Vector valued Fourier analysis on unimodular groups

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The notion of Fourier type and cotype of linear maps between operator spaces with respect to certain unimodular (possibly nonabelian and noncompact) group is defined here. We develop analogous theory compared to Fourier types with respect to locally compact abelian groups of Banach space operators. We consider the Heisenberg group as an example of nonabelian and noncompact groups and prove that Fourier type and cotype with respect to the Heisenberg group implies Fourier type with respect to classical abelian groups.

1 Introduction

The reason why we have to consider vector-valued anlaysis are apparent nowadays because they provide new points of view to several important problems such as partial differential equations, nonlinear functionals and stochastic counterparts of derterministic problems. Thus, many researchers have been extending classical theorems such as Hausdorff-Young inequality([1, 8, 16]), L_p boundedness of Hilbert transform([3, 17]), Fourier multiplier theorem([23]) and Hardy inequality([2]) to vector-valued settings. Note that all these results are based on commutative harmonic analysis. Since noncommutative harmonic analysis is getting more and more important, it is very meaningful to consider its vector-valued version.

In this paper, we are going to concentrate on vector-valued Hausdorff-Young inequality on certain unimodular groups. If we look back Banach space theory, we have Fourier type with respect to locally compact abelian groups introduced by J. Peetre in [16] for $\mathbb{R}$ and by M. Milman in [15] in general case. A Banach space X is called Fourier type p, for $1 \leq p \leq 2$, with respect to a locally compact abelian group G if, $\mathcal{F}_G^X$, the X-valued Fourier transform on G is a well-defined bounded linear operator from $L_p(G, X)$ to $L_{p'}(\widehat{G}, X)$ where $\widehat{G}$ is the dual group of G. The definition for operators is a simple extension of this. A Banach space operator $T : X \to Y$ is called Fourier type p with respect to G if the $\mathcal{F}_G \otimes T$ is extended to a bounded linear operator from $L_p(G, X)$ to $L_{p'}(\widehat{G}, Y)$ where $\mathcal{F}_G$ is the Fourier transform on G. See [1, 4, 8, 12] and [17] for further information.

For the case that the underlying group G is compact(possibly nonabelian), the notion of Fourier type and cotype with respect to G is given by J. Garsia-Cuerva and J. Parcet in [9] in the framework of operator spaces, a noncommutative analogue of Banach spaces. In [9], they used vector-valued Haussdorff-Young inequality on compact groups to measure how nice structure an operator space has with the help of representation theory for compact groups. We want to extend this definition to some noncompact groups including all locally compact abelian groups and compact groups by slightly different approach not using representation theory. However, if we restrict our definition to the case that the underlying group is compact, it is equivalent to the definition in [9].

This paper is organized as follows: In section 2, we collect well known facts about vector-valued noncommutative L_p-spaces, locally compact groups and Fourier analysis on unimodular groups. In section 3, we define Fourier type and cotype of linear maps between operator spaces with respect to certain unimodular groups and we investigate some basic properties. In section 4, we restrict our attention to abelin groups and extend results in

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Banach space setting to operator space setting. At the end of this section, we pose a compatibility problem between two Fourier notions. In the final section, we consider the Heisenberg group as an example of noncompact, nonabelian unimodular group. Since the Heisenberg group has a simple structure of representations, we give alternative equivalent definition of Fourier type and cotype using representation theory. Finally, we investigate relationship between Fourier notions with respect to the Heisenberg group and $\mathbb{Z}$.

2 Preliminaries

2.1 Noncommutative vector-valued L_p-spaces

In this section, we collect some materials we need later about noncommutative vector-valued L_p-spaces mainly adopted from [18] and some of their modifications. For the general information about operator spaces, see [6]. First, we define noncommutative vector-valued L_p-spaces in the category of operator space. These definition are based on the following decomposition of vector-valued L_p-space:

$$L_p(X) = [L_\infty(X), L_1(X)]_{\frac{1}{p}} = [L_\infty \otimes_\lambda X, L_1 \otimes_\gamma X]_{\frac{1}{p}},$$

where $\otimes_\lambda$(resp. $\otimes_\gamma$) is the injective(resp. projective) tensor product in Banach space sense.

Definition 2.1 Let E be an operator space and $n \in \mathbb{N}$.

(1) We define $S^n_\infty(E) := S^n_\infty \otimes_{min} E$ and $S^n_1(E) := S^n_1 \widehat{\otimes} E$. For $1 < p < \infty$, we define $S^n_p(E) := [S^n_\infty(E), S^n_1(E)]_{\frac{1}{p}}$.

(2) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then we define

$$L_\infty(\mu, E) := L_\infty(\mu) \otimes_{min} E$$

and

$$L_1(\mu, E) := L_1(\mu) \widehat{\otimes} E.$$

If $1 < p < \infty$, we define $L_p(\mu, E) := [L_\infty(\mu, E), L_1(E)]_{\frac{1}{p}}$.

(3) Let φ be a semi-finite normal faithful trace on an injective von Neumann algebra M. Then we define

$$L_1(\varphi, E) := L_1(\varphi) \widehat{\otimes} E,$$

and for $1 < p < \infty$, we define

$$L_p(\varphi, E) := [M \otimes_{min} E, L_1(\varphi, E)]_{\frac{1}{p}}.$$

If $p = \infty$, we denote $L^0_\infty(\varphi, E) = M \otimes_{min} E$.

The followings are basic properties related to noncommutative vector-valued L_p-spaces that will be used frequently in the sequel.

Proposition 2.2 *Let E and F be operator spaces and M and N be hyperfinite von Neumann algebras with faithful normal semi-finite traces φ and ψ respectively.*

(1) *Let $1 \leq p \leq \infty$. A linear map $T : E \to F$ is completely bounded if and only if*

$$\sup_n \left\| I_{S^n_p} \otimes T : S^n_p(E) \to S^n_p(F) \right\| < \infty,$$

and we have

$$\|T\|_{cb} = \sup_n \left\| I_{S^n_p} \otimes T \right\|_{\mathcal{L}(S^n_p(E), S^n_p(F))}.$$

(2) (Fubini type theorems) *Let* $1 \le p < \infty$. *Then for any measure space* $(\Omega, \mathcal{A}, \mu)$ *and* $n \in \mathbb{N}$, *we have completely isometric isomorphism*

$$L_p(\mu, S_p^n(E)) \cong S_p^n(L_p(\mu, E)).$$

Similarly, we have completely isometric isomorphisms

$$L_p(\varphi, L_p(\psi, E)) \cong L_p(\psi, L_p(\varphi, E)) \cong L_p(\varphi \otimes \psi, E),$$

where $\varphi \otimes \psi$ *is a faithful normal semi-finite trace of the von Neumann algebra tensor product* $M\overline{\otimes}N$ *which is hyperfinite also.*

(3) (Duality) *Let* $1 < p < \infty$. *The natural embedding from* $L_{p'}(\varphi, E^*)$ *into* $L_p(\varphi, E)^*$ *is completely isometric. Furthermore, for any* $F \in L_p(\varphi, E)$ *and* $\epsilon > 0$, *we have* $\widetilde{F} \in L_{p'}(\varphi, E^*)$ *with norm* 1 *such that*

$$\|F\|_{L_p(\varphi,E)} < (1+\epsilon)\langle F, \widetilde{F}\rangle.$$

(4) (Minkowski type inequality) *Let* $1 \le p_1 \le p_2 \le \infty$. *Then for any measure space* $(\Omega, \mathcal{A}, \mu)$ *and* $n \in \mathbb{N}$, *the natural map from* $L_{p_1}(\mu, S_{p_2}^n(E))$ *into* $S_{p_2}^n(L_{p_1}(\mu, E))$ *is complete contraction. The same statement holds if we replace* μ *into a faithful normal semi-finite trace* φ *of an injective von Neumann algebra* M.

(5) *Let* $1 \le p \le \infty$. *For any* $f \in L_p(\varphi_G, E)$ *we have*

$$\|f\|_{L_p(\varphi_G)\otimes_\lambda E} \le \|f\|_{L_p(\varphi_G,E)} \le \|f\|_{L_p(\varphi_G)\otimes_\gamma E}.$$

Particularly, for $f \otimes x \in L_p(\varphi) \otimes E$ *we have*

$$\|f \otimes x\|_{L_p(\varphi_G,E)} = \|f\|_{L_p(\varphi_G)} \|x\|_E.$$

Proof. See chapter 1, 2 and 3 of [18] for the proof of (1), (2) and (5). The hyperfiniteness of M means that $M = \overline{\cup M_\alpha}$(weak*-closure) where M_α is a net of finite dimensional $*$-subalgebra directed by inclusion. Let φ_α be the restriction of φ to M_α. Then we have a complete isometry $L_{p'}(\varphi_\alpha, E^*) \cong L_p(\varphi_\alpha, E)^*$(isometry, in the Banach space setting). Thus we get (3). For (4), we only have to recall the fact that the natural map from $E_1\widehat{\otimes}(E_2 \otimes_{min} E_3)$ into $(E_1\widehat{\otimes}E_2) \otimes_{min} E_3$ is a complete contraction(contraction, in the Banach space setting) for any operator spaces E_1, E_2 and E_3(chapter 8 of [6]). □

Remark 2.3 We need the injectivity of M in Definition 2.1 to assure that $L_\infty^0(\varphi, E)$ and $L_1(\varphi, E)$ are compatible each other for the complex interpolation. See chapter 3 of [18] for the detail. The hyperfiniteness used in the above proposition is implied by the injectivity([5]).

2.2 Weil's formula

In this section, we consider Weil's formula about quotient spaces and its modification.

Proposition 2.4 (Weil's formula) *Let* G *be a unimodular group and* H *be a closed subgroup of* G *which is unimodular also. For any Haar measures* μ_G *and* μ_H *on* G *and* H, *respectively, there exists a (unique up to constant)* G*-invariant Radon measure* $\mu_{G/H}$ *on* G/H *such that for every* $f \in L_1(G)$ *and lower semi-continuous* $f : G \to [0, \infty]$

$$\int_G f(x) d\mu_G(x) = \int_{G/H} \int_H f(xh) d\mu_H(h) d\mu_{G/H}(xH).$$

Proof. See p.57 and p.62 of [7] and [20]. □

Although the integral form of Weil's formula looks similar to Fubini's theorem, it is not exactly Fubini's theorem. Therefore, we need additional condition in order to reduce it to Fubini's theorem.

Proposition 2.5 (The Borel selection lemma) *Let G be a second countable locally compact group and H a closed subgroup of G. Then there exist a Borel set $A \subseteq G$ such that A meets each coset of H at exactly one point and the following two functions are measurable bijections:*

$$\begin{aligned} q|_A : &A \subseteq G \to G/H, \quad & \phi : &A \times H \to G. \\ &a \mapsto aH & &(a,h) \mapsto ah \end{aligned}$$

We call A a Borel selection for q where q is the canonical quotient map from G onto G/H.

Proof. See lemma 1.1 and 1.2 of [14]. □

Remark 2.6 If we give a measure μ_A on A induced by $q|_A$, then we have the following for Borel sets $B \subseteq A$ and $K \subseteq H$ with finite measure:

$$\begin{aligned} \int_{A\times H} 1_B(a)1_K(\xi)d\mu_H(\xi)d\mu_A(a) &= \int_{G/H}\int_H 1_{q(B)}(aH)1_K(\xi)d\mu_H(\xi)d\mu_{G/H}(aH) \\ &= \int_{G/H}\int_H 1_{q(B)}(aH)1_K(h\xi)d\mu_H(\xi)d\mu_{G/H}(aH) \\ &= \int_G (1_B \times 1_K)\circ\phi^{-1}(x)d\mu_G(x) \end{aligned}$$

where $1_B \times 1_K(a,\xi) = 1_B(a)1_K(\xi)$ for $a \in A$ and $\xi \in H$.

Thus ϕ becomes a measure preserving map between $(A \times H, \mu_A \times \mu_H)$ and (G, μ_G).

2.3 Fourier analysis on unimodular groups

In this section, we present summary of Fourier analysis on unimodular groups adopted in [13] and [21]. For general information about locally compact groups and abstract harmonic analysis, see [7] and [11].

For a locally compact abelian group G, the Fourier transform of $f \in L_1(G)$ is defined on the dual group $\widehat{G}$ by

$$\widehat{f}^G(\gamma) = \int_G f(x)\overline{\gamma(x)}dx$$

for $\gamma \in \widehat{G}$. When $f \in L_1(G) \cap L_2(G)$, $\Phi : f \mapsto \widehat{f}^G$ is an isometric map into $L_2(\widehat{G})$ which can be extended to an isometry between $L_2(G)$ and $L_2(\widehat{G})$. Furthermore, $M_{\widehat{f}^G}$, the multiplication by $\widehat{f}^G$ on $L_2(\widehat{G})$ is unitarily equivalent via Φ to L_f, the convolution with f on $L_2(G)$ which is given by $\Phi L_f \Phi^{-1} = M_{\widehat{f}^G}$. Thus if we identify $\widehat{f}^G$ with $M_{\widehat{f}^G}$, we get another Fourier transform L_f. Since non-abelin groups do not have their dual groups, we use this Fourier transform L_f in our formulation.

Let G be a unimodular group which means that the left Haar measure of G and the right Haar measure of G coincide. For $f \in L_1(G)$, we write L_f for the left convolution by f acting on $L_2(G)$ by:

$$L_f(g)(x) = \int_G f(y)g(y^{-1}x)dy$$

for all $g \in L_2(G)$. Let $VN(G)$ be the von Neumann algebra generated by $\{L_f\}_{f\in L_1(G)}$. This $VN(G)$ is called the group von Neumann algebra of G and is equal to the von Neumann algebra generated by $\{L_a\}_{a\in G}$, where L_a is the left translation acting on $L_2(G)$ by $L_a(g)(x) = g(a^{-1}x)$ for all $g \in L_2(G)$. Then we have a unique faithful semifinite normal trace φ_G(simply φ) on $VN(G)$ which satisfies the following: If $f \in L_1(G)$ is continuous and positive definite then we have $L_f \in L_1(\varphi)$ and $\varphi(L_f) = f(e)$ where e is the identity of G. When $f \in L_1(G) \cap L_2(G)$, $\mathcal{F}_G : f \to L_f$ is an isometric map into $L_2(\varphi)$ which can be extended to an isometry between $L_2(G)$ and $L_2(\varphi)$. Furthermore, we have fourier inverse transform defined for all $F \in L_1(\varphi)$ by

$${\mathcal{F}_G}^{-1}(F)(x) := \varphi(L_x^* F)$$

for $x \in G$. Then we have that ${\mathcal{F}_G}^{-1}(F) \in C_c(G)$ and bounded by $\|F\|_1$, and also we have Parseval's formula as follows: Let $f_1 \in L_1(G)$ and $F_2 \in L_1(\varphi)$. Set $F_1 = L_{f_1}$ and $f_2 = {\mathcal{F}_G}^{-1}(F_2)$. Then we have

$$\langle F_1, F_2 \rangle = \varphi({F_2}^* F_1) = \int_G f_1(x)\overline{f_2(x)}dx.$$

Remark 2.7 (1) In the case that G is abelian, $(VN(G), \varphi)$ is equivalent as a von Neumann algebra to $L_\infty(\widehat{G})$ with the usual integration on the dual group $\widehat{G}$ of G under the mapping $L_f \mapsto \widehat{f}^G$.

(2) In the case that G is compact, we consider the dual object $\widehat{G}$ which consist of all equivalence classes of irreducible unitary representations of G, and we define Fourier transform by

$$\widehat{f}^G(\pi) = \int_G f(x)\pi^*(x)dx$$

for $f \in L_1(G)$ and $\pi \in \widehat{G}$. Let

$$\mathcal{L}_\infty = \{F \in \prod_{\pi \in \widehat{G}} M_{d_\pi} : \sup_{\pi \in \widehat{G}} \|F^\pi\|_{S_\infty^{d_\pi}} < \infty\},$$

where d_π is the dimension of π. Then this is equivalent as a von Neumann algebra to $(VN(G), \varphi)$ under the mapping $L_f \mapsto \widehat{f}^G$ with the following trace ψ:

$$\psi(F) = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(F^\pi)$$

for appropriate positive $F \in \prod_{\pi \in \widehat{G}} M_{d_\pi}$. For the proof, see [13].

Now we present Plancherel's theorem and Hausdorff-Young inequality in the category of operator space.

Theorem 2.8 *Let G be a unimodular group.*

(1) *The Fourier transform $\mathcal{F}_G$ is a complete isometry between $L_2(G)$ and $L_2(\varphi_G)$.*

(2) *For $1 \le p \le 2$, $\mathcal{F}_G$ is a complete contraction from $L_p(G)$ into $L_{p'}(\varphi_G)$ and its inverse transform ${\mathcal{F}_G}^{-1}$ is a complete contraction from $L_p(\varphi_G)$ into $L_{p'}(G)$ where p' is the conjugate exponent of p.*

Proof. Note that it is already known that $\mathcal{F}_G$ is an isometry between $L_2(G)$ and $L_2(\varphi_G)$ in [13]. By (1) of Proposition 2.2, we need to consider

$$I_{S_2^n} \otimes \mathcal{F}_G : S_2^n(L_2(G)) \to S_2^n(L_2(\varphi_G)).$$

Since we have complete isometries $S_2^n(L_2(G)) \cong L_2(G, S_2^n)$ and $S_2^n(L_2(\varphi_G)) \cong L_2(\varphi_G, S_2^n)$, and S_2^n is a Hilbert space, we have that $I_{S_2^n} \otimes \mathcal{F}_G$ is contractive, which means that $\mathcal{F}_G$ is completely contractive. Since the same argument works for ${\mathcal{F}_G}^{-1}$ we get (1).
For the proof of (2), we consider $\mathcal{F}_G : L_1(G) \to VN(G)$, then we have that $\mathcal{F}_G$ is a complete contraction since the source space is L_1-space, which has maximum operator space structure. Similarly, ${\mathcal{F}_G}^{-1} : L_1(\varphi_G) \to L_\infty(G)$ is a complete contraction since the target space is L_∞-space, which has minimum operator space structure. □

3 Fourier type and cotype with respect to certain unimodular groups

In order to define Fourier type with respect a unimodular group G, we need some technical assumptions on G. Since we have to consider vector-valued L_p space comes from the group von neumann algebra $VN(G)$ and its natural trace φ_G, we need injectivity of $VN(G)$ by Remark 2.3. Fortunately, if G is amenable or second countable and connected then $VN(G)$ is injective([5]). Thus we can include all locally compact abelian groups, compact groups and connected Lie groups.

From now on, let G be a unimodular group with injective $VN(G)$ and $T : E \to F$ be a linear map between operator spaces, and let p be the number $1 \le p \le 2$ and p' be the conjugate exponent of p. Now we provide the definition of Fourier type and cotype.

Definition 3.1 (1) T is said to have G-Fourier type p if

$$\mathcal{F}_G \otimes T : L_p(G) \otimes E \to L_{p'}(\varphi_G) \otimes F$$

extends to a completely bounded map from $L_p(G,E)$ into $L_{p'}(\varphi_G,F)$(if $p'=\infty$ we consider $L^0_\infty(\varphi_G,F)$) and in this case we denote $\left\|T|\mathcal{FT}^G_p\right\| := \|\mathcal{F}_G \otimes T\|_{cb}$.

(2) T is said to have G-Fourier cotype p' if

$$\mathcal{F}_G^{-1} \otimes T : L_p(\varphi_G) \otimes E \to L_{p'}(G) \otimes F$$

extends to a completely bounded map from $L_p(\varphi_G,E)$ into $L_{p'}(G,F)$ and in this case we denote $\left\|T|\mathcal{FC}^G_{p'}\right\| := \left\|\mathcal{F}_G{}^{-1} \otimes T\right\|_{cb}$.

In particular, we say that an operator space E has G-Fourier type p(resp. G-Fourier cotype p') if I_E, the identity operator on E, has.

Remark 3.2 Let G be a compact group. We define

$$\mathcal{L}_r = \{F \in \prod_{\pi\in\widehat{G}} M_{d_\pi} : \Big[\sum_{\pi\in\widehat{G}} d_\pi \|F^\pi\|^r_{S^{d_\pi}_r}\Big]^{\frac{1}{r}} < \infty\}$$

and

$$\mathcal{L}_r(E) = \{F \in \prod_{\pi\in\widehat{G}} M_{d_\pi} : \Big[\sum_{\pi\in\widehat{G}} d_\pi \|F^\pi\|^r_{S^{d_\pi}_r(E)}\Big]^{\frac{1}{r}} < \infty\}$$

for any operator space E and $1 \le r < \infty$, where $\widehat{G}$ is the dual object of G. Since we have a complete isometry $L^0_\infty(\varphi_G) \cong \mathcal{L}_\infty$; $L_f \mapsto \widehat{f}^G$ (Remark 2.7), we get a complete isometry $L_1(\varphi_G) \cong \mathcal{L}_1$; $L_f \mapsto \widehat{f}^G$ by the inversion formula. Thus by the extension properties of tensor products in operator space, we have complete isometries $L^0_\infty(\varphi_G,E) \cong \mathcal{L}_\infty(E)$ and $L_1(\varphi_G,E) \cong \mathcal{L}_1(E)$; $L_f \otimes x \mapsto \widehat{f}^G \otimes x$ for $x \in E$. Consequently, we get a complete isometry $L_r(\varphi_G,E) \cong \mathcal{L}_r(E)(1 \le r \le \infty)$. This implies that the definition of Fourier type and cotype in this section is equivalent to those in [9] when the underlying group is compact.

Every linear map that has Fourier type or cotype is completely bounded and every completely bounded map has Fourier type 1 and cotype ∞ as in the usual type, cotype theory.

Proposition 3.3 (1) *If T has G-Fourier type p, then T is completely bounded with*

$$\|T\|_{cb} \le \|\mathcal{F}_G\|^{-1}_{L_p(G)\to L_{p'}(\varphi_G)} \left\|T|\mathcal{FT}^G_p\right\|,$$

and if T has G-Fourier cotype p', then T is completely bounded with

$$\|T\|_{cb} \le \left\|\mathcal{F}_G^{-1}\right\|^{-1}_{L_p(\varphi_G)\to L_{p'}(G)} \left\|T|\mathcal{FC}^G_{p'}\right\|.$$

(2) *If T is completely bounded, then T has G-Fourier type 1 and G-Fourier type ∞ with $\left\|T|\mathcal{FT}^G_1\right\| = \left\|T|\mathcal{FC}^G_\infty\right\| = \|T\|_{cb}$.*

Proof. For the proof of (1), assume that T has G-Fourier type p, then by the definition we have $\mathcal{F}_G \otimes T : L_p(G,E) \to L_{p'}(\varphi_G,F)$ is completely bounded, which means $I_{S^n_{p'}} \otimes (\mathcal{F}_G \otimes T) : S^n_{p'}(L_p(G,E)) \to S^n_{p'}(L_{p'}(\varphi_G,E))$ is uniformly bounded for all positive integer n by (1) of Proposition 2.2. Note that $S^n_{p'}(L_{p'}(\varphi_G,E))$ is completely isometric to $L_{p'}(\varphi_G,S^n_{p'}(E))$ by (2) of Proposition 2.2. Then for any $f \in L_p(G)$ and $(x_{ij}) \in M_n(E)$, $(f \otimes x_{ij})$ is mapped to $L_f \otimes (Tx_{ij})$ and we have

$$\begin{aligned}
\|L_f\|_{L_{p'}(\varphi_G)} \|(Tx_{ij})\|_{S^n_{p'}(F)} &= \|L_f \otimes (Tx_{ij})\|_{L_{p'}(\varphi_G,S^n_{p'}(F))} \\
&\le \left\|T|\mathcal{FT}^G_p\right\| \|(f \otimes x_{ij})\|_{S^n_{p'}(L_p(G,E))} \\
&\le \left\|T|\mathcal{FT}^G_p\right\| \|f \otimes (x_{ij})\|_{L_p(G,S^n_{p'}(E))} \\
&= \left\|T|\mathcal{FT}^G_p\right\| \|f\|_{L_p(G)} \|(x_{ij})\|_{S^n_{p'}(E)},
\end{aligned}$$

by (4) and (5) of Proposition 2.2. Since we take f arbitrarily, we get the desired result. The same argument applies for the cotype case also.

For the proof of (2), it is sufficient to assume that T is a complete contraction. Then since $\mathcal{F}_G : L_1(G) \to VN(G)$ is also a complete contraction, their tensor product $\mathcal{F}_G \otimes T$ extends to a complete contraction from $L_1(G)\widehat{\otimes}E$ into $VN(G)\widehat{\otimes}F$. Since the canonical embedding from $VN(G)\widehat{\otimes}F$ into $VN(G)\otimes_{min}F$ is completely contractive, we get a complete contraction $\mathcal{F}_G \otimes T : L_1(G)\widehat{\otimes}E \to VN(G)\otimes_{min}F$. Thus we can say that T has G-Fourier type 1 with $\left\|T|\mathcal{FT}_1^G\right\| \leq 1$. If we apply (1), then we get the desired equality. Similarly we can say that T has G-Fourier cotype ∞ with $\left\|T|\mathcal{FC}_\infty^G\right\| = 1$. □

Remark 3.4 By the definition, it is trivial that Fourier type and cotype norms of linear maps between operator spaces have ideal properties as follows: Let G be a unimodular group with injective $VN(G)$ and $T : E_1 \to E_2$ and $S : E_2 \to E_3$ be linear maps between operator spaces. Then we have

$$\left\|ST|\mathcal{FT}_p^G\right\| \leq \|S\|_{cb}\left\|T|\mathcal{FT}_p^G\right\|$$

and

$$\left\|ST|\mathcal{FT}_p^G\right\| \leq \|T\|_{cb}\left\|S|\mathcal{FT}_p^G\right\|.$$

We have the same inequalities for the cotype case also.

The simplest examples of spaces with Fourier type p and cotype p' is L_p-spaces, and Fourier properties get better as the exponent get closer to 2.

Proposition 3.5 (1) *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and φ be a semi-finite normal faithful trace on a von Neumann algebra M. Then $L_p(\mu)$ and $L_p(\varphi)$ has G-Fourier type p and G-Fourier cotype p'.*

(2) *Let $1 \leq p < q \leq 2$. Then G-Fourier type q(resp. G-Fourier cotype q') implies G-Fourier type p(G-Fourier cotype p).*

Proof. For (1) we consider $L_2(\mu)$(resp. $L_2(\varphi)$) first. If we give the natural OSS on $L_2(\mu)$(resp. $L_2(\varphi)$), it is G-Fourier type and cotype 2 by the same argument in the proof of Theorem 2.8. Then since $L_1(\mu)$(resp. $L_1(\varphi)$) has G-Fourier type 1 and $L_\infty(\mu)$(resp. $L_\infty(\varphi)$) has G-Fourier cotype ∞(Proposition 3.3), we get the desire result by interpolation. (2) is obtained similarly. □

Remark 3.6 For the best case, Fourier type 2 and cotype 2, we only have characterization for spaces for restricted class of groups. In [10], they proved noncommutative Kwapien's theorem, which says that for every infinite compact group G, an operator space with G-Fourier type 2 and cotype 2 is completely isomorphic to an operator Hilbert space.

We have duality relationship of type and cotype as follows.

Theorem 3.7 *Let $T : E \to F$ be a linear map and T^* is the adjoint of T. then we have the followings:*

(1) *T has G-Fourier type p if and only if T^* has G-Fourier cotype p' with the same norm;*

(2) *T has G-Fourier cotype p' if and only if T^* has G-Fourier type p with the same norm.*

Proof. We only prove 'only if' part of (1) since others are obtained similarly. Now it is sufficient to prove that

$$\left\|T^* \otimes F_G^{-1}(A_{ij})\right\|_{S_{p'}^n(L_{p'}(G,E^*))} \leq \left\|T|\mathcal{FT}_p^G\right\| \cdot \|(A_{ij})\|_{S_{p'}^n(L_p(\varphi_G,F^*))}.$$

By (3) of Proposition 2.2, for any given ϵ we have $(f_{ij}) \in S_p^n(L_p(G,E))$ with norm 1 such that

$$\left\|T^* \otimes F_G^{-1}(A_{ij})\right\|_{S_{p'}^n(L_{p'}(G,E^*))} \leq (1+\epsilon)\langle T^* \otimes F_G^{-1}(A_{ij}), (f_{ij})\rangle.$$

Since $(T \otimes \mathcal{F}^G)^* = T^* \otimes F_G^{-1}$, and $T \otimes \mathcal{F}^G(f_{ij}) \in S_p^n(L_{p'}(\varphi_G, F))$ we have

$$\begin{aligned}\left\|T^* \otimes F_G^{-1}(A_{ij})\right\|_{S_{p'}^n(L_{p'}(G,E^*))} \leq &(1+\epsilon)\langle (A_{ij}), T\otimes \mathcal{F}^G(f_{ij})\rangle \\ \leq &(1+\epsilon)\left\|(A_{ij})\right\|_{S_{p'}^n(L_p(\varphi_G,F^*))} \cdot \left\|T\otimes \mathcal{F}^G(f_{ij})\right\|_{S_p^n(L_{p'}(\varphi_G,F))} \\ \leq &(1+\epsilon)\left\|(A_{ij})\right\|_{S_{p'}^n(L_p(\varphi_G,F^*))} \cdot \left\|T|\mathcal{FT}_p^G\right\| \cdot \left\|(f_{ij})\right\|_{S_p^n(L_p(G,E))} \\ = &(1+\epsilon)\left\|(A_{ij})\right\|_{S_{p'}^n(L_p(\varphi_G,F^*))} \cdot \left\|T|\mathcal{FT}_p^G\right\|.\end{aligned}$$

Since $\epsilon > 0$ can be chosen arbitrarily, we get the desired result. □

3.1 Transference principles

In this subsection, we consider general transference principles. The first one is about direct product of groups. We begin with two lemmas, and the second one will be used later.

Lemma 3.8 *Let G_1 and G_2 be unimodular groups with injective $VN(G_1)$ and $VN(G_2)$. Then T has $(G_1 \times G_2)$-Fourier type p(resp. cotype p') if and only if $\mathcal{F}_{G_1} \otimes T$ has G_2-Fourier type p(resp. cotype p') with the same norm.*

Proof. We have $VN(G_1 \times G_2) = VN(G_1)\overline{\otimes}VN(G_2)$ and $\varphi_{G_1\times G_2} = \varphi_{G_1} \otimes \varphi_{G_2}$. Then by (2) of Proposition 2.2, we have $L_p(G_1 \times G_2, E)$ is completely isometric to $L_p(G_2, L_p(G_1, E))$ and $L_{p'}(\varphi_{G_1\times G_2}, F)$ is completely isometric to $L_{p'}(\varphi_{G_2}, L_{p'}(\varphi_{G_1}, E))$. Since $\mathcal{F}_{G_1\times G_2} = \mathcal{F}_{G_2} \otimes \mathcal{F}_{G_1}$, we have

$$\left\|T|\mathcal{FT}_p^{G_1\times G_2}\right\| = \left\|\mathcal{F}_{G_1\times G_2}\otimes T\right\|_{cb} = \left\|\mathcal{F}_{G_2}\otimes\mathcal{F}_{G_1}\otimes T\right\|_{cb} = \left\|\mathcal{F}_{G_1}\otimes T|\mathcal{FT}_p^{G_2}\right\|.$$

The proof for the cotype case is the same. □

Lemma 3.9 *Let G_1 and G_2 be unimodular groups with injective $VN(G_1)$ and $VN(G_2)$. Suppose that there exists constant $C > 0$ such that*

$$\left\|T|\mathcal{FT}_p^{G_2}\right\| \leq C\left\|T|\mathcal{FT}_p^{G_1}\right\|$$

for all T with G_1-Fourier type p($1 \leq p \leq 2$). Then we have

$$\left\|T|\mathcal{FT}_p^{G_2\times G}\right\| \leq C\left\|T|\mathcal{FT}_p^{G_1\times G}\right\|$$

for all T with $(G_1 \times G)$-Fourier type p. The same result holds for cotype case.

Proof. By our assumption and Lemma 3.8, we have that

$$\left\|T|\mathcal{FT}_p^{G_2\times G}\right\| = \left\|\mathcal{F}^G\otimes T|\mathcal{FT}_p^{G_2}\right\| \leq C\left\|\mathcal{F}^G\otimes T|\mathcal{FT}_p^{G_1}\right\| = C\left\|T|\mathcal{FT}_p^{G_1\times G}\right\|.$$

The proof for cotype case is the same. □

By Lemma 3.8, we have the following relationship between Fourier properties when we consider direct product of groups.

Theorem 3.10 *Let G_1 and G_2 be unimodular groups with injective $VN(G_1)$ and $VN(G_2)$. Let $T : E_1 \to E_2$ and $S : E_2 \to E_3$ be a completely bounded linear maps between operator spaces. Then we have*

(1) $\left\|T|\mathcal{FT}_p^{G_1}\right\| \leq \left\|\mathcal{F}_{G_2}\right\|^{-1}_{L_p(G_2)\to L_{p'}(\varphi_{G_2})}\left\|T|\mathcal{FT}_p^{G_1\times G_2}\right\|$;

(2) $\left\|T|\mathcal{FC}_{p'}^{G_1}\right\| \leq \left\|\mathcal{F}_{G_2}^{-1}\right\|^{-1}_{L_p(\varphi_{G_2})\to L_{p'}(G_2)}\left\|T|\mathcal{FC}_{p'}^{G_1\times G_2}\right\|$;

(3) $\left\|ST|\mathcal{FT}_p^{G_1\times G_2}\right\| \leq \left\|T|\mathcal{FT}_p^{G_1}\right\|\left\|S|\mathcal{FT}_p^{G_2}\right\|$;

(4) $\left\|ST|\mathcal{FC}_{p'}^{G_1\times G_2}\right\| \leq \left\|T|\mathcal{FC}_{p'}^{G_1}\right\|\left\|S|\mathcal{FC}_{p'}^{G_2}\right\|$.

Proof. By the previous lemma, we have

$$\left\|T|\mathcal{FT}_p^{G_1}\right\| = \|\mathcal{F}_{G_1}\otimes T\|_{cb} \le \|\mathcal{F}_{G_2}\|^{-1}_{L_p(G_2)\to L_{p'}(\varphi_{G_2})}\left\|\mathcal{F}_{G_1}\otimes T|\mathcal{FT}_p^{G_2}\right\|$$
$$= \|\mathcal{F}_{G_2}\|^{-1}_{L_p(G_2)\to L_{p'}(\varphi_{G_2})}\left\|T|\mathcal{FT}_p^{G_1\times G_2}\right\|,$$

and we have

$$\left\|ST|\mathcal{FT}_p^{G_1\times G_2}\right\| = \left\|\mathcal{F}_{G_1}\otimes ST|\mathcal{FT}_p^{G_2}\right\| = \left\|S(\mathcal{F}_{G_1}\otimes T)|\mathcal{FT}_p^{G_2}\right\|$$
$$\le \left\|S|\mathcal{FT}_p^{G_2}\right\|\|\mathcal{F}_{G_1}\otimes T\|_{cb} = \left\|S|\mathcal{FT}_p^{G_2}\right\|\left\|T|\mathcal{FT}_p^{G_1}\right\|,$$

by Theorem 3.4. This proves (1) and (3). The proof for (2) and (4) is similar. □

Theorem 3.10 says that Fourier properties behave well with respect to direct product of group. That leads us to further investigation about the case that groups are combined by weaker relationship, but we only get the following restricted results.

Theorem 3.11 *Let G be a unimodular group with injective $VN(G)$ and $T : E \to F$ has G-Fourier type p(resp. cotype p').*

(1) *Let H be an open subgroup of G with injective $VN(H)$. Then T has H-Fourier type p(resp. cotype p') with*

$$\left\|T|\mathcal{FT}_p^{H}\right\| \le \left\|T|\mathcal{FT}_p^{G}\right\| \,(resp.\, \left\|T|\mathcal{FC}_{p'}^{H}\right\| \le \left\|T|\mathcal{FC}_{p'}^{G}\right\|).$$

(2) *Suppose that G is second countable and let H be a compact normal subgroup of G with injective $VN(G/H)$. Then T has G/H-Fourier type p(resp. cotype p') with*

$$\left\|T|\mathcal{FT}_p^{G/H}\right\| \le \left\|T|\mathcal{FT}_p^{G}\right\| \,(resp.\, \left\|T|\mathcal{FC}_{p'}^{G/H}\right\| \le \left\|T|\mathcal{FC}_{p'}^{G}\right\|).$$

Proof. (1) First of all, since the restriction of a Haar measure of G to H is also a Haar measure of H, H is also unimodular. Now we prove our theorem by showing that several specific maps are complete contractions. Let Φ_r and Ψ_r for $1 \le r \le \infty$ be given by:

$$\begin{array}{ll}\Phi_r : L_r(H) \to L_r(G), & \Psi_r : L_r(G) \to L_r(H)\\ \qquad f \mapsto \widetilde{f} & \qquad g \mapsto g|_H\end{array}$$

where $\widetilde{f}$ is the extension of f to whole G by giving 0 outside H. Let ϕ_r and ψ_r be given by:

$$\begin{array}{ll}\phi_r : L_r(\varphi_H) \to L_r(\varphi_G), & \psi_r : L_r(\varphi_G) \to L_r(\varphi_H)\\ \qquad L_f \mapsto L_{\widetilde{f}} & \qquad L_g \mapsto L_{g|_H}\end{array}$$

where $\widetilde{f}$ is the extension of f as the above. If we can show that $\Phi_p\otimes I_E$, $\Psi_{p'}\otimes I_F$, $\phi_p\otimes I_E$ and $\psi_{p'}\otimes I_F$ are complete contractions between corresponding vector-valued Lebesgue spaces then the proof is over since $\mathcal{F}_H\otimes T$ and $\mathcal{F}_H^{-1}\otimes T$ factorizes as follows:

$$\begin{array}{ccc}L_p(H,E) & \xrightarrow{\mathcal{F}_H\otimes T} & L_{p'}(\varphi_H,F)\\ {\scriptstyle\Phi_p\otimes I_E}\big\downarrow & & \big\uparrow{\scriptstyle\psi_{p'}\otimes I_F}\\ L_p(G,E) & \xrightarrow[\mathcal{F}_G\otimes T]{} & L_{p'}(\varphi_G,F)\end{array}$$

and

$$\begin{array}{ccc}L_p(\varphi_H,E) & \xrightarrow{\mathcal{F}_H^{-1}\otimes T} & L_{p'}(H,F)\\ {\scriptstyle\phi_p\otimes I_E}\big\downarrow & & \big\uparrow{\scriptstyle\Psi_{p'}\otimes I_F}\\ L_p(\varphi_G,E) & \xrightarrow[\mathcal{F}_G^{-1}\otimes T]{} & L_{p'}(G,F).\end{array}$$

First we consider $\Phi_p \otimes I_E$ and $\Psi_{p'} \otimes I_F$. It is easily seen that Φ_∞, Ψ_∞, Ψ_1 and Ψ_1 are contractive. Since their source and target spaces are L_∞ and L_1-spaces, we have that they are complete contractions, and consequently so are $\Phi_\infty \otimes I_E$, $\Psi_\infty \otimes I_F$, $\Phi_1 \otimes I_E$ and $\Psi_1 \otimes I_F$ by extension properties of injective and projective tensor products in operator spaces. Then by interpolation, we get complete contractions $\Phi_p \otimes I_E$ and $\Psi_{p'} \otimes I_F$.

For the next, we consider $\phi_p \otimes I_E$ and $\psi_{p'} \otimes I_F$. Note that $S_\infty^n \subseteq \mathcal{L}(S_2^n)$. Then we have for $f \in C_c(H, S_\infty^n)$ that

$$\begin{aligned}
\left\|I_{S_\infty^n} \otimes \phi_\infty(L_f)\right\|^2_{M_n(VN(G))} &= \left\|L_{\widetilde{f}}\right\|^2_{L^0_\infty(\varphi_G, S_\infty^n)} = \sup_{\|g\|_{L_2(G,S_2^n)}\le 1} \left\|L_{\widetilde{f}}(g)\right\|^2_{L_2(G)} \\
&= \sup_{\|g\|_2 \le 1} \int_G \left\| \int_G \widetilde{f}(y) g(y^{-1}x) dy \right\|^2_{S_2^n} dx \\
&= \sup_{\|g\|_2 \le 1} \int_G \left\| \int_H f(y) g(y^{-1}x) dy \right\|^2_{S_2^n} dx \\
&= \sup_{\|g\|_2 \le 1} \sum_{Hx} \int_H \left\| \int_H f(y) g(y^{-1}\xi x) dy \right\|^2_{S_2^n} d\xi,
\end{aligned}$$

where $M_n(\cdot)$ means n-th matrix level([6]), and the last equality is obtained when we apply Weil's formula for the right Haar measure which is the same with left Haar measure by the unimodularity of G. Thus if we set $g_x(y) = g(yx)$, then we get

$$\begin{aligned}
\left\|I_{S_\infty^n} \otimes \phi_\infty(L_f)\right\|^2_{M_n(VN(G))} &= \sup_{\|g\|_2 \le 1} \sum_{Hx} \int_H \left\| \int_H f(y) g_x(y^{-1}\xi) dy \right\|^2_{S_2^n} d\xi \\
&\le \sup_{\|g\|_2 \le 1} \|L_f\|^2_{M_n(VN(H))} \sum_{Hx} \int_H \|g_x(y)\|^2_{S_2^n} dy \\
&= \sup_{\|g\|_2 \le 1} \|L_f\|^2_{M_n(VN(H))} \|g\|_2^2 \le \|L_f\|^2_{M_n(VN(H))}.
\end{aligned}$$

Since such L_f's are dense in $M_n(VN(H))$, we get a complete contraction ϕ_∞ by (1) of Proposition 2.2 which means that so is $\phi_\infty \otimes I_E$.

Also we have for $g \in C_c(G, S_\infty^n)$ that

$$\begin{aligned}
\left\|I_{S_\infty^n} \otimes \psi_\infty(L_g)\right\|^2_{M_n(VN(H))} &= \left\|L_{g|_H}\right\|^2_{L^0_\infty(\varphi_H, S_\infty^n)} = \sup_{\|h\|_{L_2(H,S_2^n)}\le 1} \left\|L_{g|_H}(h)\right\|^2_{L_2(H,S_2^n)} \\
&= \sup_{\|h\|_2 \le 1} \int_H \left\| \int_H g|_H(y) h(y^{-1}x) dy \right\|^2_{S_2^n} dx \\
&= \sup_{\|h\|_2 \le 1} \int_G \left\| \int_G g(y) \widetilde{h}(y^{-1}x) dy \right\|^2_{S_2^n} dx \\
&\le \sup_{\|h\|_2 \le 1} \|L_g\|^2_{M_n(VN(G))} \left\|\widetilde{h}\right\|^2_{L_2(G,S_2^n)} \\
&\le \|L_g\|^2_{M_n(VN(G))},
\end{aligned}$$

where $\widetilde{h}$ is the extension of h with 0 outside H. Thus we get ψ_∞ is a complete contraction, and so is $\psi_\infty \otimes I_F$.

In the case of ϕ_1, for any $f \in C_c(H, S_1^n)$, we have

$$\begin{aligned}
\left\|I_{S_1^n} \otimes \phi_1(L_f)\right\|_{L_1(\varphi_G, S_1^n)} &= \left\|L_{\widetilde{f}}\right\|_{L_1(\varphi_G, S_1^n)} = \sup_{\|L_g\|_{L^0_\infty(\varphi_G, S_\infty^n)}\le 1} \left|\langle L_{\widetilde{f}}, L_g\rangle\right| \\
&\le \sup_{\left\|L_{g|_H}\right\|_{L^0_\infty(\varphi_H, S_\infty^n)}\le 1} \left|\langle L_f, L_{g|_H}\rangle\right| \le \|L_f\|_{L_1(\varphi_H, S_1^n)}.
\end{aligned}$$

The first inequality is by the above result about ψ_∞. Thus we get a complete contraction $\phi_1 \otimes I_E$ by the same argument as above which leads by interpolation to a complete contraction $\phi_p \otimes I_E$.

Similarly, we have for $g \in C_c(G, S_1^n)$ that

$$\left\|I_{S_1^n} \otimes \psi_1(L_g)\right\|_{L_1(\varphi_H, S_1^n)} = \left\|L_{g|_H}\right\|_{L_1(\varphi_H, S_1^n)} = \sup_{\|L_h\|_{L^0_\infty(\varphi_H, S^n_\infty)} \leq 1} \left|\langle L_{g|_H}, L_h\rangle\right|$$
$$\leq \sup_{\left\|L_{\widetilde{h}}\right\|_{L^0_\infty(\varphi_G, S^n_\infty)} \leq 1} \left|\langle L_g, L_{\widetilde{h}}\rangle\right| \leq \|L_g\|_{L_1(\varphi_G, S_1^n)},$$

where $\widetilde{h}$ is the extension of h to whole G in the same manner we extend f. The first inequality is by the above result about ϕ_∞. Thus we get a complete contraction $\psi_1 \otimes I_F$ by the same argument as above which leads by interpolation to the complete contraction $\psi_{p'} \otimes I_F$.

(2) Since H is compact, it is unimodular. Thus by Weil's formula we can easily show that G/H is also unimodular. Now we follow the same procedure as the above. Let $\Phi_r : L_r(G/H) \to L_r(G)$ be given by $\Phi_r(f)(x) = \widetilde{f}(x) = f(xH)$ for $x \in G$, and let $\Psi_r : L_r(G) \to L_r(H)$ be given by

$$\Psi_r(\widetilde{f})(xH) = f(xH) = \int_H \widetilde{f}(xh)dh$$

for $1 \leq r \leq \infty$ and $x \in G$. Let $\phi_r : L_r(\varphi_H) \to L_r(G)$ be given by $\phi_r(L_f) = L_{\widetilde{f}}$ where $\widetilde{f}(x) = f(xH)$ for $x \in G$ and $\psi_r : L_r(\varphi_G) \to L_r(\varphi_H)$ be given by $\psi_r(L_{\widetilde{f}}) = L_f$ where

$$f(xH) = \int_H \widetilde{f}(xh)dh$$

for $1 \leq r \leq \infty$ and $x \in G$.

We can easily see that Φ_∞, Ψ_∞, Φ_1 and Ψ_1 are contractive. Thus we get complete contractions $\Phi_p \otimes I_E$ and $\Psi_{p'} \otimes I_F$ by the same reason as in the proof of (1).

Next, we consider $\phi_p \otimes I_E$ and $\psi_{p'} \otimes I_F$. Let $A \subseteq G$ be the Borel selection of the canonical quotient map $q : G \to G/H$ as in Lemma 2.5. By Remark 2.6 and Proposition 2.4, we have for $f \in C_c(H, S^n_\infty)$ that

$$\begin{aligned}
&\left\|I_{S^n_\infty} \otimes \phi_\infty(L_f)\right\|^2_{M_n(VN(G))} \\
&= \left\|L_{\widetilde{f}}\right\|^2_{L^0_\infty(\varphi_G, S^n_\infty)} = \sup_{\|g\|_{L_2(G, S^n_2)} \leq 1} \left\|L_{\widetilde{f}}(g)\right\|^2_{L_2(G)} \\
&= \sup_{\|g\|_2 \leq 1} \int_G \left\|\int_G \widetilde{f}(y) g(y^{-1}x)dy\right\|^2_{S^n_2} dx \\
&= \sup_{\|g\|_2 \leq 1} \int_A \int_H \left\|\int_A \int_H f \circ q(a) g(h^{-1}a^{-1}a'h')dhda\right\|^2_{S^n_2} dh'da' \\
&= \sup_{\|g\|_2 \leq 1} \int_H \int_A \left\|\int_A f \circ q(a) \Big[\int_H g(h^{-1}a^{-1}a'h')dh\Big] da\right\|^2_{S^n_2} da'dh' \\
&\leq \sup_{\|g\|_2 \leq 1} \int_H \|L_f\|^2_{M_n(VN(G/H))} \Big[\int_A \left\|\int_H g(h^{-1}ah')dh\right\|^2_{S^n_2} da\Big] dh' \\
&\leq \|L_f\|^2_{M_n(VN(G/H))} \sup_{\|g\|_{L_2} \leq 1} \int_G \left\|\int_H g(h^{-1}x)dh\right\|^2_{S^n_2} dx.
\end{aligned}$$

By applying Proposition 2.4, Lemma 2.5 and Remark 2.6 for right cosets, we have, for the Borel selection $B \in G$ of $q' : G \to G/H, x \mapsto Hx$

$$\begin{aligned}
&\left\|I_{S^n_\infty} \otimes \phi_\infty(L_f)\right\|^2_{M_n(VN(G))} \\
&\leq \|L_f\|^2_{M_n(VN(G/H))} \sup_{\|g\|_2 \leq 1} \int_B \Big[\int_H \left\| \int_H g(h^{-1}h'b)dh \right\|^2_{S^n_2} dh' \Big] db \\
&= \|L_f\|^2_{M_n(VN(G/H))} \sup_{\|g\|_2 \leq 1} \int_B \Big[\int_H \left\| \int_H g_b(h)dh \right\|^2_{S^n_2} dh' \Big] db \\
&\leq \|L_f\|^2_{M_n(VN(G/H))} \sup_{\|g\|_2 \leq 1} \int_B \Big[\int_H \|g_b(h)\|^2_{S^n_2} dh \Big] db \\
&= \|L_f\|^2_{M_n(VN(G/H))} \sup_{\|g\|_2 \leq 1} \int_B \Big[\int_H \|g(hb)\|^2_{S^n_2} dh \Big] db \\
&\leq \|L_f\|^2_{M_n(VN(G/H))} .
\end{aligned}$$

The first equality is due to the fact that $h \mapsto h'h^{-1}$ is measure preserving, and the second inequality is by standard Minkowski inequality. Thus we get a complete contraction ϕ_∞ by (1) of Proposition 2.2 so that $\phi_\infty \otimes I_E$ is completely contractive also.

For ψ_∞, we consider $\widetilde{f} \in C_c(G, S^n_\infty)$. Then we have

$$\begin{aligned}
&\left\|I_{S^n_\infty} \otimes \phi_\infty(L_{\widetilde{f}})\right\|^2_{M_n(VN(G/H))} \\
&= \|L_f\|^2_{L^0_\infty(\varphi_{G/H}, S^n_\infty)} = \sup_{\|g\|_{L_2(G/H, S^n_2)} \leq 1} \|L_f(g)\|^2_{L_2(G/H)} \\
&= \sup_{\|g\|_2 \leq 1} \int_{G/H} \left\| \int_{G/H} f(xH) g(x^{-1}x'H) dxH \right\|^2_{S^n_2} dx'H \\
&= \sup_{\|g\|_2 \leq 1} \int_{G/H} \left\| \int_{G/H} \Big[\int_H \widetilde{f}(xh)dh \Big] g(x^{-1}x'H) dxH \right\|^2_{S^n_2} dx'H \\
&= \sup_{\|g\|_2 \leq 1} \int_{G/H} \int_H \left\| \int_{G/H} \int_H \widetilde{f}(xh) \widetilde{g}(h^{-1}x^{-1}x'h') dh dxH \right\|^2_{S^n_2} dh' dx'H,
\end{aligned}$$

where $\widetilde{g}(x) = g(xH)$. The last line is by the fact that H is compact and $\widetilde{g}(h^{-1}x^{-1}x'h') = g(x^{-1}x'H)$. Since $\|\widetilde{g}\|_{L_2(G, S^n_2)} = \|g\|_{L_2(H, S^n_2)}$, by Weil's formula, we have

$$\begin{aligned}
\left\|I_{S^n_\infty} \otimes \phi_\infty(L_{\widetilde{f}})\right\|^2_{M_n(VN(G/H))} &= \sup_{\|\widetilde{g}\|_{L_2(G, S^n_2)} \leq 1} \int_G \left\| \int_G \widetilde{f}(y) \widetilde{g}(y^{-1}x) dy \right\|^2_{S^n_2} dx \\
&\leq \left\| L_{\widetilde{f}} \right\|^2_{M_n(VN(G))} .
\end{aligned}$$

Thus we get another complete contraction $\psi_\infty \otimes I_F$.

For $\phi_1 \otimes I_E$ and $\psi_1 \otimes I_F$, we can show that they are completely contractive by the same argument as in the proof of (1) using the previous results about ϕ_∞ and ψ_∞. Then by the interpolation again, we get complete contractions $\phi_p \otimes I_E$ and $\psi_{p'} \otimes I_F$. This proves (2). □

4 Fourier types with respect to abelian groups

In this section, we are going to focus on abelian groups. Since we have many results for locally compact abelian groups in Banach space setting([1], [8] and [12]), our main theme in this section would be to extend those results into operator space setting. After that, a question about compatibility between operator space case and Banach space case is presented. Note that we don't need to consider Fourier cotype when we deal with abelian groups by Theorem 3.7 and the fact that Fourier inverse transform is essentially the same with Fourier transform with respect to dual group. Furthermore, we only mention that we can prove stronger duality theorem analogous to that in [12] by the same approach. Also note that by Remark 2.7, if G is abelian then $L_p(\varphi_G, E) \cong L_p(\widehat{G}, E)$ completely isometrically under the mapping $L_f \mapsto \widehat{f}^G$, where $\widehat{G}$ is the dual group of G.

We present equivalence theorems between Fourier types with respect to classical abelian groups $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{T}$. In the proof, we use the same idea as in Banach space case and the same constant

$$B_r = \inf_{\theta\in\mathbb{R}} \Big(\sum_{k\in\mathbb{Z}} \left| \frac{\sin\theta}{\theta + k\pi} \right|^r \Big)^{\frac{1}{r}}$$

for $1 \le r < \infty$, which is found in [8]. Note that $B_r > 0$ for $1 \le r < \infty$ and $B_r \le B_2 = 1$ for $r \ge 2$.

Theorem 4.1 *Let $T : E \to F$ be a linear map between operator spaces. Then we have for $d \in \mathbb{N}$,*

(1) $\left\| T|\mathcal{FT}_p^{\mathbb{R}^d} \right\| \le \left\| T|\mathcal{FT}_p^{\mathbb{Z}^d} \right\| \le B_{p'}^{-d} \left\| T|\mathcal{FT}_p^{\mathbb{R}^d} \right\|.$

(2) $\left\| T|\mathcal{FT}_p^{\mathbb{R}^d} \right\| \le \left\| T|\mathcal{FT}_p^{\mathbb{T}^d} \right\| \le B_{p'}^{-d} \left\| T|\mathcal{FT}_p^{\mathbb{R}^d} \right\|.$

P r o o f. We only prove (1) and the case $d = 1$, because (2) is implied by (1) and Theorem 3.7. For general d, we can apply Lemma 3.9. Suppose that T has $\mathbb{Z}$-Fourier type p. In order to check that T has $\mathbb{R}$-Fourier type p, we have to consider uniform boundedness of

$$I_{S^n_{p'}} \otimes (\mathcal{F}^{\mathbb{R}} \otimes T) : S^n_{p'}(L_p(\mathbb{R}, E)) \to S^n_{p'}(L_{p'}(\mathbb{R}, F)).$$

Let $f_{ij}(t) = \sum_{m\in\mathbb{Z}} 1_{[m\delta,(m+1)\delta)}(t) x^{ij}_m$ where $x^{ij}_m \in E$ and $\delta > 0$. Then we have

$$\widehat{f_{ij}}^{\mathbb{R}}(s) = \sum_{m\in\mathbb{Z}} e^{-2\pi i m\delta s} \frac{1 - e^{-2\pi i\delta s}}{2\pi i s} x^{ij}_m$$

and

$$\begin{aligned}
\left\| (T\widehat{f_{ij}}^{\mathbb{R}}) \right\|^{p'}_{L_{p'}(\mathbb{R}, S^n_{p'}(F))} &= \delta^{p'-1} \int_{\mathbb{R}} \left| \frac{\sin \pi s}{\pi s} \right|^{p'} \left\| \sum_{m\in\mathbb{Z}} (Tx^{ij}_m) e^{-2\pi i m s} \right\|^{p'}_{S^n_{p'}(F)} ds \\
&= \delta^{p'-1} \int_0^1 \sum_{k\in\mathbb{Z}} \left| \frac{\sin \pi s}{\pi(s+k)} \right|^{p'} \left\| \sum_{m\in\mathbb{Z}} (Tx^{ij}_m) e^{-2\pi i m(s+k)} \right\|^{p'}_{S^n_{p'}(F)} ds \\
&\le \delta^{p'-1} \int_0^1 \left\| \sum_{m\in\mathbb{Z}} (Tx^{ij}_m) e^{-2\pi i m s} \right\|^{p'}_{S^n_{p'}(F)} ds \\
&\le \delta^{p'-1} \left\| T|\mathcal{FT}^{\mathbb{Z}}_p \right\|^{p'} \left\| (x^{ij}_m) \right\|^{p'}_{S^n_{p'}(L_p(\mathbb{Z},E))}.
\end{aligned}$$

Thus we get $\left\| (T\widehat{f_{ij}}^{\mathbb{R}}) \right\|_{L_{p'}(\mathbb{R}, S^n_{p'}(E))} \le \left\| T|\mathcal{FT}^{\mathbb{Z}}_p \right\| \left\| (f_{ij}) \right\|_{S^n_{p'}(L_p(\mathbb{R},E))}$ if we prove

$$\left\| (f_{ij}) \right\|_{S^n_{p'}(L_p(\mathbb{R},E))} = \delta^{\frac{1}{p}} \left\| (x^{ij}_m) \right\|_{S^n_{p'}(L_p(\mathbb{Z},E))}. \tag{4.1}$$

Since functions like (f_{ij}) is dense in $S^n_{p'}(L_p(\mathbb{R}, E))$ we get the left inequality of the theorem.

Now we prove (4.1). Consider $\Phi_r : L_r(\mathbb{Z}) \to L_r(\mathbb{R})$ and $\Psi_r : L_r(\mathbb{R}) \to L_r(\mathbb{Z})$ given by $\Phi_r((a_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} 1_{[n\delta,(n+1)\delta)} a_n$ and $\Psi_r(f)_n = \frac{1}{\delta} \int_{n\delta}^{(n+1)\delta} f(t)dt$ for all $1 \le r \le \infty$. It is easy to see that $\|\Phi_\infty\| =$

$\|\Psi_\infty\| = 1$, $\|\Phi_1\| \le \delta$ and $\|\Psi_1\| \le \frac{1}{\delta}$ so that they are completely bounded with the same c.b. norm since source and target spaces are L_∞ and L_1. Thus we have that $\Phi_\infty \otimes I_E$, $\Psi_\infty \otimes I_E$, $\Phi_1 \otimes I_E$ and $\Psi_1 \otimes I_E$ are completely bounded with the same c.b. norm so that by the interpolation we get completely bounded map $\Phi_p \otimes I_E$ and $\Psi_p \otimes I_E$ with $\|\Phi_p \otimes I_E\|_{cb} \le \delta^{\frac{1}{p}}$ and $\|\Psi_p \otimes I_E\|_{cb} \le \delta^{-\frac{1}{p}}$. This proves (4.1).

For the right inequality, we consider

$$I_{S^n_{p'}} \otimes (\mathcal{F}^{\mathbb{Z}} \otimes T) : S^n_{p'}(L_p(\mathbb{Z}, E)) \to S^n_{p'}(L_{p'}(\mathbb{T}, F)).$$

Let $f_{ij}(t) = \sum_{m\in\mathbb{Z}} 1_{[m,(m+1))}(t) x^{ij}_m$. Then we have

$$\widehat{f_{ij}}^{\mathbb{R}}(s) = \sum_{m\in\mathbb{Z}} e^{-2\pi i m s} \frac{1-e^{-2\pi i s}}{2\pi i s} x^{ij}_m$$

and

$$\begin{aligned}
\left\|(T\widehat{f_{ij}}^{\mathbb{R}})\right\|^{p'}_{L_{p'}(\mathbb{R},S^n_{p'}(F))} &= \int_{\mathbb{R}} \left|\frac{\sin \pi s}{\pi s}\right|^{p'} \left\|\sum_{m\in\mathbb{Z}} (Tx^{ij}_m) e^{-2\pi i m s}\right\|^{p'}_{S^n_{p'}(F)} ds \\
&= \int_0^1 \sum_{k\in\mathbb{Z}} \left|\frac{\sin \pi s}{\pi(s+k)}\right|^{p'} \left\|\sum_{m\in\mathbb{Z}} (Tx^{ij}_m) e^{-2\pi i m(s+k)}\right\|^{p'}_{S^n_{p'}(F)} ds \\
&\ge B^{p'}_{p'} \int_0^1 \left\|\sum_{m\in\mathbb{Z}} (Tx^{ij}_m) e^{-2\pi i m s}\right\|^{p'}_{S^n_{p'}(F)} ds.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
\left\|\widehat{(Tx^{ij}_m)}^{\mathbb{Z}}\right\|_{L_{p'}(\mathbb{R},S^n_{p'}(F))} &\le B^{-1}_{p'} \left\|(T\widehat{f_{ij}}^{\mathbb{R}})\right\|^{p'}_{L_{p'}(\mathbb{R},S^n_{p'}(F))} \\
&\le B^{-1}_{p'} \left\|T|\mathcal{FT}^{\mathbb{R}}_p\right\| \|(f_{ij})\|_{S^n_{p'}(L_p(\mathbb{R},E))} \\
&= B^{-1}_{p'} \left\|T|\mathcal{FT}^{\mathbb{R}}_p\right\| \left\|(x^{ij}_m)\right\|_{S^n_{p'}(L_p(\mathbb{Z},E))},
\end{aligned}$$

where the last inequality is by (4.1) with $\delta = 1$. This proves the second inequality. □

There is another equivalence relationship about classical groups.

Theorem 4.2 *Let $T : E \to F$ be a linear map between operator spaces. Let p be $1 \le p \le 2$ and p' is the conjugate exponent of p. Then we have for $d \in \mathbb{N}$,*

(1) $\left\|T|\mathcal{FT}^{\mathbb{Z}^d}_p\right\| = \left\|T|\mathcal{FT}^{\mathbb{Z}}_p\right\|$.

(2) $\left\|T|\mathcal{FT}^{\mathbb{T}^d}_p\right\| = \left\|T|\mathcal{FT}^{\mathbb{T}}_p\right\|$.

P r o o f. We only consider (1) and $d = 2$ case for the same reason as in the previous theorem. Since $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ we have $\left\|T|\mathcal{FT}^{\mathbb{Z}}_p\right\| \le \left\|T|\mathcal{FT}^{\mathbb{Z}^2}_p\right\|$ by Theorem 3.10. For the converse inequality, consider

$$I_{S^n_{p'}} \otimes (\mathcal{F}_{\mathbb{Z}^2} \otimes T) : S^n_{p'}(L_p(\mathbb{Z}^2, E)) \to S^n_{p'}(L_{p'}(\mathbb{T}^2, F)).$$

Then for any $(x^{lm}_{ij}) \in S^n_{p'}(L_p(\mathbb{Z}^2, E))$ where $1 \le i, j \le n$ and $1 \le |l|, |m| \le N$ for $N \in \mathbb{N}$ we have

$$\left\|\widehat{(Tx^{lm}_{ij})}^{\mathbb{Z}^2}\right\|^{p'}_{L_{p'}(\mathbb{T}^2,S^n_{p'}(F))} = \int_0^1\int_0^1 \left\|\sum_{1\le|l|,|m|\le N} (Tx^{lm}_{ij}) e^{-2\pi i(ls+mt)}\right\|^{p'}_{S^n_{p'}(F)} dtds$$

If we set $M = 2N+1$, then $l + mM$ are distinct for all $1 \le |l|, |m| \le N$ and $(e^{-2\pi imt})_{|m|\le N}$ has the same distribution with $(e^{-2\pi imMt})_{|m|\le N}$. Thus by the translation invariance and the previous observation we get

$$\begin{aligned}\left\|\widehat{(Tx_{ij}^{lm})}^{\mathbb{Z}^2}\right\|^{p'}_{L_{p'}(\mathbb{T}^2, S^n_{p'}(F))} &= \int_0^1\int_0^1 \left\|\sum_{1\le|l|,|m|\le N} (Tx_{ij}^{lm}) e^{-2\pi i(ls+mMt)}\right\|^{p'}_{S^n_{p'}(F)} dtds \\ &= \int_0^1\int_0^1 \left\|\sum_{1\le|l|,|m|\le N} (Tx_{ij}^{lm}) e^{-2\pi ils} e^{-2\pi i(l+mM)t}\right\|^{p'}_{S^n_{p'}(F)} dtds \\ &\le \int_0^1 \left\|T|\mathcal{FT}^{\mathbb{Z}}_p\right\|^{p'} \left\|(x_{ij}^{lm} e^{-2\pi ils})\right\|^{p'}_{S^n_{p'}(L_p(\mathbb{Z},E))} ds \\ &= \left\|T|\mathcal{FT}^{\mathbb{Z}}_p\right\|^{p'} \left\|(x_{ij}^{lm})\right\|^{p'}_{S^n_{p'}(L_p(\mathbb{Z}^2,E))}.\end{aligned}$$

The last line holds since we have complete isometry $L_p(\mathbb{Z}^2, E) \simeq L_p(\mathbb{Z}, E)$; $(x^{lm}) \mapsto (y^k)$ where $y^k = x^{\rho(k)}$ for a bijection $\rho : \mathbb{Z} \to \mathbb{Z}^2$. This complete isometry is obtained as in the proof of (4.1) in Theorem 4.1. Since we take N arbitrarily we get the desired result. □

Now we compare two Fourier types with respect to a locally compact abelian group G, the first one is Fourier type of Banach space and the second one is Fourier type of operator space. Let E be a operator space which has G-Fourier type p in operator space sense. If we denote $E^{(1)}$ as the first matrix level of E, then

$$L^{(1)}_\infty(G, E) = (L_\infty(G) \otimes_{min} E)^{(1)} \cong L_\infty(G) \otimes_\lambda E^{(1)} = L_\infty(G, E^{(1)})$$

and

$$L^{(1)}_1(G, E) = (L_1(G)\widehat{\otimes}E)^{(1)} \cong L_1(G) \otimes_\gamma E^{(1)} = L_1(G, E^{(1)})$$

isometrically(chapter 8 of [6]), and consequently $L^{(1)}_p(G, E) \cong L_p(G, E^{(1)})$ isometrically by interpolation. Thus we have that $E^{(1)}$ has G-Fourier type p in Banach space sense. This naturally leads us to the question that whether the converse can be obtained or not.

Problem *Let G be a locally compact abelian group and X be a Banach space which has G-Fourier type p in Banach space sense. Can we give an operator space structure on X which has G-Fourier type p in operator space sense?*

We have trivial answer for $p = 2$ case. If we give an operator space structure on X by $(\min X, \max X)_{\frac{1}{2}}$, then for X, G-Fourier type 2 in Banach space sense means G-Fourier type 2 in operator space sense. This is by the fact that X has G-Fourier type 2 in Banach space sense if and only if X is isomorphic to a Hilbert space and for a Hilbert space $\mathcal{H}$, we have a complete isometry $(\min \mathcal{H}, \max \mathcal{H})_{\frac{1}{2}} \simeq OH_{\mathcal{H}}$ where $OH_{\mathcal{H}}$ is the operator Hilbert space([19]). In the case that $p < 2$, we could not answer at the time of this writing.

5 Fourier type and cotype with respect to the Heisenberg group

In this section, we concentrate on the Heisenberg group as an example of nonabelian and noncompact group. Since the Heisenberg group is unimodular and connected Lie group, we have the definition of Fourier type and cotype given in Definition 3.1. However representations of the Heisenberg group are well-known and easy to describe, so that we can present another equivalent definition of Fourier type and cotype using representation theory. The materials about the Heisenberg group which you will see in this section are mainly adopted from [22], and $S_r(1 \le r \le \infty)$ means the Schatten-von Neumann class defined on $L_2(\mathbb{R}^n)$ from now on.

We define the Heisenberg group H_n on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the group law given by

$$(x, y, t)(v, w, s) = (x + v, y + w, t + s + \frac{1}{2}(x \cdot w - y \cdot v)),$$

where $\cdot$ means the usual inner product in $\mathbb{R}^n$ and $n \in \mathbb{N}$. It is easily seen that the Lebesgue measure $dxdydt$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is both left and right translation invariant. This makes H_n unimodular. By the Stone-von Neumann theorem, we have a complete list of all irreducible unitary representations of H_n, but we need only a part of it here. For each nonzero real λ, we define a unitary representation on H_n by

$$\pi_\lambda(x,y,t)\varphi(\xi) = e^{i\lambda t} e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y)$$

for $\varphi \in L_2(\mathbb{R}^n)$.

Now we define Fourier transform on H_n as follows: for each nonzero real λ and $f \in L_1(H_n)$, $\widehat{f}^{H_n}(\lambda)$ is the operator acting on $L_2(\mathbb{R}^n)$ by

$$\widehat{f}^{H_n}(\lambda)\varphi = \int_{H_n} f(z,t)\pi_\lambda(z,t)\varphi \, dzdt,$$

where $z = (x,y) \in \mathbb{C}^n$.

The followings are Plancherel's theorem and the Hausdorff-Young inequality for the Heisenberg group.

Theorem 5.1 (1) *The Fourier transform on the Heisenberg group is a complete isometry from $L_2(H_n)$ onto $L_2(d\mu, S_2)$ where $d\mu(\lambda) = (2\pi)^{-n-1} |\lambda|^n \, d\lambda$ on the set of nonzero reals $\mathbb{R}^*$.*

(2) *The Fourier transform on the Heisenberg group is a complete contraction from $L_p(H_n)$ into $L_{p'}(d\mu, S_{p'})$, and the inverse Fourier transform is a complete contraction from $L_p(d\mu, S_p)$ into $L_{p'}(H_n)$.*

Proof. Note that it is already known that the Fourier transform on the Heisenberg group is an isometry from $L_2(H_n)$ onto $L_2(d\mu, S_2)$ in [22, 13]. For the extension to the operator space setting, we follow the same procedure as in Theorem 2.8. □

One of the usual technic in analysis on the Heisenberg group is taking Fourier transform with respect to the last variable t. Then we get the following integral transform:

If we denote $f^\lambda(z) = \int_{\mathbb{R}} e^{i\lambda t} f(z,t)dt$, then we have

$$\widehat{f}^{H_n}(\lambda)\varphi = \int_{\mathbb{C}^n} f^\lambda(z)\pi_\lambda(z)\varphi \, dz,$$

where $\pi_\lambda(x,y)\varphi(\xi) = e^{i\lambda(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y)$, and this leads for us to consider another operator of the form

$$W_\lambda(g) = \int_{\mathbb{C}^n} g(z)\pi_\lambda(z) \, dz$$

for functions on $\mathbb{C}^n$.

When $\lambda = 1$, we call this the Weyl transform and denote it by $W(g)$. We have the Plancherel theorem for the Weyl transform as follows:

$$\|W(g)\|_{S_2} = (2\pi)^{\frac{n}{2}} \|g\|_{L_2(\mathbb{C}^n)}.$$

Then by the change of variables, we get

$$\|W_\lambda(g)\|_{S_2} = (2\pi)^{\frac{n}{2}} |\lambda|^{-\frac{n}{2}} \|g\|_2. \tag{5.1}$$

Furthermore, we have for $\varphi, \psi \in L_2(\mathbb{R}^n)$,

$$\langle W_\lambda(g)\varphi, \psi\rangle = \int_{\mathbb{C}^n} g(z)\langle \pi_\lambda(z,t)\varphi, \psi\rangle \, dz.$$

Since $\pi_\lambda(z)$ is unitary, it follows that

$$|\langle \pi_\lambda(z)\varphi, \psi\rangle| \leq \|\varphi\|_2 \|\psi\|_2,$$

so that we have

$$|\langle W_\lambda(g)\varphi, \psi\rangle| \leq \|\varphi\|_2 \|\psi\|_2 \|g\|_1.$$

This means that

$$\|W_\lambda(g)\|_{S_\infty} \leq \|g\|_1. \tag{5.2}$$

Combining (5.1) and (5.2) by interpolation with the parameter $\theta = \frac{2}{p'}$, we get the following lemma.

Lemma 5.2 *For any nonzero real λ and $g \in L_p(\mathbb{C}^n)$, we have*

$$\|W_\lambda(g)\|_{S_{p'}} \leq (2\pi)^{\frac{n}{p'}} |\lambda|^{-\frac{n}{p'}} \|g\|_p .$$

Now we give another definition for Fourier type and cotype on the Heisenberg group.

Definition 5.3 Let $T : E \to F$ be a linear map between operator spaces and $\mathcal{F}_{H_n}$ be the Fourier transform mapping f to $\widehat{f}^{H_n}$.

(1) T is said to have H_n-Fourier type p if

$$\mathcal{F}_{H_n} \otimes T : L_p(H_n) \otimes E \to L_{p'}(d\mu, S_{p'}) \otimes F$$

extends to a completely bounded map from $L_p(H_n, E)$ into $L_{p'}(d\mu, S_{p'}(F))$ and in this case we denote $\left\|T|\mathcal{FT}_p^{H_n}\right\| := \|\mathcal{F}_{H_n} \otimes T\|_{cb}$.

(2) T is said to have H_n-Fourier cotype p' if

$$\mathcal{F}_{H_n}^{-1} \otimes T : L_p(d\mu, S_p) \otimes E \to L_{p'}(H_n) \otimes F$$

extends to a completely bounded map from $L_p(d\mu, S_p(E))$ into $L_{p'}(H_n, F)$ and in this case we denote $\left\|T|\mathcal{FC}_{p'}^{H_n}\right\| := \left\|\mathcal{F}_{H_n}{}^{-1} \otimes T\right\|_{cb}$.

The definition for spaces is straightforward.

Remark 5.4 The Fourier transform on H_n takes convolution into products as in the commutative case, that is

$$\widehat{(f * g)}^{H_n}(\lambda) = \widehat{f}^{H_n}(\lambda)\widehat{g}^{H_n}(\lambda).$$

Since $f \mapsto \widehat{f}^{H_n}$ is a complete isometry, we have complete isometry $L^0_\infty(\varphi_{H_n}) \cong L_\infty(d\mu, S_\infty)$; $L_f \mapsto \widehat{f}^{H_n}$. Then by the same observation in Remark 3.2, we have complete isometries $L_r(\varphi_{H_n}, E) \cong L_r(d\mu, S_r(E))(1 \leq r \leq \infty)$; $L_f \otimes x \mapsto \widehat{f}^{H_n} \otimes x$ for any operator space E and $x \in E$. This implies that the definition of Fourier type and cotype on the Heisenberg group in this section is equivalent to those in Definition 3.1.

The above Fourier properties are related with classical Fourier type. The following transference theorem provides an example of partial connection between commutative and noncommutative case.

Theorem 5.5 *Let $T : E \to F$ be a linear map between operator spaces which has H_n-Fourier type p or H_n-Fourier cotype p'. Then T has $\mathbb{Z}$-Fourier type p.*

Proof. Consider $(x^k_{ij})_{k=1,2,\cdots} \in L_p(\mathbb{Z}, M_n(E))$, where $1 \leq i,j \leq n$ and define $f_{ij}(t) = \sum_{k\in\mathbb{Z}} 1_{[2\pi k, 2\pi(k+1))}(t)x^k_{ij}$. We extend f_{ij} to whole H_n by $\widetilde{f_{ij}}(x,y,t) = g(x,y)f_{ij}(t)$, where $g(x,y) = 1_{[0,1]^{2n}}(x,y)$ means characteristic function on $[0,1]^{2n} \subseteq \mathbb{C}^n$. Then we have

$$\widehat{\widetilde{f_{ij}}}^{H_n}(\lambda) = W_\lambda(g)\int_{\mathbb{R}} e^{i\lambda t} f_{ij}(t)dt = \varphi(\lambda)W_\lambda(g)\sum_{k\in\mathbb{Z}} e^{2\pi i\lambda k}x^k_{ij},$$

where

$$\varphi(\lambda) = \frac{e^{2\pi i\lambda} - 1}{i\lambda}.$$

Thus we get

$$\left\|(T\widehat{\widetilde{f_{ij}}}^{H_n}(\lambda))\right\|_{S^n_{p'}(S_{p'}(F))} = |\varphi(\lambda)| \, \|W_\lambda(g)\|_{S_{p'}} \left\|(\sum_{k\in\mathbb{Z}} e^{2\pi i\lambda k}Tx^k_{ij})\right\|_{S^n_{p'}(F)}$$

and consequently

$$\begin{aligned}\left\| (\widehat{Tf_{ij}}^{H_n}) \right\|^{p'}_{L_{p'}(d\mu, S^n_{p'}(S_{p'}(F)))} &= \int_{\mathbb{R}} \Phi(\lambda) \left\| (\sum_{k\in\mathbb{Z}} e^{2\pi i\lambda k} Tx^k_{ij}) \right\|^{p'}_{S^n_{p'}(F)} d\lambda \\ &= \int_0^1 \sum_{l\in\mathbb{Z}} \Phi(\lambda + l) \left\| (\sum_{k\in\mathbb{Z}} e^{2\pi i\lambda k} Tx^k_{ij}) \right\|^{p'}_{S^n_{p'}(F)} d\lambda,\end{aligned}$$

where $\Phi(\lambda) = (2\pi)^{-n-1} |\lambda|^n |\varphi(\lambda)|^{p'} \|W_\lambda(g)\|^{p'}_{S_{p'}}$.

We want to show that there is a constant C such that $0 < C \le \sum_{l\in\mathbb{Z}} \Phi(\lambda + l) < \infty$ for almost all $\lambda \in [0, 1]$. Now we claim that $\lambda \mapsto \sum_{|l|>1} \Phi(\lambda + l)$ is continuous on a compact interval $[0, 1]$. For each $|l| > 1$, it is trivial that $\lambda \mapsto \Phi(\lambda + l)$ is continuous on $[0, 1]$. Furthermore, since

$$|\varphi(\lambda)| \le \left|\frac{2}{\lambda}\right|,$$

we have by Lemma 5.2

$$|\Phi(\lambda)| \le (2\pi)^{-1} \left|\frac{2}{\lambda}\right|^{p'} \|g\|^{p'}_{L_p(\mathbb{C}^n)} = (2\pi)^{-1} \left|\frac{2}{\lambda}\right|^{p'}$$

for any $\lambda \in \mathbb{R}$. Thus we get

$$\sum_{|l|\ge N} |\Phi(\lambda + l)| \le \frac{2^{p'}}{2\pi} \sum_{|l|\ge N} \frac{1}{|(\lambda + l)|^{p'}} \le \frac{2^{p'}}{2\pi} \sum_{|l|\ge N-1} \frac{1}{|l|^{p'}}$$

for any $N \ge 2$ and $\lambda \in [0, 1]$. Since $p' \ge 2$, $\sum_{|l|>1} |\Phi(\lambda + l)|$ converges uniformly and this proves our claim. By the continuity of $\lambda \mapsto \sum_{|l|>1} \Phi(\lambda + l)$, we have $\lambda_0 \in [0, 1]$ such that

$$C := \sum_{|l|>1} \Phi(\lambda_0 + l) = \inf_{\lambda\in[0,1]} \sum_{|l|>1} \Phi(\lambda + l).$$

If we suppose that $C = 0$, then we have $\|W_{\lambda_0+l}(g)\|_{S_{p'}} = 0$, which means

$$\|W_{\lambda_0+l}(g)\|_{S_2} = (2\pi)^{\frac{n}{2}} |\lambda_0 + l|^{-\frac{n}{2}} \|g\|_{L_2(\mathbb{C}^n)} = 0,$$

and this is contradictory, so that

$$\sum_{l\in\mathbb{Z}} \Phi(\lambda + l) \ge C > 0$$

for almost all $\lambda \in [0, 1]$. Since φ is bounded on $[0, 1]$, we have by Lemma 5.2 again that $|\Phi(\lambda)|$ is uniformly bounded on $\mathbb{R} - \{0\}$, so that

$$\sum_{l\in\mathbb{Z}} \Phi(\lambda + l) = \sum_{|l|>1} \Phi(\lambda + l) + \sum_{|l|\le 1} \Phi(\lambda + l)$$

is finite almost all $\lambda \in [0, 1]$.

Now we have

$$\begin{aligned}\Big\|(\sum_{k\in\mathbb{Z}} e^{2\pi i\lambda k}Tx_{ij}^k)\Big\|_{S_{p'}^n(L_{p'}(\mathbb{T},F))} &= \Big[\int_0^1 \Big\|(\sum_{k\in\mathbb{Z}} e^{2\pi i\lambda k}Tx_{ij}^k)\Big\|_{S_{p'}^n(F)}^{p'} d\lambda\Big]^{\frac{1}{p'}} \\ &\le C^{-\frac{1}{p'}} \Big\|(T\widehat{\widetilde{f_{ij}}}^{H_n})\Big\|_{L_{p'}(d\mu,S_{p'}^n(S_{p'}(F)))} \\ &\le C^{-\frac{1}{p'}} \big\|T|\mathcal{FT}_p^{H_n}\big\| \,\Big\|(\widetilde{f_{ij}})\Big\|_{S_{p'}^n(L_p(H_n,E))} \\ &\le C^{-\frac{1}{p'}} \big\|T|\mathcal{FT}_p^{H_n}\big\| \,\|(f_{ij})\|_{S_{p'}^n(L_p(\mathbb{R},E))} \\ &= C^{-\frac{1}{p'}}(2\pi)^{\frac{1}{p}} \big\|T|\mathcal{FT}_p^{H_n}\big\| \,\|(x_{ij})\|_{S_{p'}^n(L_p(\mathbb{Z},E))}.\end{aligned}$$

Last two lines are by the fact that the extension $f \mapsto \widetilde{f} = 1_{[0,1]^{2n}} \otimes f$ is a complete contraction from $L_p(\mathbb{R},E)$ into $L_p(H_n,E)$ and (4.1) with $\delta = 2\pi$. Now we get the desired result with the following inequality:

$$\big\|T|\mathcal{FT}_p^{\mathbb{Z}}\big\| \le C^{-\frac{1}{p'}}(2\pi)^{\frac{1}{p}} \big\|T|\mathcal{FT}_p^{H_n}\big\|.$$

For the case that T has H_n-Fourier cotype p', we have that T^* has H_n-Fourier type p by the duality(Theorem 3.7). Then by the previous result, T^* has $\mathbb{Z}$-Fourier type p. By the duality again, T has $\mathbb{Z}$-Fourier cotype p' which is equivalent to $\mathbb{T}$-Fourier type p, and consequently T has $\mathbb{Z}$-Fourier type p by the equivalence of classical groups(Theorem 4.1).

Remark 5.6 By the previous theorem, Kwapien's theorem for the Heisenberg group follows; an operator space with H_n-Fourier type 2 or H_n-Fourier cotype 2 is completely isomorphic to an operator Hilbert space. This is by the fact that $\mathbb{Z}$-Fourier type 2 is equivalent to $\mathbb{T}$-Fourier type 2 and $\mathbb{T}$ is an infinite compact group.

□

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