $k > 3$, by minimality of k there must be a neighbour $t' \in \mathcal{S}^*$ of t on the way to one of s_1, s_2, s_3, that is, $t' \in [t, s_i]$ for some i. Let $\Delta(t, t') = \{e\}$. Then t' is the median of the three images of s_1, s_2, s_3 under the push down relative to e, yielding a smaller distance sum, contrary to the minimality of k. $\square$

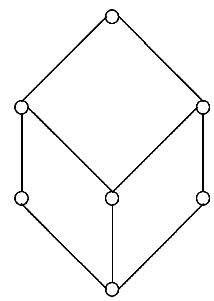

FIGURE 5. Obstruction to medianness in lopsided sets

Conformality of $\mathcal{X}(\mathcal{S})$ along with properties of lopsided sets (such as $\#\mathcal{S} = \#\mathcal{X}(\mathcal{S})$) constitute the gist of the results from [4, 8] on median sets $\mathcal{S}$.

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Figure legends

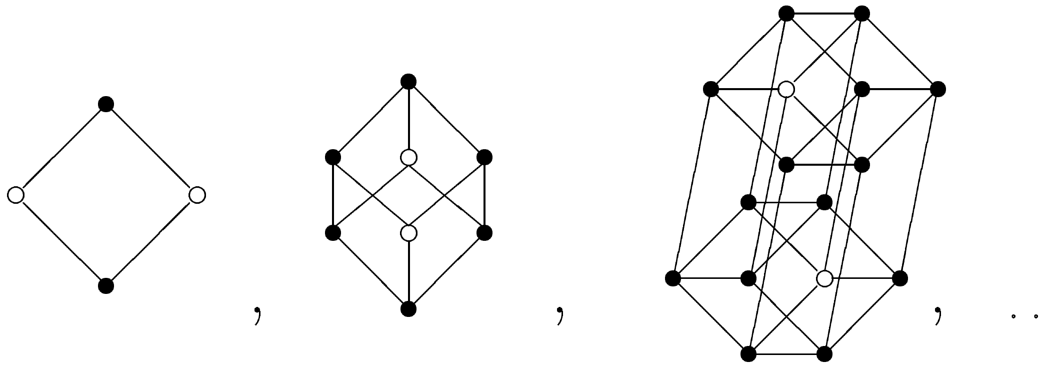
FIGURE 1. Obstructions to lopsidedness.

FIGURE 2. Two complementary isometric 8-cycles that are not lopsided.

FIGURE 3. The conditional antimatroid of all partial orders on $\{1, 2, 3\}$ represented by their characteristic sign maps; cf [5, Fig.3].

FIGURE 4. Schedule for the proof of Theorem 2.

FIGURE 5. Obstruction to medianness in lopsided sets.



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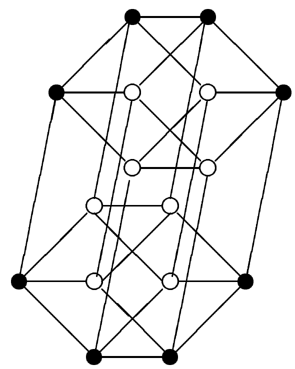
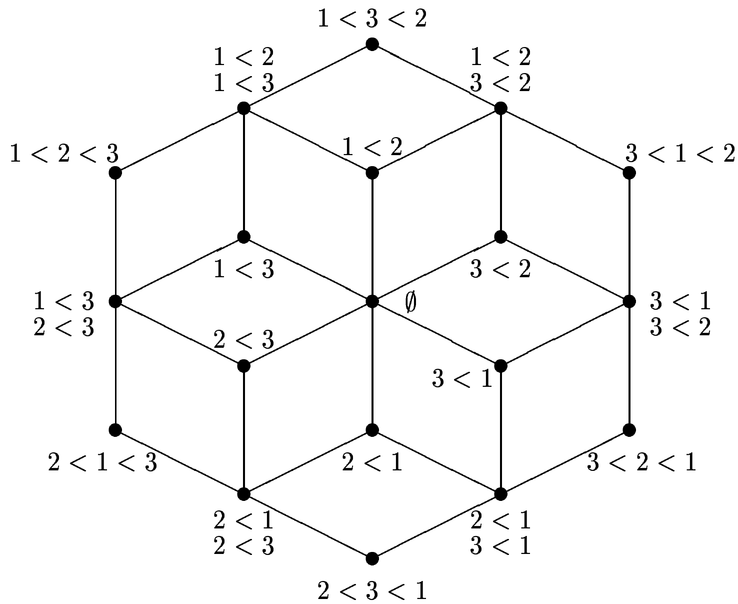


Figure 2. Two complementary isometric 8-cycles that are not lopsided.



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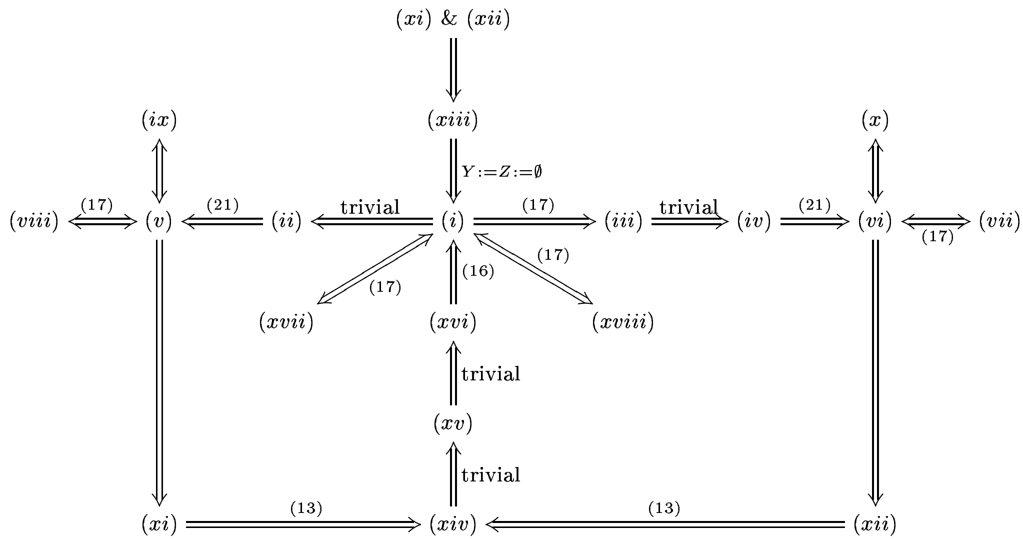

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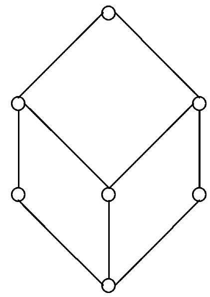

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